ON A CLASS OF EQUIVALENT SYSTEMS OF LINEAR INEQUALITIES

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0. Summary. This note is concerned with the question of when a matrix A of mn columns and rank $m + n - 1$ is transformable into a matrix of a special class known as the constraint matrices of an m by n transportation program. The question is of practical significance in the solution of linear programs. The main result, a set of necessary and sufficient conditions on the matrix A, is formulated in § 3 (Theorem 3.3), and proved in §§ 4–5. As application, § 6 outlines a method of testing for the conditions of the theorem and of effectuating the transformation when the conditions are satisfied.

1. Introduction. A finite system of linear inequalities can, in general, be reexpressed in the form

$$\mathrm{A}x = b,\ x \geqq 0. \tag{1.6}$$

The objective is to characterize among the systems (1.6) those that are equivalent (in a sense to be defined in § 2) to the systems occurring as constraints of a special type of linear optimization programs, known heuristically as "transportation" programs, which admit relatively simple and efficient algorithms of solution (see for instance Dantzig [1] and [2] and Ford and Fulkerson [3]).

We shall refer to a system of the form

$$\sum_{i=1}^{m} \varepsilon_{ij} x_{ij} = c_j, \quad \sum_{j=1}^{n} \varepsilon_{ij} x_{ij} = r_i, \qquad x_{ij} \geqq 0 \tag{1.7}$$

where $\varepsilon_{ij} = \pm 1$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$, as "the constraints of a transportation program," the special case $\varepsilon_{ij} = 1$ representing the constraints of a "standard" transportation program.

Interpreting $x = (x_{ij})$ as vector in R^{mn}, (1. 7) can also be written in the form

$$Dx = c,\ x \geqq 0 \tag{1.8}$$

where $c' = (r_1, r_2, \cdots, r_m,\ c_1, c_2, \cdots, c_n)$, hence, $c \in R^{m+n}$ and D is a matrix of $m + n$ rows and mn columns d_ν $(\nu = 1, 2, \cdots, mn)$, also denotable as d_{ij} $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$ in lexicographic order, with

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(1.9) $$d_{ij} = d_\nu = \varepsilon_{ij}(e_i + e_{m+j}) \text{ where } \nu = n(i-1) + j$$

e_μ denoting the μth unit vector in R^{m+n}.

It should be noted that D is of rank $m + n - 1$: first, the subset of columns d_{i1}, d_{1j} $(i = 1, 2, \cdots, m;\ j = 2, 3, \cdots, n)$ is independent, hence, the rank $\geqq m + n - 1$; second, if $e = e_1 + \cdots + e_m - e_{m+1} - \cdots - e_{m+n}$, then consideration of the inner products

$$e'd_{ij} = 0,\ e'e_i = 1 \text{ when } i \leqq m,\ e'e_i = -1 \text{ when } i > m$$

shows that, say, e_1 is not representable in terms of columns of D, since $e_1 = \sum_{i,j} \alpha_{ij} d_{ij}$ would imply

$$1 = e'e_1 = \sum_{i,j} \alpha_{ij} e' d_{ij} = 0.$$

For later reference we also mention the geometric interpretation of the set S of columns of D. Setting

$$f_j = -e_{m+j},$$

the e_i and f_j $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$, interpreted as points in affine space, are the $m + n$ vertices of an $(m + n - 1)$—simplex and appear thus partitioned into two disjoint sets

$$E = \{e_i \mid i = 1, 2, \cdots, m\},\ F = \{f_j \mid j = 1, 2, \cdots, n\}.$$

By (1.9) S consists of all vectors of the form $\varepsilon_{ij}(e_i - f_j)$, that is of all those edges of the simplex that connect a vertex of E with a vertex of F, the orientation being determined by ε_{ij}. In the special case where all $\varepsilon_{ij} = 1$ the orientation is always from F to E.

2. Equivalence. When it is necessary to distinguish between a matrix and the ordered set of its columns, and this distinction is not clear from the context, we shall write (A) for the ordered set of columns of the matrix A.

Denoting by V_A the linear span of (A) we pose for the purpose of this note the following

(2.1) DEFINITION Two systems of linear inequalities

$$\begin{array}{lll} \text{(i)} & Ax = b, & x \geqq 0 \\ \text{(ii)} & Cy = d, & y \geqq 0 \end{array}$$

are strongly equivalent, in symbols

$$(A, b) \sim (C, d)$$

if and only if there exist a permutation matrix P, a positive diagonal matrix Q, and a nonsingular linear mapping $T: V_{(A\ b)} \to V_{(C\ d)}$ such that

$$T(APQ) = (C), \quad Tb = d.$$

The last two relations in the definition should be read to mean: the indicated operations are meaningful and equality holds. By a positive diagonal matrix is meant a matrix $Q = (q_{\mu\nu})$ such that

$$q_{\mu\nu} > 0 \text{ when } \mu = \nu, \ q_{\mu\nu} = 0 \text{ when } \mu \neq \nu.$$

Clearly, if the two systems in (2.1) are strongly equivalent, then A and C necessarily have the same number of columns and the same rank, and the substitution

$$x = PQy$$

establishes a one-to-one correspondence between the solutions of (i) and (ii).

In order to see that the equivalence relation defined in (2.1) is reflexive, symmetric and transitive, it is sufficient to observe that for a given order n all matrices R of the form $R = PQ$ constitute a group, since, first the P and the Q each form a group, second $PQP^{-1} = Q_1$, hence $PQ = Q_1P$, and

$$\begin{aligned} R_1R_2^{-1} = P_1Q_1(P_2Q_2)^{-1} &= P_1Q_1Q_2^{-1}P_2^{-1} = P_1Q_3P_2^{-1} \\ &= P_1P_2^{-1}Q_4 = P_3Q_4 = R_3. \end{aligned}$$

If the sets (A, b) and (C, d) are in the same R^n, the existence of a nonsingular linear T: $V_{(A,b)} \to V_{(C\ d)}$ satisfying the last relation in (2.1) is equivalent to the existence of a nonsingular linear transformation on R^n satisfying the same relation.

