THE METHOD OF INTERIOR PARALLELS APPLIED TO POLYGONAL OR MULTIPLY CONNECTED MEMBRANES

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1. Introduction.

1.1. The scope of this paper is (a) to discuss the possibilities of the method of interior parallels (*Makai, Pólya, Payne-Weinberger*) by considering the case of polygonal membranes (§ 2); (b) to extend it to multiply connected domains in a more satisfactory manner than has hitherto been proposed (§ 3); to this end we use a result of H. F. Weinberger [7] on the existence of an "effectless cut", published immediately after the present paper.

1.2. We consider the problem of a vibrating membrane covering a plane domain G and fixed along the boundary Γ. We are interested in the first eigenvalue λ_1 of the problem $\Delta u + \lambda u = 0$ in G, $u = 0$ along Γ; by Rayleigh's principle,

$$\lambda_1 \leqq R[v] \equiv \frac{D(v)}{\iint_G v^2 dA} \qquad \text{if } v = 0 \text{ along } \Gamma .$$

$dA = dxdy$ is the element of area; $D(v) = \iint_G grad^2 v dA$, Dirichlet's integral; $R[v]$, Rayleigh's quotient.

The method of interior parallels consists in using trial functions v whose level lines are parallel to Γ. It was first introduced by E. Makai [2, 3]: using the trial function $v(Q) = \delta_{Q\Gamma}$ ($Q \in G$, δ = Euclidean distance), he obtained, for every simply or doubly connected membrane G of area A, fixed along its boundary Γ of total length L_Γ, the bound

$$\lambda_1 \leqq 3\frac{L_\Gamma^2}{A^2} . \tag{1}$$

His proof makes use of B. Sz.-Nagy's [6] inequality

$$q(\delta) \leqq L_\Gamma \tag{2}$$

bounding the total length $q(\delta)$ of the "interior parallel at distance δ" in a simply or doubly connected domain; as Sz.-Nagy proved, this length exists for almost all values of δ.

Received August 23, 1962.

1.3. Refining Makai's method, G. Pólya [5] admits *a priori* for v any regular function $v(\delta_{Q\Gamma})$ satisfying $v(0) = 0$.

Let us call $a = a(\delta)$ the area of the subdomain $\{Q \mid Q \in G, \delta_{Q\Gamma} < \delta\}$ of G; $q(\delta) = da/d\delta$. By Rayleigh's principle,

$$(3) \qquad \lambda_1 \leqq R[v] = \frac{\int_{a=0}^{A} \left(\frac{dv}{d\delta}\right)^2 da}{\int_{a=0}^{A} v^2 da} = \frac{\int_{a=0}^{A} q^2 \left(\frac{dv}{da}\right)^2 da}{\int_{a=0}^{A} v^2 da} \quad \text{if } v(0) = 0 .$$

Let $\lambda_1^+ = \mathrm{Min}_{v(\delta)} R[v]$; $\lambda_1 \leqq \lambda_1^+$; if G is simply or doubly connected inequality (2) gives

$$(4) \qquad \lambda_1 \leqq \lambda_1^+ \leqq \underset{\text{Pólya}}{\lambda_1^{++}} \equiv L_\Gamma^2 \, \mathrm{Min}_{v(0)=0} \frac{\int_{a=0}^{A} \left(\frac{dv}{da}\right)^2 da}{\int_{a=0}^{A} v^2 da} = \left(\frac{\pi}{2} \cdot \frac{L_\Gamma}{A}\right)^2 ;$$

this is *Pólya's inequality* (sharper than (1)).

