ANOTHER CONFORMAL STRUCTURE ON IMMERSED SURFACES OF NEGATIVE CURVATURE

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1. Everyone is familiar with the ordinary conformal structure on oriented surfaces immersed smoothly in E^3. This standard structure is obtained by using the first fundamental form as metric tensor. It is possible, however, to define very different conformal structures which are still vitally connected with the geometry of a surface's immersion in E^3.

Consider, for instance, the conformal structure induced upon a strictly convex surface (oriented so that mean curvature $H > 0$) by using its positive definite second fundamental form as metric tensor. (See [3] and [4].) This particular structure coincides with the usual one only on spheres.

The present paper is devoted, principally, to a description of the corresponding non-standard conformal structure on oriented surfaces of negative Gaussian curvature immersed smoothly in E^3. This new structure is obtained by using a specific linear combination of the first and second fundamental forms as metric tensor. It will be seen that our new structure coincides with the usual one only on minimal surfaces.

Also included below is a section describing the arithmetic of certain expressions associated with the various fundamental forms on an immersed surface. These expressions become quadratic differentials whenever *any* paticular conformal structure is introduced on a surface.

The paper closes with a theorem which generalizes a well known fact about minimal surfaces. For investigations related to the material which follows, see [5].

2. Consider an oriented surface S which is C^3 immersed in E^3. We may number the principal curvatures so that

$$(1) \qquad k_1 \geqq k_2$$

holds over all of S. For convenience of notation, lines of curvature coordinates x, y will always be chosen so that k_1 is the principal curvature in the $y \equiv$ constant direction, while k_2 is the principal curvature in the $x \equiv$ constant direction.

We now define the function

$$H' = \frac{k_2 - k_1}{2} = -\sqrt{H^2 - K}\ .$$

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Note that the zeros of H' coincide with the umbilics on S, that H' is never positive, and that the value of H' (unlike H) is independent of the orientation on S. If we set

$$H'II' = KI - HII , \tag{2}$$

we obtain a new "fundamental form" II', defined at all non-umbilic points on S.[1] Of course, I is determined by II and II' wherever $K \neq 0$, and II is determined by I and II' wherever $H \neq 0$. Moreover, II' (unlike II) is independent of the orientation on S. The relationship of II to II' can be more clearly seen in the following lemma.

LEMMA 1. *If x, y are lines of curvature coordinates on S, so that*

$$I = Edx^2 + Gdy^2 ,$$
$$II = k_1Edx^2 + k_2Gdy^2 , \tag{3}$$

then, where II' is defined,

$$II' = k_1Edx^2 - k_2Gdy^2 . \tag{4}$$

Proof. Suppose $II' = L'dx^2 + 2M'dxdy + M'dy^2$. Then (2) yields

$$H'L' = k_1k_2E - \left(\frac{k_1 + k_2}{2}\right)k_1E = H'k_1E .$$

If II' is defined at all, $H' \neq 0$, so that $L' = k_1E$. Similarly, $M' = 0$, and $N' = -k_2G$, as claimed.

COROLLARY. *II' is positive definite on S if and only if $K < 0$.*

Proof. Wherever II' is defined or $K < 0$, there are no umbilics to consider. Thus (4) applies, and, using (1), the result is obvious.

LEMMA 2. *Just as $H' = 0$ charcterizes points where $I \propto II$, $H = 0$ characterizes points where $I \propto II'$, and $K = 0$ characterizes points where $II \propto II'$.*

Proof. The first fact merely recalls the definition of an umbilic point. The remaining facts follow easily from (2), recalling that $H' \neq 0$ wherever II' is defined.

REMARK. Elementary application of (2) reveals that lines of

[1] By continuity, II' could be sensibly defined at certain umbilics. We assume throughout this paper, however, that II' is defined only at non-umbilics.

curvature coordinates are characterized by $M' = F = 0$ wherever $H \neq 0$, and by $M' = M = 0$ wherever $K \neq 0$.

3. Suppose now that $K < 0$ on the surface S discussed above, so that II' is a C^1 positive definite form on S. Then C^2 coordinates x, y may be found in the neighborhood of any point on S in terms of which

$$II' = \mu'(x, y)\{dx^2 + dy^2\} ,$$

with $\mu' > 0$. (See § 4 of [1].) Such coordinates will be called disothermal. It is well known that distinct pairs of disothermal coordinates are related by the Cauchy Riemann equations, and that coordinates so related to a pair of disothermals are themselves disothermal. Thus we obtain on S the structure of a Riemann surface R_2', with conformal parameters $z = x + iy$ corresponding to disothermals x, y.

Of course, there is still the usual structure of a Riemann surface R_1 on S, determined by using conformal parameters $z = x + iy$ corresponding to isothermal coordinates x, y in terms of which

$$I = \lambda(x, y)\{dx^2 + dy^2\} .$$

By Lemma 2, R_1 and R_2' coincide on S if and only if S is a minimal surface. (We will also have occasion to mention below the Riemann surface R_2 determined on strictly convex surfaces by using conformal parameters $z = x + iy$ corresponding to bisothermal coordinates x, y in terms of which

$$II = \mu(x, y)\{dx^2 + dy^2\}$$

with $\mu > 0$.)