If the two sets are not in the same R^n and $\bar{A}, \bar{b}, \bar{C}, \bar{d}$ denote the sets and vectors obtained from A, b, C, d by adjoining zero rows to A and b or to C and d such that $(\bar{A}, \bar{b})$ and $(\bar{C}, \bar{d})$ are in the same space, then, obviously

(2.2) $$(A, b) \sim (C, d) \Longleftrightarrow (\bar{A}, \bar{b}) \sim (\bar{C}, \bar{d})$$

and hence

(2.3) $(A, b) \sim (C, d)$ if and only if there exist a permutation matrix P, a positive diagonal matrix Q and a nonsingular matrix T such that

$$T\bar{A}PQ = \bar{C}, \ T\bar{b} = \bar{d}.$$

3. Conditions. The question whether a given system (1.6) is strongly equivalent to the constraints of some transportation program as described in (1.7) through (1.9), amounts to the question of existence of T, P and Q in the sense of (2.3) such that

$$T\bar{A}PQ = \bar{D} \tag{3.1}$$

for some D of (1.8–9).

One trivially necessary condition for such equivalence is that there exist a D of (1.8–9) with the same rank and the same number of columns as A. Hence, if A has γ columns and rank ρ, then it is necessary that there exist two positive integers m and n such that

$$mn = \gamma \qquad m + n = \rho + 1\,;$$

or, equivalently, that the expression for $(m - n)^2$ be the square of an integer, that is

$$(\rho + 1)^2 - 4\gamma = k^2\,, \qquad k \text{ an integer}\,. \tag{3.2}$$

The values for m and n are then

$$m = \frac{1}{2}(\rho + 1 + k)\,, \qquad n = \frac{1}{2}(\rho + 1 - k)\,;$$

that the expressions in parentheses are even numbers follows from

$$(\rho + 1)^2 - k^2 = (\rho + 1 + k)(\rho + 1 - k) = 4\gamma \equiv 0 \pmod 2$$

and

$$\rho + 1 + k \equiv \rho + 1 - k \pmod 2\,.$$

We therefore restrict our consideration to matrices which satisfy (3.2), that is, matrices of mn columns and rank $m + n - 1$.

(3.3) THEOREM. *Let* $m \geqq 3$, $n \geqq 3$. *A system of the form* (1.6), *where the matrix* A *has* mn *columns and rank* $m + n - 1$, *is strongly equivalent to the system* (1.8–9) *if and only if the set* S *of columns of* A *satisfies the following conditions:*

(i) *every three distinct elements of* S *are linearly independent*

(ii) *for every two distinct elements* a, c *of* S *the equation*

$$a + \zeta c + \xi x + \eta y = 0 \tag{a}$$

has a nontrivial solution in S, *that is there exist elements* x, y *in* S *and real numbers* ξ, η, ζ *such that* (a) *is satisfied and* $\{-a, -\zeta c\} \neq \{\xi x, \eta y\}$

(iii) *if four distinct elements* a_1, a_2, a_3, a_4 *of* S *satisfy a nontrivial relation*

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 = 0\,, \tag{b}$$

and ρ_ν $(\nu = 2, 3, 4)$ *is the number of nontrivial solutions in* S *to the equation*

$$\alpha_1 a_1 + \zeta_\nu a_\nu + \xi x + \eta y = 0 , \tag{c}$$

then

$$\{\rho_2, \rho_3, \rho_4\} = \{1, m-1, n-1\} \tag{d}$$

$$\zeta_\nu = \alpha_\nu \text{ in each solution of (c)} . \tag{e}$$

Note that when (i) holds, then for every nontrivial solution of equation (a) in (ii) it follows that the four vectors a, c, x, y are distinct. Also note that in (iii) two solutions of (c) related to each other by exchange of x and y are considered as the same solution and counted only once.

Proof. That the conditions are necessary is proved in § 4. The sufficiency, which is the essential part of the theorem, is proved in § 5.

4. Proof of necessity. Let the two systems be strongly equivalent, that is, for some T and Q,

$$T(AQ) = (D). \tag{4.1}$$

We observe first that Q merely multiplies each column a_ν of A by a number $q_\nu \neq 0$, and hence a glance at the conditions in (3.3) shows that each of these conditions is satisfied by the set $S = (A)$ if it is satisfied by the set (AQ).

Second, since the nonsingular linear T preserves linear relations both ways, the conditions of the theorem are satisfied by (AQ) if and only if they are satisfied by (D).

Finally, if D^* denotes the matrix obtained from D by setting $\varepsilon_{ij} = 1$ in (1.9), then the argument used in our first observation shows that the conditions are satisfied by (D) if and only if they are satisfied by (D^*).

Therefore it is sufficient to prove that (D^*) satisfies the conditions of (3.3).

That (D^*) satisfies the conditions is almost obvious from the geometric interpretation of (D^*) outlined at the end of § 1. However, for the sake of completeness we shall give a detailed proof.

For brevity we write

$$e_{m+j} = f_j$$

hence for the elements of (D^*)

$$e_i + e_{m+j} = e_i + f_j \qquad (i = 1, 2, \cdots, m ; \; j = 1, 2, \cdots, n) ,$$

and make first an observation regarding the set (D^*).