1.4. For a *simply connected* domain G, L. E. Payne and H. F. Weinberger [4] made use of the sharp inequality of B. Sz.-Nagy [6]:

$$(5) \qquad q(\delta) \leqq L_\Gamma - 2\pi\delta ;$$

it follows by integration that $q^2 \leqq L_\Gamma^2 - 4\pi a$ (see also [1]), whence by (3):

$$(6) \qquad \begin{aligned} &\lambda_1 \leqq \lambda_1^+ \leqq \underset{P-W}{\lambda_1^{++}} \\ &\equiv \lambda_{1\text{ext}}^{++}(A, L_\Gamma) \equiv \mathrm{Min}_{v(0)=0} \frac{\int_{a=0}^{A} (L_\Gamma^2 - 4\pi a)\left(\frac{dv}{da}\right)^2 da}{\int_{a=0}^{A} v^2 da} \left(\leqq \underset{\text{Pólya}}{\lambda_1^{++}}\right) . \end{aligned}$$

Payne and Weinberger remarked that *all inequalities* (1), (2), (3), (4), (5), (6) remain valid if G is allowed to have *also interior boundary curves* γ *along which the membrane is free* ("holes"): L_Γ is then the total length of the "fixed" boundaries Γ, A the area of G (without the holes); $q(\delta)$ *is the length of that part of the "interior parallel" to* Γ (not γ!) *which lies inside* G.

Inequality (4) is valid if Γ is formed by the outer boundary Γ_0 and at most one inner boundary curve Γ_1; along the other interior boundary curves $\gamma_2, \gamma_3, \cdots, \gamma_n$ the membrane is free; $L_\Gamma = L_{\Gamma_0} + L_{\Gamma_1}$. —(5) and (6) are valid only if $\Gamma = \Gamma_0$ and all inner boundaries are free.

If G is a circular ring fixed along its outer boundary Γ_0 and free along its inner boundary γ_1, its first eigenfunction $u_1 = u_1(r)$, whence

$\lambda_1 = \lambda_1^+$, and $q^2 = L_{\Gamma_0}^2 - 4\pi a$, whence $\lambda_1^+ = \underset{P-W}{\lambda_1^{++}}$. Therefore $\underset{P-W}{\lambda_1^{++}} \equiv \lambda_{1\text{ext}}^{++}(A, L_{\Gamma_0})$ *is equal to the first eigenvalue of an annular membrane fixed along* Γ_0, *free along* γ_1.

$\lambda_{1\text{ext}}^{++}(A, L_{\Gamma_0})$ is the root of an equation involving Bessel functions; its solution is indicated graphically in Jahnke-Emde's Tables of functions, pp. 207–8.

The inequality $\lambda_1 \leqq \underset{P-W}{\lambda_1^{++}}$ thus expresses an "*isoperimetric*" extremal property of such annular membranes.

1.5. In another paper [1] one can find a "unified" and more detailed discussion of Makai's, Pólya's and Payne-Weinberger's methods; and furthermore the proof of an analogous "isoperimetric" theorem, which we shall essentially *use* in § 3:

Of all multiply connected membranes of given area A, fixed along one inner boundary Jordan curve Γ_1 *of given length* L_{Γ_1} *and free along all others* (γ_0 *exterior*; $\gamma_2, \gamma_3, \cdots, \gamma_n$ *interior*), *the annulus has highest* λ_1.

Let $\delta = \delta_{Q\Gamma_1}$ (Euclidean distance), and $q = q(\delta)$ as before; the proof of our theorem becomes easy once we introduce the new parameter

$$t(\delta) = \int_0^\delta \frac{d\delta}{q} \tag{7}$$

instead of $a(\delta) = \int_0^\delta q d\delta$ (see 1.3 and 1.4). We then have, instead of (3),

$$\lambda_1 \leqq R[v] = \frac{\int_{t=0}^{T} \left(\frac{dv}{dt}\right)^2 dt}{\int_{t=0}^{T} q^2 v^2 dt} \; ; \qquad \lambda_1 \leqq \lambda_1^+ \equiv \text{Min}_v\, R[v] \, . \tag{3'}$$

(Often $T = \infty$.) This is the Rayleigh quotient of a vibrating string, fixed at its extremity $t = 0$ and of total mass $\int_{t=0}^{T} q^2 dt = A$.