Assume now that x, y are disothermals on S. Then (2) becomes

$$\begin{aligned} HL + H'\mu' &= KE , \\ HN + H'\mu' &= KG , \\ HM &= KF . \end{aligned} \tag{5}$$

Thus, for instance, the equation for the directions of principal curvature reads

$$-Fdx^2 + (E - G)\,dxdy + Fdy^2 = 0 . \tag{6}$$

This is, incidentally, exactly the form which the same equation takes when using bisothermal coordinates on a strictly convex surface. (See [3] or [4].) Note that (6) depends only on I.

REMARK. Recalling the remark which closes § 2, we see that all conjugate disothermals are lines of curvature coordinates. And, where $H \neq 0$, all orthogonal disothermals are lines of curvature coordinates.

It will be helpful to note that since

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}, \tag{7}$$

asymptotic coordinates, which are characterized by $L = N = 0$, always yield the third equation of (5), so that $M' = 0$. Moreover, by (2), coordinates are asymptotic if and only if $H'L' = KE$, $H'N' = KG$. Thus the following can be said.

REMARK. Asymptotic coordinates are disothermal if and only if $E = G$.

The previous remark and the two results which follow characterize disothermal coordinates on S in terms of the coefficients of I and II. Lemma 3 is a trivial consequence of (2), (5), and the fact that $K < 0$ on S.

LEMMA 3. *Coordinates are disothermal on S if and only if*

$$H(L - N) = K(E - G), \quad HM = KF.$$

LEMMA 4. *Nonasymptotic coordinates are disothermal on a C^3 immersed surface if and only if*

$$L = -N \neq 0, \quad HM = KF. \tag{8}$$

Proof. Using (7), we obtain

$$2H(LN - M^2) = K(EN - 2FM + GL). \tag{9}$$

If x, y are disothermals, (5) holds. Thus $HM = KF$, and we may multiply the equations of (5) by N, L and M respectively, and combine the resulting expressions so as to obtain the right side of (9). This yields

$$2H(LN - M^2) = 2H(LN - M^2) + H'\mu'(L + N).$$

Since $H'\mu' \neq 0$, assuming x, y to be nonasymptotic, we have $L = -N \neq 0$. Suppose, on the other hand, that (8) holds when using coordinates x, y. Then multiplying the equations of (2) by L, N and M respectively, and proceeding as above, we obtain

$$2H(LN - M^2) = -2H(L^2 + M^2) + LH'(N' - L') .$$

Since $L = -N \neq 0$ while $H' \neq 0$, it follows that $L' = N'$. But $M' = 0$ by (8) and (2), so that x, y are disothermals.

Many formulas may be simplified, of course, by using disothermals and (8). We note here only that

$$L' = N' = \mu' = \frac{K(E+G)}{2H'} ,$$

while

$$L = -N = \frac{K(E-G)}{2H} .$$

4. In [2] and [6], use is made of the quadratic differential $\Omega_2 = \phi_2 dz^2$ on R_1, where

$$\phi_2 = \frac{L-N}{2} - iM .$$

In [4], use is made of the quadratic differential $\Omega_1 = \phi_1 dz^2$ on R_2 (for a strictly convex surface), where

$$\phi_1 = \frac{E-G}{2} - iF .$$

We prove the following lemma in order to fascilitate the definition of similar quadratic differentials on the various Riemann surfaces of interest here.

LEMMA 5. *Let $\Lambda = Adx^2 + 2Bdxdy + Cdy^2$ be a quadratic form on an oriented C^1 surface S. Suppose R is a Riemann surface defined on S. Then $\Omega = \phi\, dz^2$ with*

$$\phi = \frac{A-C}{2} - iB$$

is a quadratic differential on R.

Proof. Let $z = x + iy$ and $w = u + iv$ be conformal parameters on R. Then

$$\Lambda = \hat{A}du^2 + 2\hat{B}dudv + \hat{C}dv^2$$

where

$$\hat{A} = Ax_u^2 + 2Bx_u y_u + Cy_u^2 ,$$
$$\hat{B} = Ax_u x_v + B(x_u y_v + x_v y_u) + Cy_u y_v ,$$

$$\hat{C} = Ax_v^2 + 2Bx_vy_v + Cy_v^2 ;$$

Since

$$x_u = y_v ,$$
$$x_v = -y_u ,$$

while,

$$\frac{dz}{dw} = x_u - ix_v ,$$

simple computation yields

$$\left\{\frac{\hat{A} - \hat{C}}{2} - i\hat{B}\right\} = \left\{\frac{A - C}{2} - iB\right\}\left(\frac{dz}{dw}\right)^2 ,$$

as required.