Let $\{d_\nu\}$ be a subset of (D^*), $d_\nu = e_{i_\nu} + f_{j_\nu}$. Then

$$\sum \lambda_\nu d_\nu = 0 \Rightarrow \tag{4.2}$$

$$\text{(a)} \quad \sum \lambda_\nu e_{i_\nu} = 0 , \qquad \text{(b)} \quad \sum \lambda_\nu f_{j_\nu} = 0 , \qquad \text{(c)} \quad \sum \lambda_\nu = 0 .$$

The first two implications are obvious. To see the third, let

$$g = \sum_{i=1}^{m} e_i + \sum_{j=1}^{n} f_j .$$

Then the inner product

$$(g, d_\nu) = 2 ,$$

hence

$$0 = (g, 0) = (g, \sum \lambda_\nu d_\nu) = 2 \sum \lambda_\nu .$$

We now proceed to prove that (D^*) satisfies the conditions of (3.3).

Condition (i). Assume that three distinct elements of (D^*),

$$d_\nu = e_{i_\nu} + f_{j_\nu} \qquad (\nu = 1, 2, 3)$$

satisfy

$$\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 = 0 \qquad (\lambda_1 \neq 0) .$$

If, say, $\lambda_3 = 0$, then by (4.2), $\lambda_1 = -\lambda_2$, hence $d_1 = d_2$, contradicting that the d_ν were distinct.

If $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$, then, by (4.2)

$$\lambda_1 e_{i_1} + \lambda_2 e_{i_2} + \lambda_2 e_{i_3} = 0$$

and, obviously, the e_{i_ν} cannot be distinct. If, say, $e_{i_1} = e_{i_2}$, then

$$(\lambda_1 + \lambda_2) e_{i_1} + \lambda_3 e_{i_3} = 0$$

implies $e_{i_3} = e_{i_1}$. By the same argument on the f_{j_ν} we obtain $f_{j_1} = f_{j_2} = f_{j_3}$. Hence $d_1 = d_2 = d_3$, again contradicting the assumption that the d_ν were distinct.

Condition (ii). Let $a = e_i + f_j$, $c = e_\mu + f_\nu$, and $a \neq c$. Then $(\mu, \nu) \neq (i, j)$. Say $\mu \neq i$, and consider separately the two possibilities for ν, j.

If $\nu = j$, then for $j^* \neq j$, $j^* \leqq n$ (which exists, since $n \geqq 2$), $x = e_\mu + f_{j^*}$ and $y = e_i + f_{j^*}$, the vectors a, c, x, y are distinct and

$$a - c + x - y = 0 .$$

If $\nu \neq j$, then, for $x = e_i + f_\nu$ and $y = e_\mu + f_j$, the vectors a, c, x, y are distinct and

$$a + c - x - y = 0 .$$

Condition (iii). Let four distinct vectors $a_\nu = e_{i_\nu} + f_{j_\nu}$ of (D^*) satisfy (iiib). Then, since (D^*) satisfies condition (i),

$$\alpha_\nu \neq 0 \qquad (\nu = 1, 2, 3, 4) . \tag{4.3}$$

First, among the three vectors a_2, a_3, a_4 there is one, say a_λ, which satisfies

$$i_\lambda \neq i_1 , \quad j_\lambda \neq j_1 . \tag{4.4}$$

To see this, let a_2 not satisfy (4.4). Say $j_2 = j_1$. Then $a_2 \neq a_1 \Rightarrow i_2 \neq i_1$ and (4.2–3) $\Rightarrow \{i_3, i_4\} = \{i_1, i_2\}$. Say $i_3 = i_1$, $i_4 = i_2$. Then $a_4 \neq a_2 \Rightarrow j_4 \neq j_2$. Thus $j_4 \neq j_2 = j_1$ and $i_4 = i_2 \neq i_1$, hence $i_4 \neq i_1$ and $j_4 \neq j_1$, which proves (4.4).

Second, keeping a_1 fixed, we may assume the index 2 so assigned to one of the other three vectors that a_1, a_2 satisfy (4.4), that is

$$i_1 \neq i_2 , \quad j_1 \neq j_2 .$$

Then (4.2–3) $\Rightarrow \{i_3, j_4\} = \{i_1, i_2\}$, and we assume the numbering of a_3, a_4 such that

$$i_3 = i_1 , \quad i_4 = i_2 .$$

Then $a_3 \neq a_1 \Rightarrow j_3 \neq j_1$, $a_4 \neq a_2 \Rightarrow j_4 \neq j_2$, and (4.2–3) $\Rightarrow \{j_1, j_2\} = \{j_3, j_4\}$. Hence

$$j_3 = j_2 , \quad j_4 = j_1 .$$

Thus

$$\begin{gathered} a_1 = e_{i_1} + f_{j_1} , \qquad a_2 = e_{i_2} + f_{j_2} \\ a_3 = e_{i_1} + f_{j_2} , \qquad a_4 = e_{i_2} + f_{j_1} \\ i_1 \neq i_2,\ j_1 \neq j_2 ;\ \alpha_1 = \alpha_2 = -\alpha_3 = -\alpha_4 . \end{gathered} \tag{4.5}$$

We now consider the equation (iiic)

$$\alpha_1 a_1 + \zeta_\nu a_\nu + \xi x + \eta y = 0$$

separately for $\nu = 2, 3, 4$ denoting by

$$x = e_i + f_j , \quad y = e_{i^*} + f_{j^*}$$

the two vectors of a nontrivial solution.

For $\nu = 2$, (4.2), (4.3) and (4.5) imply

$$\begin{aligned} \{\xi e_i, \eta e_{i^*}\} &= \{-\alpha_1 e_{i_1}, -\zeta e_{i_2}\} \\ \{\xi f_j, \eta f_{j^*}\} &= \{-\alpha_1 f_{j_1}, -\zeta f_{j_2}\} . \end{aligned} \tag{4.6}$$

Since the solution is assumed nontrivial, (4.6) leaves, apart from an interchange of x and y, the only possibility

$$\begin{aligned} \xi e_i &= -\alpha_1 e_{i_1}, \quad \eta e_{i^*} = -\zeta e_{i_2} \\ \xi f_j &= -\zeta f_{j_2}, \quad \eta f_{j^*} = -\alpha_1 f_{j_1} . \end{aligned}$$

Hence

$$\xi = -\alpha_1 = -\zeta = \eta$$
$$e_i = e_{i_1}, \quad e_{i^*} = e_{i_2}, \quad f_j = f_{j_2}, \quad f_{j^*} = f_{j_1} .$$

is the only nontrivial solution. Thus $\rho_2 = 1$ and $\zeta_2 = \alpha_1 = \alpha_2$.