B. Sz.-Nagy proved that here $q(\delta) \leqq L_{\Gamma_1} + 2\pi\delta$; whence by integration:

$$q(t) \leqq L_{\Gamma_1} e^{2\pi t} \quad \text{for} \quad t \leqq t_1 = \frac{1}{4\pi} \ln\left(1 + \frac{4\pi A}{L_{\Gamma_1}^2}\right) \qquad \text{(see [1])} \; ;$$

the proof is completed by a discussion of the effect of displacing the masses along the vibrating string.—We thus have

$$\lambda_1 \leqq \lambda_1^+ \leqq \lambda_{1\text{int}}^{++}(A, L_{\Gamma_1}) \, ,$$

where $\lambda_{1\text{int}}^{++}(A, L_{\Gamma_1})$ is the first eigenvalue of an annular membrane of area A, fixed along its interior boundary Γ_1 of length L_{Γ_1}, free along γ_0. To determine $\lambda_{1\text{int}}^{++}$, use again Jahnke-Emde's Tables of functions, pp. 207–8.

2. Membranes with fixed polygonal outer boundary.

2.1. For (simply or multiply connected) membranes, fixed along their polygonal outer boundary Γ_0 but free along the (possible) inner boundaries $\gamma_1, \gamma_2, \cdots, \gamma_n$, we shall sharpen Payne-Weinberger's upper bound (§ 1.4).—Also, the new bounds obtained will give us a glimpse of *the limits* of the method's possibilities.

2.2. The regular polygon with m sides, which is circumscribed to the unit circle, has *perimeter* $K_m = 2m tg(\pi/m)$, and area $K_m/2$.

Any regular m-polygon with area A and perimeter L, is circumscribed to a circle of radius $r_i = L/K_m$; $L^2 = (K_m r_i)^2 = 2K_m(K_m r_i^2/2) = 2K_m(Lr_i/2) = 2K_m A$. Therefore, by the isoperimetric property of regular polygons, *any* m-polygon with area A and perimeter L satisfies

$$L^2 \geqq 2K_m A \,. \tag{8}$$

In particular, every m-polygon (whether convex or not), which is circumscribed to a circle of radius r_i, satisfies $A = Lr_i/2$; therefore

$$L \geqq K_m r_i \,. \tag{8'}$$

Let $p \leqq m$; a regular p-polygon *is* an irregular m-polygon, circumscribed to the same circle, thus $K_p r_i = L \geqq K_m r_i$, whence $K_p \geqq K_m$; K_m is a decreasing function of m (which can be verified directly); when $m \to \infty$, $K_m \searrow 2\pi$.

2.3. Let the membrane cover a plane domain G and be fixed only along the m-polygonal outer boundary Γ_0; let us call $\tilde{G}(\supset G)$ the polygonal domain bounded by Γ_0; the line $\tilde{\Gamma}_0^{(\delta)}$ parallel to Γ_0 *in* $\tilde{G}$ is composed of $p \leqq m$ straight segments and possibly (if Γ_0 is not convex) some circular arcs of radius δ. The length $\tilde{q}(\delta)$ of $\tilde{\Gamma}_0^{(\delta)}$ is a piecewise differentiable function of δ; $\tilde{q}(\delta) \geqq q(\delta)$. If the domain $\tilde{G}^{(\delta)}$, bounded by $\tilde{\Gamma}_0^{(\delta)}$, is convex, it is readily seen that, for $\varepsilon > 0$, $\tilde{q}(\delta) - \tilde{q}(\delta + \varepsilon)$ is equal to the perimeter of a convex p-polygon (of sides parallel to Γ_0) with $p \leqq m$, circumscribed to a circle of radius ε; whence $\tilde{q}(\delta) - \tilde{q}(\delta + \varepsilon) \geqq K_p \varepsilon \geqq K_m \varepsilon$. This remains true if $\tilde{G}^{(\delta)}$ is non-convex: indeed, $\tilde{q}(\delta) - \tilde{q}(\delta + \varepsilon)$ is then *larger* than the perimeter of a non-convex p-polygon with $p \leqq m$, circumscribed to a circle of radius ε. We thus have always