REMARK. The quadratic differential Ω on R associated with $\Lambda \not\equiv 0$ on S is identically zero if and only if R is determined by choosing conformal parameters $z = x + iy$ on S corresponding to coordinates x, y in terms of which $\Lambda = \lambda(x, y)\{dx^2 + dy^2\}$. Thus, for example, there is no R on which $\Omega \equiv 0$ if Λ is indefinite somewhere on S.

REMARK. Let R and $\hat{R}$ be Riemann surfaces defined on S. Suppose the identity mapping on S is not conformal from R to $\hat{R}$ at p. Then $\Omega = 0$ at p on both R and $\hat{R}$ if and only if $\Lambda = 0$ at p.

We are now free to discuss Ω_1 and Ω_2 on R_1, R_2 or R_2'. We may also define the quadratic differential $\Omega_2' = \phi_2' dz$ on any umbilic free portion of R_1, R_2 or R_2' with

$$\phi_2' = \frac{L' - N'}{2} - iM' .$$

And, for the sake of completeness, we will consider Ω_3, the quadratic differential associated with

$$III = 2HII - KI \tag{10}$$

on S. As is well known, III yields, wherever $K \neq 0$, the first fundamental form on the unit spherical image of S. The relation (2) implies that

$$III = HII - H'II' = KI - 2H'II' , \tag{11}$$

so that III is determined by any pair of the forms I, II and II'.

All linear relations, such as (2), (10) or (11), among the forms I, *II*, *II'* and *III* hold also for their associated quadratic differentials. However, simplified versions of these relations are satisfied by Ω_1, Ω_2, Ω'_2 and Ω_3 on R_1, R_2, R'_2, and on the Riemann surface R_3 determined on S by using the spherical image map (where $K \neq 0$) to carry back onto S the ordinary conformal structure of the sphere. For on each of the Riemann surfaces R_1, R_2, R'_2 or R_3, at least one of the quadratic differentials Ω_1, Ω_2, Ω'_2 or Ω_3 vanishes identically.

LEMMA 6. *Let R be a Riemann surface on the oriented surface S immersed C^3 in E^3. Suppose K, H, $H' \neq 0$. Then $R = R_1$ if and only if*

$$2H\Omega_2 = -2H'\Omega'_2 = \Omega_3$$

on R; $R = R_2$ if and only if

$$K\Omega_1 = H'\Omega'_2 = -\Omega_3$$

on R; $R = R'_2$ if and only if

$$K\Omega_1 = H\Omega_2 = \Omega_3$$

on R; and $R = R_3$ if and only if

$$K\Omega_1 = 2H\Omega_2 = 2\mathrm{H}'\Omega'_2$$

on R.

Proof. Use (2), (10), (11), and the remarks which follow Lemma 5.

REMARK. By Lemma 4, $\Omega_2 = (L - iM)dz^2$ on R'_2. By (2) and (7), $\Omega'_2 = (L' - iM')dz^2$ on R_2.

5. In [2] it is shown that Ω_2 is holomorphic on R_1 if and only if S is of constant mean curvature. In [3] it is shown that Ω_1 is holomorphic on R_2 if and only if S is of constant (positive) Gaussian curvature. We have the following (somewhat less satisfying) result of a similar nature.

THEOREM. *Ω_1 is holomorphic on R'_2 if and only if the vector X describing the immersion of S in E^3 is a harmonic function of disothermal coordinates.*

Proof. By the definition of *I*, it is easily checked, using

$$\frac{\partial}{\partial z} = \frac{1}{2}\left\{\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right\},$$

that

$$\phi_1 = 2X_z \cdot X_z . \tag{12}$$

But Ω_1 is holomorphic on R_2' if and only if

$$(\phi_1)_{\bar{z}} = 0 ,$$

that is, by (12), if and only if

$$X_z \cdot X_{z\bar{z}} = 0 , \tag{13}$$

where $z = x + iy$ is any conformal parameter on R_2', and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left\{\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right\} .$$

And (13) is equivalent to

$$(X_{xx} + X_{yy}) \cdot X_x = (X_{xx} + X_{yy}) \cdot X_y = 0 . \tag{14}$$

On the other hand, by Lemma 4, the Gauss equations in disothermals yield

$$X_{xx} = \Gamma^1_{11} X_x + \Gamma^2_{11} X_y + L\overset{\perp}{X} ,$$
$$X_{yy} = \Gamma^1_{22} X_x + \Gamma^2_{22} X_y - L\overset{\perp}{X} ,$$

where $\overset{\perp}{X}$ is the unit normal to S. Thus (14) holds if and only if X is a harmonic function of disothermal coordinates.

Note, however, that disothermal coordinates are isothermal on a minimal surface. Thus $\Omega_1 \equiv 0$ is trivially holomorphic on R_2' if S is minimal. Our theorem therefore includes the well known fact that the vector immersing a minimal surface is a harmonic function of isothermal coordinates. It would be nice, of course, to have a geometric characterization of all surfaces for which X is a harmonic function of disothermals.

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