For $\nu = 3$, (4.2), (4.3) and (4.5) imply

$$\begin{aligned} (\alpha_1 + \zeta) e_{i_1} &= -(\xi e_i + \eta e_{i^*}) \\ \{\alpha_1 f_{j_1}, \zeta f_{j_2}\} &= \{-\xi f_j, -\eta f_{j^*}\} . \end{aligned}$$

If $\alpha_1 + \zeta \neq 0$, then the first relation implies $e_i = e_{i^*} = e_{i_1}$, and the second relation above then implies that the solution is trivial. Hence, the only nontrivial solutions (apart from interchange of x and y) are

$$\alpha_1 = -\zeta = -\xi = \eta$$
$$f_j = f_{j_1}, \quad f_{j^*} = f_{j_2}, \quad e_i = e_{i^*} \neq e_{i_1}$$

and the last relation shows that their number $\rho_3 = m - 1$.

For $\nu = 4$, the argument used for $\nu = 3$, with interchange of role between e_μ and f_μ, shows that (iiic) has $n - 1$ nontrivial solutions in (D^*), and again with constant ζ. This completes the proof that (D^*) satisfies condition (iii), and hence that all conditions of (3.3) are necessary.

5. Sufficiency.

Outline of sufficiency proof. In this proof we pursue a two-fold objective: besides establishing the truth of the theorem we wish to obtain a practical method of testing for possible equivalence and of actually finding the transformations T, P and Q when equivalence holds. For this reason, the proof will be by construction.

The idea of the proof is roughly as follows. Let S be the set of columns of A. Visualizing an (ordered) basis D_0 among the columns of D, we seek a basis S_0 in S which has the same structural (linear) relation to the rest of S as D_0 has to the rest of D; in other words S_0 should be such that among the transformations T, Q that map S_0 onto D_0 there is at least one pair that maps S onto D.

The scheme of construction is the following. After choosing a set B_0 of two distinct vectors in S, the conditions of the theorem are used to first extend B_0 to a certain set B_1 of 4 distinct vectors in S and subsequently extend B_1 to a set B_2 of $m + n - 1$ distinct vectors in S. The next objective is to prove that B_2 is a basis for S. This is done by first extending B_2 to a set B_3 of mn vectors in S in such a way that each vector of B_3 appears represented in terms of B_2, and subsequently showing that the vectors in B_3 are all distinct, so that $B_3 = S$ and hence B_2 is a basis for S. Implicitly B_2 has been so modeled after a particular basis D_0 in D as to insure that transformations T, Q which adequately map B_2 onto D_0 will also map S onto D. The construction of T, Q and P is then straightforward.

From the above it is clear that much of the proof is concerned with existence and number of solutions, with strong emphasis on distinctness of certain sets of vectors obtained from solutions.

REMARK. The basis D_0 which implicitly serves as model for the construction of $B_2 = S_0$, is one whose linear relation to D shows a maximum of symmetry; in other words, D_0 has been so chosen, that among the tranformations that leave D invariant the number of those that also leave D_0 invariant is a maximum. In the geometric interpretation D_0 represents what in graph theoretical terms may be called a twin star, that is two stars with a common link. Heuristically, the columns of D_0 correspond to shipping routes which connect a fixed source with all destinations plus those which connect a fixed destination with all sources.

Proof of sufficiency. We first show a particular consequence of (i), (ii) and (iii), which will be of frequent use in the subsequent proof.

(5.1) Let $a_1 \in S$, $c_1 \in S$, $a_1 \neq c_1$, $\alpha_1 \neq 0$, $k \geqq 2$. If the equation

(a) $$\alpha_1 a_1 + \zeta c_1 + \xi x + \eta y = 0$$

has $k - 1$ distinct nontrivial solutions in S given by

(b) $$\alpha_1 a_1 + \gamma_1 c_1 + \alpha_i a_i + \gamma_i c_i = 0 \qquad (i = 2, \cdots, k)\,,$$

then the $2k$ vectors

$$a_1, a_2, \cdots, a_k\,, \quad c_1, c_2, \cdots, c_k$$

are distinct and

$$\alpha_i \neq 0,\ \gamma_i \neq 0 \qquad (i = 1, 2, \cdots k)\,.$$

Indeed, by (i), for every fixed $i > 1$ the four vectors a_1, c_1, a_i, c_i

are distinct and their coefficients $\neq 0$. Hence

$$\alpha_i \neq 0\,, \quad \gamma_i \neq 0 \qquad (i = 1, 2 \cdots, k)$$
$$a_i \neq c_i \qquad (i = 1, 2, \cdots, k)$$
$$a_i \neq a_1 \neq c_i\,, \quad a_i \neq c_1 \neq c_i \qquad (i = 2, 3, \cdots, k)\,.$$

Further, for $2 \leqq i < \nu \leqq k$, the relation (b) implies

$$\alpha_i a_i + \gamma_i c_i = \alpha_\nu a_\nu + \gamma_\nu c_\nu\,.$$

This is a nontrivial relation, since the two sides belong to distinct solutions of (a). Therefore by (i) the four vectors are distinct, hence, in particular

$$a_i \neq a_\nu\,, \quad a_i \neq c_\nu\,, \quad c_i \neq a_\nu\,, \quad c_i \neq c_\nu \qquad (2 \leqq i < \nu \leqq k)\,.$$

This completes the verification of (5.1).