$$-\frac{d\tilde{q}}{d\delta} \geqq K_m \,. \tag{9}$$

As in 1.3 and 1.4, we use as parameter the area $a(\delta) = \int_0^\delta q d\delta$ of the subdomain $\{Q \mid Q \in G, \delta_{Q\Gamma_0} < \delta\}$; $da/d\delta = q$;

$$\frac{-d(\tilde{q}^2)}{da} = 2\tilde{q}\left(\frac{-d\tilde{q}}{da}\right) \geqq 2q\left(\frac{-d\tilde{q}}{da}\right) = 2\frac{da}{d\delta}\left(\frac{-d\tilde{q}}{da}\right)$$
$$= 2\left(\frac{-d\tilde{q}}{d\delta}\right) \geqq 2K_m ;$$

whence by integration from 0 to a: $L_{\Gamma_0}^2 - \tilde{q}^2 \geqq 2K_m a$;

$$q^2 \leqq \tilde{q}^2 \leqq L_{\Gamma_0}^2 - 2K_m a \,, \tag{10}$$

with equality if $G = \tilde{G}$ = regular m-polygon.—This evaluation (valid for m-polygons) is sharper than $q^2 \leqq \tilde{q}^2 \leqq L_{\Gamma_0}^2 - 4\pi a$ (always valid), which is the basis of Payne-Weinberger's method (see [1]).

Using (3), we thus may write (instead of (6)):

$$\lambda_1 < \lambda_1^+ \leqq \underset{(m)}{\lambda_1^{++}} \,, \quad \text{where} \quad \underset{(m)}{\lambda_1^{++}} = \text{Min}_{v(0)=0} \frac{\int_{a=0}^{A} (L_{\Gamma_0}^2 - 2K_m a)\left(\frac{dv}{da}\right)^2 da}{\int_{a=0}^{A} v^2 da} \,. \tag{11}$$

Note that for polygons λ_1 is *always* smaller than λ_1^+: this limits the sharpness obtainable by the method of interior parallels.

When $m \to \infty$, $K_m \searrow 2\pi$; thus

$$\underset{(m)}{\lambda_1^{++}} \nearrow \underset{(\infty)}{\lambda_1^{++}} = \underset{P-W}{\lambda_1^{++}} \,. \tag{12}$$

We shall construct an annular membrane having exactly the first eigenvalue $\underset{(m)}{\lambda_1^{++}}$:

Instead of a, we introduce a new independent variable r by

$$L_{\Gamma_0}^2 - 2K_m a = K_m^2 r^2 \,, \quad \text{i.e.} \quad a = \frac{L_{\Gamma_0}^2}{2K_m} - \frac{1}{2}K_m r^2 \,; \quad \text{then} \quad \frac{da}{dr} = -K_m r \,;$$

$$\underset{(m)}{\lambda_1^{++}} = \text{Min}_{v(R_0)=0} \frac{\int_{r=r_1}^{R_0} \left(\frac{dv}{dr}\right)^2 K_m r\, dr}{\int_{r=r_1}^{R_0} v^2 K_m r dr} = \text{Min}_{v(R_0)=0} \frac{\int_{r=r_1}^{R_0} \left(\frac{dv}{dr}\right)^2 2\pi r\, dr}{\int_{r=r_1}^{R_0} v^2 2\pi r\, dr} \tag{13}$$

with $R_0 = L_{\Gamma_0}/K_m$ and $r_1^2 = R_0^2 - 2A/K_m$.

This is the annular membrane we wanted: fixed along its outer circle of radius R_0, free along its inner circle of radius r_1.—Consider two homothetic *regular* m-polygons, the outer one of length L_{Γ_0}, the inner one such that the area comprised between them be A: *the first is circumscribed to the circle of radius* R_0, *the second to the circle of radius* r_1.

REMARK. The fact that $\underset{(m)}{\lambda_1^{++}}$ increases with m thus expresses a property of Bessel functions.

2.4. *More precise evaluations in terms of A, L_{Γ_0} and the* interior *angles* $\pi - \alpha_1, \pi - \alpha_2, \cdots, \pi - \alpha_m$ of Γ_0, when $\tilde{G}$ is convex.