Further we note

(5.2) If the equation

(a) $$\alpha_1 a_1 + \zeta a_2 + \xi x + \eta y = 0$$

has the unique solution in S

(b) $$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 = 0\,,$$

then the equation

(c) $$\alpha_3 a_3 + \zeta a_4 + \xi x + \eta y = 0$$

has a unique solution in S.

Indeed, if (c) had a solution in S in addition to and distinct from the one exposed in (b), say

$$\alpha_3 a_3 + \alpha_4 a_4 + \gamma c + \delta d = 0\,,$$

this would imply

$$\alpha_1 a_1 + \alpha_2 a_2 - \gamma c - \delta d = 0$$

and thus contradict the premise on (a).

Now let x_{11} be an arbitrary but fixed element of S.

By condition (ii) and the assumption on m and n there are elements x_{12}, x_{21}, x_{22} in S and real number $\alpha_{12}, \alpha_{21}, \alpha_{22}$ such that

(5.3) $$x_{11} + \alpha_{12} x_{12} + \alpha_{21} x_{21} + \alpha_{22} x_{22} = 0\,.$$

By (5.1) the x_{ij} are distinct, and the $a_{ij} \neq 0$.

In view of (iii) we assume the numbering of the last three terms on the left hand side of (5.3) so chosen that

(5.4) $x_{11} + \zeta_{22}x_{22} + \xi x + \eta y = 0$ has a unique solution in S

(5.5) $x_{11} + \zeta_{12}x_{12} + \xi x + \eta y = 0$ has $(m - 1)$ solutions in S

(5.6) $x_{11} + \zeta_{21}x_{21} + \xi x + \eta y = 0$ has $(n - 1)$ solutions in S.

While the unique solution of (5.4) is given by (5.3), let the solutions to (5.5) and (5.6) be given by

$$x_{11} + \alpha_{12}x_{12} + \alpha_{i1}x_{i1} + \alpha_{i2}x_{i2} = 0 \qquad (i = 2, 3, \cdots, m) \tag{5.5*}$$

$$x_{11} + \alpha_{21}x_{21} + \alpha_{1j}x_{1j} + \alpha_{2j}x_{2j} = 0 \qquad (j = 2, 3, \cdots, n) \tag{5.6*}$$

where the coefficients of x_{12} and x_{21} have been taken from (5.3), since, for $i = 2$ and for $j = 2$, (5.5*) and (5.6*) specialize to (5.3), and the four vectors in (5.3) are all distinct, so that (iii) and hence in particular (iiie) applies.

We note that by condition (i) the four vectors in (5.5*) are distinct for fixed i and those in (5.6*) are distinct for fixed j.

Application of (iii) to (5.5*) asserts, in view of (5.5), that exactly one of the two equations

$$x_{11} + \zeta x_{i1} + \xi x + \eta y = 0$$
$$x_{11} + \zeta x_{i2} + \xi x + \eta y = 0$$

has a unique solution in S. Application of (iii) to (5.6*) yields, in view of (5.6), a similar assertion. We assume the notation in (5.5*) and (5.6*) so chosen that

(5.7) each of the $m + n - 2$ equations

$$x_{11} + \zeta x_{i2} + \xi x + \eta y = 0$$
$$x_{11} + \zeta x_{2j} + \xi x + \eta y = 0$$

has a unique solution in S.

These solutions are obviously given by (5.5*) and (5.6*), and we note that, then, by (5.2), also

(5.8) each of the $m + n - 2$ equations

$$\alpha_{12}x_{12} + \zeta x_{i1} + \xi x + \eta y = 0$$
$$\alpha_{21}x_{21} + \zeta x_{1j} + \xi x + \eta y = 0$$

has a unique solution in S,

(which is again given by (5.5*) and (5.6*)).

Next, using (5.5*) as premise of (iii), (iiid) in conjunction with assumptions (5.5) and (5.7) implies that, for each i such that $2 \leqq i \leqq m$, the equation

(5.9) $x_{11} + \zeta x_{i1} + \eta y + \xi x = 0$ has $n - 1$ solutions in S

and (iiie) implies that the solutions are of the form

(5.9*) $\quad x_{11} + \alpha_{i1}x_{i1} + \beta_{ij}y_{ij} + \alpha_{ij}x_{ij} = 0 \qquad (i = 2, 3, \cdots, m;\ j = 2, 3, \cdots, n)$

where

(5.10)
$$\begin{aligned} \beta_{i2} &= \alpha_{12}\,, \quad y_{i2} = x_{12} && (i = 2, 3, \cdots, m) \\ \beta_{2j} &= \alpha_{1j}\,, \quad y_{2j} = x_{1j} && (j = 2, 3, \cdots, n)\,. \end{aligned}$$

By (iiid) and (5.2) we may assume the notation in (5.9*) so chosen that

(5.11)
$$\begin{aligned} &x_{11} + \zeta x_{ij} + \xi x + \eta y = 0 \quad \text{has a unique solution in } S \\ &\alpha_{i1}x_{i1} + \zeta y_{ij} + \xi x + \eta y = 0 \quad \text{has a unique solution in } S \end{aligned}$$

for each (i, j) such that $2 \leqq i \leqq m,\ 2 \leqq j \leqq n$.

The set of vectors in (5.9*) shall now be investigated for distinctness.

Let F denote the formal set of vectors exposed in (5.9*), that is the set of those symbols in (5.9*) which denote vectors in S, and let F_i denote the subset of F for a fixed i in (5.9*). When G is a subset of F such that distinct symbols in G denote distinct vectors in S, we shall say briefly that "the vectors in G are distinct."

For easy reference we note the following implications of (5.1) when applied to (5.9*) and (5.6*).

(5.12)
(a) The vectors in F_i are distinct
(b) $x_{11} \neq y_{ij}\,;\quad x_{11} = x_{ij} \Rightarrow i = j = 1$
(c) $\alpha_{ij} \neq 0,\ \beta_{ij} \neq 0$

and consider the sets

(5.13)
$$\begin{aligned} E_1 &= \{x_{i1}/1 \leqq i \leqq m\} \\ E_2 &= \{y_{2j}/2 \leqq j \leqq n\} \\ G_2 &= \{y_{ij}/2 \leqq i \leqq m,\ 2 \leqq j \leqq n\} \\ E_3 &= \{x_{ij}/2 \leqq i \leqq m,\ 2 \leqq j \leqq n\} \\ E &= E_1 \cup E_2 \cup E_3\,. \end{aligned}$$

Formally E consists of $m + (n-1) + (m-1)(n-1) = mn$ vectors, that is E consists of mn distinct symbols denoting vectors in S. We shall prove

(5.14)
(a) In each E_α the vectors are distinct ($\alpha = 1, 2, 3$)
(b) E_1, G_2, E_3 are pairwise disjoint
(c) the vectors in E are distinct

where (c), as obvious consequence of (a) and (b), is merely noted for reference.

Proof of (5.14a).

From (5.1) it follows that the x_{i1} are distinct, since they are among the vectors of (5.5*); that the y_{2j} are distinct and that the x_{ij} are distinct for each fixed i follows from (5.12a). That the x_{ij} are distinct for distinct i's, we show indirectly. Assume

$$x_{ij} = x_{\mu\nu}$$

for some particular i, j, μ, ν with $i \neq \mu$. Then by (5.9*)

$$\begin{aligned} x_{11} + \alpha_{i1}x_{i1} + \beta_{ij}y_{ij} + \alpha_{ij}x_{ij} &= 0 \\ x_{11} + \alpha_{\mu1}x_{\mu1} + \beta_{\mu\nu}y_{\mu\nu} + \alpha_{\mu\nu}x_{ij} &= 0 \, . \end{aligned}$$

Since these two relations can be interpreted as representing two solution in S to the equation

$$x_{11} + \zeta x_{ij} + \xi x + \eta y = 0 \, ,$$

the first assumption in (5.11) implies

$$\begin{aligned} \alpha_{ij} &= \alpha_{\mu\nu} \\ \alpha_{i1}x_{i1} + \beta_{ij}y_{ij} &\equiv \alpha_{\mu1}x_{\mu1} + \beta_{\mu\nu}y_{\mu\nu} \end{aligned} \tag{5.15}$$

where $\equiv$ is to signify that the vectors and their coefficients are the same on both sides.

Certainly $x_{i1} \neq x_{\mu1}$, since both belong to E_1 and $i \neq \mu$. Therefore the only possibility is

$$\alpha_{i1}x_{i1} = \beta_{\mu\nu}y_{\mu\nu} \, , \quad \alpha_{\mu1}x_{\mu1} = \beta_{ij}y_{ij} \, . \tag{5.16}$$

On the other hand, subtraction of the μth from the ith relation in (5.5*) yields

$$\alpha_{i1}x_{i1} - \alpha_{\mu1}x_{\mu1} + \alpha_{i2}x_{i2} - \alpha_{\mu2}x_{\mu2} = 0 \, , \tag{5.17}$$

which relation, after replacement of the second term by its expression from (5.16), reads

$$\alpha_{i1}x_{i1} - \beta_{ij}y_{ij} + \alpha_{i2}x_{i2} - \alpha_{\mu2}x_{\mu2} = 0 \, . \tag{5.18}$$

By (5.12a, c) $x_{i1} \neq y_{ij}$, $\alpha_{i1} \neq 0$, $\beta_{ij} \neq 0$. Hence (5.18) and (5.9*) can each be interpreted as representing a solution in S to the equation

$$\alpha_{i1}x_{i1} + \zeta y_{ij} + \xi x + \eta y = 0 \, . \tag{5.19}$$

The solution (5.18) is not trivial, since the first three vectors are distinct by (5.12a). Then (iiie) implies $\beta_{ij} = -\beta_{ij}$, contradicting $\beta_{ij} \neq 0$. (*Remark.* In order to dissipate a possible feeling of uneasiness about hinging the rather boresome proof on a mere—possibly erroneous!—sign, we still note this alternative argument: By (5.11), the equation (5.19) has a unique solution in S; this solution is exposed in (5.9*); hence either $x_{i2} = x_{11}$ or $x_{\mu 2} = x_{11}$; either case contradicts (5.12b)). This completes the proof of (5.14a).

Proof of (5.14b). First, to see that $E_1 \cap G_2 = \varnothing$ assume that for some $i \geqq 2$, $j \geqq 2$ and μ,

$$x_{\mu 1} = y_{ij} .$$

Then, by (5.12b), $\mu \geqq 2$.

Substitution in (5.9*) yields

$$x_{11} + \alpha_{i1}x_{i1} + \beta_{ij}x_{\mu 1} + \alpha_{ij}x_{ij} = 0 . \tag{5.20}$$

Since x_{11}, x_{i1}, x_{ij} are in F_i, they are by (5.12a) distinct. Then by (i) and (5.12c) all four vectors in (5.20) are distinct and (5.20) can be interpreted as exposing a nontrivial solution in S to the equation

$$x_{11} + \zeta x_{\mu 1} + \xi x + \eta y = 0 .$$

Hence by (iiie) and (5.9*)

$$\beta_{ij} = \alpha_{\mu 1}$$

and subtraction of (5.20) from the μth relation in (5.5*) yields

$$\alpha_{12}x_{12} - \alpha_{i1}x_{i1} + \alpha_{\mu 2}x_{\mu 2} - \alpha_{ij}x_{ij} = 0 . \tag{5.21}$$

The first 3 vectors in (5.21) are distinct since they are among the vectors of (5.5*) which, by (5.1), are all distinct. Hence (5.21) exposes a nontrivial, by (5.2), (5.7) and (5.5*) unique, solution in S to the equation

$$\alpha_{12}x_{12} + \zeta x_{i1} + \xi x + \eta y = 0 .$$

Therefore either $x_{\mu 2} = x_{11}$ or $x_{ij} = x_{11}$; either case contradicts (5.12b), since $i \geqq 2$. This proves $E_1 \cap G_2 = \varnothing$.