We consider a membrane G fixed only along its convex polygonal outer boundary Γ_0. We have $\alpha_1 + \cdots + \alpha_m = 2\pi,\ 0 < \alpha_i < \pi$.

Let us call $F(\alpha_1, \cdots, \alpha_m) = 2\sum_{i=1}^m tg(\alpha_i/2)$ *the perimeter of the* (convex) *polygon C with interior angles $\pi - \alpha_1, \cdots, \pi - \alpha_m$ (in this order), circumscribed to the unit circle.* The area of C is $F(\alpha_1, \cdots, \alpha_m)/2$. By (8′), $F(\alpha_1, \cdots, \alpha_m) \geqq K_m$; with equality if $\alpha_1 = \cdots = \alpha_m = 2\pi/m$.

Every interior parallel $\tilde{\Gamma}_0^{(\delta)}$ to Γ_0 in $\tilde{G}$ is a polygon with $p \leqq m$ sides (parallel to those of Γ_0) and inner angles $\pi - \beta_1, \cdots, \pi - \beta_p$, where $\beta_1 + \cdots + \beta_p = 2\pi$ and each β_j is equal either to an α_i or to the sum of several consecutive α_i. For a sufficiently small $\varepsilon > 0$, $\tilde{q}(\delta) - \tilde{q}(\delta + \varepsilon)$ is equal to the length of a (convex) p-polygon with angles $\pi - \beta_1, \cdots, \pi - \beta_p$ (in this order), circumscribed to a circle of radius ε; whence $\tilde{q}(\delta) - \tilde{q}(\delta + \varepsilon) = F(\beta_1, \cdots, \beta_p)\cdot\varepsilon$;

$$\frac{-d\tilde{q}}{d\delta} = F(\beta_1, \cdots, \beta_p) = 2\sum_{j=1}^{p} tg\frac{\beta_j}{2}\ ;$$

since $\tilde{G}$ is by hypothesis convex, $0 < \alpha_i < \pi,\ 0 < \beta_j < \pi$, thus each $tg(\alpha_i/2) > 0$ and

$$tg\frac{\alpha_i + \alpha_{i+1}}{2} = \frac{tg\dfrac{\alpha_i}{2} + tg\dfrac{\alpha_{i+1}}{2}}{1 - tg\dfrac{\alpha_i}{2}tg\dfrac{\alpha_{i+1}}{2}} > tg\frac{\alpha_i}{2} + tg\frac{\alpha_{i+1}}{2}\ ;$$

therefore $F(\beta_1, \cdots, \beta_p) \geqq F(\alpha_1, \cdots, \alpha_m)$ (which is also geometrically clear) and always

$$-\frac{d\tilde{q}}{d\delta} \geqq F(\alpha_1, \cdots, \alpha_m)\ ; \tag{9′}$$

whence

$$q^2 \leqq \tilde{q}^2 \leqq L_{\Gamma_0}^2 - 2F(\alpha_1, \cdots, \alpha_m)a \tag{10′}$$

and the inequality

$\lambda_1 < \lambda_1^+ \leqq \lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++}$, where

(11′)
$$\lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++} = \mathrm{Min}_{v(0)=0} \frac{\int_{a=0}^{A} [L_{\Gamma_0}^2 - 2F(\alpha_1, \cdots, \alpha_m)a]\left(\frac{dv}{da}\right)^2 da}{\int_{a=0}^{A} v^2 da}.$$

$\lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++} \leqq \lambda_{1\,(m)}^{++}$; equality only if $\alpha_1 = \cdots = \alpha_m = 2\pi/m$.

Let us now introduce another independent variable r instead of a: $L_{\Gamma_0}^2 - 2F(\alpha_1, \cdots, \alpha_m)a = [F(\alpha_1, \cdots, \alpha_m)]^2 \cdot r^2$; we then obtain a formula like (13) with $F(\alpha_1, \cdots, \alpha_m)$ instead of K_m, now $R_0 = L_{\Gamma_0}/F(\alpha_1, \cdots, \alpha_m)$ and $r_1^2 = R_0^2 - 2A/F(\alpha_1, \cdots, \alpha_m)$. *The annular membrane with fixed outer circle of radius* R_0 *and free inner circle of radius* r_1 *has first eigenvalue* $\lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++}$.

Let us construct two homothetic m-polygons, circumscribed to concentric circles, with sides parallel to those of Γ_0 (and in the same order), *the outer polygon of length* L_{Γ_0}, *the inner polygon such that the area comprised between them be* A; *the outer circle has then radius* R_0, *the inner circle radius* r_1: *this is our auxiliary annulus.*

2.5. *Remark on the limits of the possibilities of the method of interior parallels.*—As follows from the above discussion, if $G = \tilde{G}$ is itself a convex polygon circumscribed to a circle, we have $L_{\Gamma_0}^2 = 2F(\alpha_1, \cdots, \alpha_m)A$, whence $r_1 = 0$; $R_0 = r_{\text{inscr}}$;

$$\lambda_1 < \lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++} = \frac{j_0^2}{r_{\text{inscr}}^2} \leqq \lambda_{1\,(m)}^{++} < \lambda_{1\,(\infty)}^{++} = \lambda_{1\,P-W}^{++} < \lambda_{1\,\text{Pólya}}^{++}$$
$$= \left(\frac{\pi}{2}\frac{L_{\Gamma_0}}{A}\right)^2 = \frac{\pi^2}{r_{\text{inscr}}^2} < \lambda_{1\,\text{Makai}}^{++} = 3\left(\frac{L_{\Gamma_0}}{A}\right)^2 = \frac{12}{r_{\text{inscr}}^2}.$$

Observe that here $d\tilde{q}/d\delta = -F(\alpha_1, \cdots, \alpha_m)$ and $q^2 = \tilde{q}^2 = L_{\Gamma_0}^2 - 2F(\alpha_1, \cdots, \alpha_m)a$, i.e. $\lambda_1^+ = \lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++}$ is the first eigenvalue of the inscribed circle; *the inequality* $\lambda_1 < j_0^2/r_{\text{inscr}}^2$ *is trivial* (*monotony*), *but the method of interior parallels is* (*in this case of a circumscribed polygon*) *unable to give any sharper bound.*

It may be noted that Pólya's bound—and therefore Payne-Weinberger's bound as well as $\lambda_{1\,(m)}^{++}$ and $\lambda_{1\,(\alpha_1,\cdots,\alpha_m)}^{++}$—become sharp for the infinite strip *considered as the limit of a long rectangle*: let b be its breadth, $\lambda_{1\,\text{Pólya}}^{++} \approx (\pi/b)^2$; but, if we consider the strip as the limit of a long rhombus (i.e. circumscribed to a circle), $\lambda_{1\,\text{Pólya}}^{++} \approx (2\pi/b)^2$ and $\lambda_{1\,(\varepsilon,\pi-\varepsilon,\varepsilon,\pi-\varepsilon)}^{++} = j_0^2/r_{\text{inscr}}^2 \approx (2j_0/b)^2$, which is trivial by monotony.

3. Multiply connected membranes.

3.1. Let us consider e.g. a doubly connected membrane G, fixed both along its outer boundary Γ_0 and its inner boundary Γ_1.

(i) Given the area A of G and the lengths L_{Γ_0} and L_{Γ_1}, we are looking for a bound $\lambda_1 \leqq \lambda_1^{++}(A; L_{\Gamma_0}, L_{\Gamma_1})$ such that, when Γ_1 reduces to a point, $\lambda_1^{++}(A; L_{\Gamma_0}, 0) = \lambda_{1\text{ext}}^{++}(A, L_{\Gamma_0})$ (exact bound of Payne-Weinberger); indeed, such *is* the case for the true λ_1.