To see that $E_1 \cap E_3 = \varnothing$, assume that for a particular triple μ, i, j, such that $i \geqq 2$, $j \geqq 2$,

$$x_{\mu 1} = x_{ij} .$$

Then, by (5.12b), $\mu \geqq 2$. By (5.11) the equation

$$x_{11} + \zeta x_{ij} + \xi x + \eta y = 0$$

has a unique solution in S, whereas by (iiid), (5.7) and (5.5*) the same equation has $n-1$ solutions, thus contradicting the assumption $n \geqq 3$.

To see that $G_2 \cap E_3 = \varnothing$, assume

$$x_{ij} = y_{\mu\nu}$$

for some particular i, j, μ, ν, all $\geqq 2$. By (5.11) the equation

$$x_{11} + \zeta x_{ij} + \xi x + \eta y = 0$$

has a unique solution in S, whereas by (iiid), (5.9), (5.9*) and (5.11) the same equation has $m-1$ solutions in S, thus contradicting that $m \geqq 3$. This completes the proof of (5.14b).

Proof of (5.14c). Obvious, from (5.14a, b) and $E_2 \subset G_2$. This completes the proof of (5.14).

Since $E \subset S$ and each consists of mn distinct elements it follows that

$$E = S \,. \tag{5.22}$$

Considering now the set $G_2 - E_2$, that is

$$\{y_{ij}/3 \leqq i \leqq m\,, \quad 2 \leqq j \leqq n\} \tag{5.23}$$

it is clear from (5.14b) that each vector of (5.23) must equal some vector of E_2, that is

$$i \geqq 3, j \geqq 2 \Rightarrow \begin{cases} y_{ij} = y_{2\nu} \text{ for some } \nu \geqq 2 \\ \beta_{ij} = \beta_{2\nu} \end{cases} \tag{5.24}$$

where the second implication is, in view of (5.9*) and (iiie), a direct consequence of the first.

Further, noting that for every fixed $i \geqq 2$ the y_{ij} belong to F_i and therefore by (5.12a) are distinct, we have

$$\text{If } y_{ij} = y_{2\nu} \text{ and } y_{ij^*} = y_{2\nu^*}, \text{ then} \quad j = j^* \Longleftrightarrow \nu = \nu^*. \tag{5.25}$$

Thus by (5.24) and (5.25), for each i such that $3 \leqq i \leqq m$, there is a permutation π_i on the set of integers $\{2, 3, \cdots, n\}$ such that

$$y_{ij} = y_{2\nu} \Longleftrightarrow j = \pi_i(\nu) \,.$$

Noting that besides assumption (5.10) for the case $i = 2$ or $j = 2$ no determination has been made concerning the numbering of the solutions to (5.9) in (5.9*), we shall now, in order to obtain a convenient notation, assume that for each $i \geqq 3$

(5.26) the numbering in (5.9*) is so chosen that

$$y_{ij} = y_{2j} \qquad (j = 2, 3, \cdots, n)$$

with the obvious consequence

$$(5.27) \qquad \beta_{ij} = \beta_{2j} ,$$

and this is consistent with the choice (5.10) for $j = 2$, namely $y_{i2} = x_{12} = y_{22}$. Renaming these vectors and their coefficients in (5.9*) by

$$(5.28) \qquad y_{2j} = x_{1j}, \quad \beta_{2j} = \alpha_{1j}$$

(5.9*) can be re-written in the form

$$(5.29) \quad x_{11} + \alpha_{i1}x_{i1} + \alpha_{1j}x_{1j} + \alpha_{ij}x_{ij} = 0 \quad (i = 2, \cdots, m;\ j = 2, \cdots, n) .$$

By (5.22) the mn vectors in (5.29) are the mn distinct vectors of S, and by (5.29) they appear represented in terms of the set

$$(5.30) \qquad B = E_1 \cup E_2 = \{x_{i1}/1 \leqq i \leqq m\} \cup \{x_{1j}/2 \leqq j \leqq n\} .$$

Hence

$$(5.31) \qquad B \text{ is a basis in } S .$$

It is now a simple matter to construct the matrices T, P and Q such that $TAPQ = D$.

First we set, for $i = 2, 3, \cdots, m$ and $j = 2, 3, \cdots, n$,

$$(5.32) \qquad \begin{aligned} \gamma_{11} &= 1 \\ \gamma_{i1} &= -\alpha_{i1} \\ \gamma_{1j} &= -\alpha_{1j} \\ \gamma_{ij} &= \alpha_{ij} \end{aligned}$$

so that (5.29) reads

$$(5.33) \qquad \gamma_{ij}x_{ij} = \gamma_{i1}x_{i1} + \gamma_{1j}x_{1j} - \gamma_{11}x_{11} .$$

Then, assuming that the $m + n + k$ rows of $\bar{A}$ have been so arranged that

(5.34) the rows numbered $2, 3, \cdots, m + n$ are linearly independent,

and denoting by e_ν the νth unit vector in R^{m+n+k}, we define $m + n + k$ vectors:

$$(5.35) \qquad \begin{aligned} u_1 &= e_1 \\ v_1 &= x_{11} - e_1 \\ u_i &= \gamma_{i1}x_{i1} - u_1 && (i = 2, 3, \cdots, m) \\ v_j &= \gamma_{1j}x_{1j} - u_1 && (j = 2, 3, \cdots, n) \\ w_r &= e_{m+n+r} && (r = 1, 2, \cdots, k) . \end{aligned}$$