This requirement is not fulfilled by Pólya's -or Makai's- bounds (even if Γ_1 is very small, they consider trial functions depending only on the distance to $\Gamma = \Gamma_0 \cup \Gamma_1$, which does not correspond, qualitatively, to the behavior of the true first eigenfunction of G); nor is it fulfilled by Payne-Weinberger's suggestion to make G simply connected by adding between Γ_0 and Γ_1 a rectilinear constraint (length c), thus replacing L_Γ by $L_\Gamma + 2c$: indeed, when Γ_1 reduces to a point, this constraint would remain and the bound $\lambda_{1\text{ext}}^{++}(A, L_{\Gamma_0} + L_{\Gamma_1} + 2c)$ would become $\lambda_{1\text{ext}}^{++}(A, L_{\Gamma_0} + 2c)$ instead of $\lambda_{1\text{ext}}^{++}(A, L_{\Gamma_0})$. Any small boundary component Γ_1 has then a disproportionate effect on the bound.—In particular, consider a fixed annular membrane with radii 1 and $\varepsilon \to 0$; the true λ_1 tends to $j_0^2 \cong 5.78$; λ_1^{++} (Payne-Weinberger) tends to $\lambda_{1\text{ext}}^{++}(\pi, 2\pi + 2)$, which is larger than the first eigenvalue of the unit circular sector of aperture 360^0, i.e. larger than π^2; Pólya's inequality gives (as for the circle) $\lambda_1 \to \leqq ((\pi/2)(2\pi/\pi))^2 = \pi^2$.

(ii) We look for a bound which, for any fixed annular membrane, should coincide with the exact value λ_1.

3.2. From *H. F. Weinberger*'s paper [7], which is printed immediately after the present one, it follows that: Given a multiply connected membrane G which is *fixed* along its outer boundary Γ_0 and its inner boundary components $\Gamma_1, \Gamma_2, \cdots, \Gamma_p$, and free along its other inner boundaries $\gamma_{p+1}, \gamma_{p+2}, \cdots, \gamma_n$ (the Γ_i are assumed to have continuous normals and the γ_j to be analytic), then *there exists an "effectless cutting"* of the membrane G into $p + 1$ sub-membranes $G_0, G_1, \cdots, G_p$, where each G_i has Γ_i as a fixed boundary component and is otherwise free, such that $\lambda_1^{G_0} = \lambda_1^{G_1} = \cdots = \lambda_1^{G_p} = \lambda_1^G$. In other words: *The domain G can be cut into $G_0, \cdots, G_p$ by means of a system of analytic arcs along which $\partial u_1/\partial n = 0$, where u_1 is the first eigenfunction of G*; u_1 is then also the first eigenfunction of each G_i (membrane fixed along Γ_i, free along the cuts and the γ_j). We use essentially this result in the following.

3.3. Let A_i be the area of the partial domain G_i; $A_0 + A_1 + \cdots + A_p = A$; the lengths $L_{\Gamma_0}, L_{\Gamma_1}, \cdots, L_{\Gamma_p}$ are known, but *not* the individual A_i!—We know that $\lambda_1 \leqq \lambda_{1\text{ext}}^{++}(A_0, L_{\Gamma_0})$ and $\lambda_1 \leqq \lambda_{1\text{int}}^{++}(A_i, L_{\Gamma_i})$

for $i = 1, 2, \cdots, p$. Therefore:

$$\lambda_1 \leqq \min \{\lambda_{1\mathrm{ext}}^{++}(A_0, L_{\Gamma_0}); \lambda_{1\mathrm{int}}^{++}(A_1, L_{\Gamma_1}); \cdots; \lambda_{1\mathrm{int}}^{++}(A_p, L_{\Gamma_p})\}$$

and hence

$$\lambda_1 \leqq \max \left\{ {\text{choice of } \hat{A}_0 \geqq 0 \ \cdots, \ \hat{A}_p \geqq 0 \atop \text{satisfying } \hat{A}_0 + \cdots + \hat{A}_p = A} \right\} \min \{\lambda_{1\mathrm{ext}}^{++}(\hat{A}_0, L_{\Gamma_0}); \lambda_{1\mathrm{int}}^{++}(\hat{A}_1, L_{\Gamma_1}); \cdots\} .$$

Since each of the $\lambda_{1\mathrm{ext}}^{++}$, $\lambda_{1\mathrm{int}}^{++}$ is a monotonous decreasing function of the corresponding $\hat{A}_i$, *the* max min *is attained when* $\hat{A}_0, \cdots, \hat{A}_p$ are chosen such that *all those* λ_1^{++} *are equal*:

$$\lambda_{1\mathrm{ext}}^{++}(\hat{A}_0, L_{\Gamma_0}) = \lambda_{1\mathrm{int}}^{++}(\hat{A}_1, L_{\Gamma_1}) = \cdots = \lambda_{1\mathrm{int}}^{++}(\hat{A}_p, L_{\Gamma_p}) ; \tag{14}$$

those are p transcendental equations, which together with

$$\hat{A}_0 + \hat{A}_1 + \cdots + \hat{A}_p = A \tag{15}$$

determine $\hat{A}_0, \cdots, \hat{A}_p$; *these values are in general NOT equal to the true* $A_0, \cdots, A_p$ *corresponding to Weinberger's "effectless cutting"*; but *the common value*

$$\lambda_1^{++}(\hat{A}_i, L_{\Gamma_i}) = \lambda_1^{++}(A; L_{\Gamma_0}, L_{\Gamma_1}, \cdots, L_{\Gamma_p}) \tag{16}$$

is the upper bound we were looking for.

Indeed: (i) If an inner boundary component Γ_p reduces to a point, i.e. $L_{\Gamma_p} \to 0$, then the corresponding $\hat{A}_p \to 0$ (and also $A_p \to 0$); there remain $p-1$ transcendental relations in (14) between $\hat{A}_0, \cdots, \hat{A}_{p-1}$, which together with (15) determine these p quantities; therefore $\lambda_1^{++}(A; L_{\Gamma_0}, \cdots, L_{\Gamma_{p-1}}, 0) = \lambda_1^{++}(A; L_{\Gamma_0}, \cdots, L_{\Gamma_{p-1}})$ as we wanted.

In the special case $p = 1$, we have $\lambda_1^{++}(A; L_{\Gamma_0}, 0) = \lambda_{1\mathrm{ext}}^{++}(A, L_{\Gamma_0})$.

(ii) If $p = 1$ and $L_{\Gamma_0}^2 - L_{\Gamma_1}^2 = 4\pi A$, there exists a circular ring with area A, outer perimeter L_{Γ_0} and inner perimeter L_{Γ_1}; its first eigenvalue is precisely equal to $\lambda_1^{++}(A; L_{\Gamma_0}, L_{\Gamma_1})$. (Here $\hat{A}_0 = A_0$ and $\hat{A}_1 = A_1$, G_0 and G_1 are separated by the "maximum line" of the annular membrane's first eigenfunction.)—Whence the isoperimetric inequality:

Of all (doubly or multiply connected) *membranes which are fixed along their outer boundary* Γ_0 *and one inner boundary component* Γ_1 (and otherwise free), *with given* A, L_{Γ_0} *and* L_{Γ_1} *satisfying* $L_{\Gamma_0}^2 - L_{\Gamma_1}^2 = 4\pi A$, *the annular membrane has maximal* λ_1.

EXAMPLE. A doubly connected fixed membrane, bounded by two circles of given radii, has maximum λ_1 when the circles are *concentric*.

REMARKS. (a) If Γ_0 is a polygon, $\lambda_{1\text{ext}}^{++}$ in (14) can be advantageously replaced by $\underset{(m)}{\lambda_1^{++}}$ or by $\underset{(\alpha_1,\cdots,\alpha_m)}{\lambda_1^{++}}$.

(b) If the considered membrane has a *free outer boundary* γ_0, the above discussion remains valid, the first term in (14) disappears from the formula, as disappear A_0, $\hat{A}_0$ and L_{Γ_0}.

My best thanks are due to *H. F. Weinberger* for his proof [7] of the existence of an "effectless cutting", which allowed the very simple proof given in this § 3; without both Weinberger's kindness and skill, a long and delicate construction and discussion of a continuous trial function in the whole domain G (with level lines consisting of arcs parallel to different Γ_i) would have been necessary to get the same (14), (15) and (16) finally.

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