Then

$$\gamma_{ij}x_{ij} = u_i + v_j \qquad (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)\,. \tag{5.36}$$

By (5.31) and (5.34) the set

$$e_1, x_{11}, x_{21}, \cdots, x_{m1}, x_{12}, x_{13}, \cdots, x_{1n}, e_{m+n+1}, \cdots, e_{m+n+k}$$

is a basis in R^{m+n+k}. Hence by (5.12c) the set defined in (5.35) is also a basis. Therefore the transformation T^{-1} defined by

$$\begin{aligned} T^{-1}e_i &= u_i && (i = 1, 2, \cdots, m)\\ T^{-1}e_{m+j} &= v_j && (j = 1, 2, \cdots, n)\\ T^{-1}e_{m+n+r} &= e_{m+n+r} && (r = 1, 2, \cdots, k) \end{aligned} \tag{5.37}$$

is nonsingular.

It follows from (5.35) and (5.36) that

$$\gamma_{ij}x_{ij} = T^{-1}(e_i + e_{m+j}) = T^{-1}d_{ij}\,, \tag{5.38}$$

where d_{ij} denotes the (ij)-column of D, that is the column of D in position $n(i-1)+j$.

If the vector x_{ij} of S is the μth column of A, and

$$\mu = \varphi(n(i-1)+j) \tag{5.39}$$

denotes this permutation on the set of numbers $\{1, 2, \cdots, mn\}$, then let P denote the matrix of the transformation on R^{mn} which carries the unit vector e_ν into $e_{\varphi(\nu)}$ $(\nu = 1, 2, \cdots, mn)$, that is, P is a permutation matrix defined by

$$\begin{aligned} P = (P_{\mu\nu});\ p_{\mu\nu} &= 1 \text{ when } \mu = \varphi(\nu)\\ p_{\mu\nu} &= 0 \text{ otherwise}\,. \end{aligned} \tag{5.40}$$

Finally let Q denote the nonsingular diagonal matrix of mn rows and mn columns

$$\begin{aligned} Q &= (q_{\mu\nu})\\ q_{\nu\nu} &= |\gamma_{ij}| \text{ where } (n-1)i + j = \nu\\ q_{\mu\nu} &= 0 \text{ when } \mu \neq \nu\,. \end{aligned} \tag{5.41}$$

Then, obviously, AP is the matrix whose columns are the vectors x_{ij} arranged in lexicographic order. APQ is the matrix with the columns $|\gamma_{ij}|x_{ij}$ in the same order. Lastly, by (5.38) the transformation T whose inverse was defined in (5.37), carries each column $|\gamma_{ij}|x_{ij}$ into $\varepsilon_{ij}d_{ij}$ where

(5.42 $$\varepsilon_{ij} = |\gamma_{ij}|/\gamma_{ij}\,.$$

Hence

$$TA\bar{P}Q = \bar{D}\,. \tag{5.43}$$

This completes the proof of the theorem (3.3).

6. **Remarks.** First, it is clear from (5.41) and (5.42) that the equivalence is to the constraints of the standard transportation program if and only if the γ_{ij} are all of the same sign:

(6.1) In (5.43) $\bar{D}$ is of the form $\{e_i + e_{m+j}\}$ if and only if the γ_{ij} are all of the same sign

where obviously $-T$ is used when they are all negative.

Second, we remark that in applications the following method of test for equivalence and construction of the transformations ensues directly from the proof.

1. Find 4 columns in A satisfying (5.3), denote them so as to satisfy (5.4–6), and expose the solutions (5.5*) and (5.6*) to (5.5) and (5.6), denoted to satisfy (5.7).
2. Establish that the set $x_{11}, x_{21}, \cdots, x_{m1}, x_{12}, \cdots, x_{1n}$ is independent.
3. Establish that each vector $-x_{11} + \gamma_{i1}x_{i1} + \gamma_{1j}x_{1j} = z_{ij}$ is a multiple of a column of A, that is $z_{ij} = \gamma_{ij}x_{ij}$ with x_{ij} a column of A.
4. Construct T^{-1}, P and Q as defined in (5.32) to (5.41).
5. Invert T^{-1} and compute $T\bar{A}PQ$ to obtain $\bar{D}$.

If either of the steps 1 through 3 can logically not be performed, there is no equivalence in the sense of (2.1).

The computational algorithm is discussed in detail in a forthcoming self-contained note where the direct proof of the algorithm is naturally considerably shorter due to the significantly stronger conditions of the algorithm as compared to those of the theorem in the present note.

Finally, it is obvious that a concept of equivalence in a more general sense, where it is merely required that the optimal vectors of the two linear-optimization programs be in one-to-one linear relation, is of greater practical significance; it includes the case where A in (1.5) has less than mn columns and hence can be mapped only onto a proper subset D_0 of D. Methodologically, the treatment of this case requires a different mechanism: in particular, the choice of a structurally simple basis in D as used in the present note is not generally possible in the general case since such basis may not exist in D_0, and moreover D_0 is not given. In other words, whereas in the case of the present note there is equivalence if and only if the linear structure of A is the same as the linear structure of DQ

for a given D and some diagonal matrix Q, equivalence in the other case will exist whenever A has the same linear structure as D_0Q_0, for some subset D_0 of D of rank $m+n-1$ and some diagonal Q_0; there are thus as many structurally different matrices A equivalent to a subset of D as there are structurally different subsets D_0 in D (of the same rank as D); therefore, whereas in the present case the method is simply a mechanism for testing whether A has the structure of DQ, the first objective in the study of the general case is a characterization of the structure of A (in a form suitable for comparison with the structures of the sets D_0Q), and hence requires an altogether different method. A study of the general case will be presented in a forthcoming note.

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