SEMIGROUPS OF MATRICES DEFINING LINKED OPERATORS WITH DIFFERENT SPECTRA

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1. Introduction. The concept of "linked operators" was introduced by A. E. Taylor and the author in [1]. This concept was originally suggested by work involving bounded linear operators on the sequence spaces l_p. For example, if the infinite matrix (t_{ij}) defines operators T_p and T_q that are bounded on l_p and l_q, respectively, then these operators are linked. The somewhat complicated general definition of linked operators is deferred until § 2 of this paper. In [1] an isolated, specific example of linked operators with different spectra was given. The purpose of this paper is to exhibit three infinite semigroups of infinite matrices (t_{ij}), with complex coefficients, such that each of their elements defines linked operators with different spectra.

In the next section we give some preliminary definitions and notation and in the final section we prove a basic lemma and our principal theorems.

2. Preliminary definitions and notation. We first give the definition of linked operators.

DEFINITION. *Let X, Y be complex linear spaces, and Z a non-void complex linear space contained in both X and Y. Let X be a Banach space X_1, Y a Banach space Y_2 under the norms n_1, n_2 respectively. Let Z be a Banach space Z_N under the norm N defined by $N(z) = \max[n_1(z), n_2(z)]$. With the usual uniform norms let T_1, T_2 be bounded linear operators on X_1, Y_2 respectively, such that $T_1 z = T_2 z \in Z$ when $z \in Z$. Operators satisfying these conditions are said to be "linked."*

Our basic notation will be as follows: If T denotes the infinite matrix (t_{ij}), with complex coefficients, then T^t will denote its transpose, and $\bar{T}$ the matrix $(\bar{t}_{ij})$, where $\bar{z}$ is the complex conjugate of z. Let T_p denote the operator defined on l_p by the matrix T, $\| T_p \|$ its norm, and $[l_p]$ the algebra of bounded linear operators on l_p. Also let $\rho(T_p)$ denote the resolvent set of T_p, consisting of all complex λ such that $\lambda I - T_p$ defines a one-to-one correspondence of l_p onto l_p; $\sigma(T_p)$ denote the spectrum of T_p, consisting of all λ not in $\rho(T_p)$; and $|\sigma(T_p)|$ the spectral radius of T_p.

The matrix (t_{ij}) is said to be "regular" in case for every convergent sequence $[\zeta_n]$, $\lim_{n\to\infty} \zeta_n = \zeta$, each of the series $\sum_{k=1}^{\infty} t_{ik}\zeta_k$ is convergent

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and $\lim_{i\to\infty}\sum_{k=1}^{\infty} t_{ik}\zeta_k = \zeta$. It is well known that a set of necessary and sufficient conditions for a matrix to be regular are:

$$\sup_i \sum_{k=1}^{\infty} |t_{ik}| < \infty \tag{1}$$

$$\lim_{i\to\infty} t_{ik} = 0 \text{ for } k = 1, 2, \cdots \tag{2}$$

$$\lim_{i\to\infty} \sum_{k=1}^{\infty} t_{ik} = 1\ . \tag{3}$$

3. Principal theorems.

LEMMA. *Suppose that $C = (c_{ij})$ and $D = (d_{ij})$ define elements of $[l_1]$, and $C^t/\|C_1\|$ and $D^t/\|D_1\|$ are regular. Then $(CD)^t/\|(CD)_1\|$ is regular and $\|(CD)_1\| = \|C_1\|\,\|D_1\|$.*

Proof. Since the product of regular matrices exists and is regular, we have,

$$\lim_{i\to\infty} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{d_{ik}^t}{\|D_1\|}\frac{c_{kj}^t}{\|C_1\|} = 1\ ,$$

whence,

$$1 = \lim_{i\to\infty}\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{d_{ik}^t\,c_{kj}^t}{\|D_1\|\,\|C_1\|}\right| \leqq \overline{\lim_{i\to\infty}}\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\frac{c_{jk}\,d_{ki}}{\|C_1\|\,\|D_1\|}\right| \leqq \frac{\|(CD)_1|}{\|C_1\|\,\|D_1\|} \leqq 1\ .$$

Therefore $\|(CD)_1\| = \|C_1\|\,\|D_1\|$, and $D^tC^t/\|D_1\|\,\|C_1\| = (CD)^t/\|(CD)_1\|$ is regular. The following result is a simple consequence of this lemma, coupled with the well known fact that

$$\lim_{n\to\infty} \|T^n\|^{1/n} = |\sigma(T)|\ ,$$

whenever $T \in [X]$, where X is a complex Banach space.

COROLLARY. *If $T\in[l_1]$ and $T^t/\|T_1\|$ is regular, then $|\sigma(T_1)| = \|T_1\|$.*

We are now ready for our principal theorems.

THEOREM 1. *Suppose that both $T = (t_{ij})$ and $T^t = (t_{ij}^t)$ define elements of $[l_1]$, $T^t/\|T_1\|$ is regular, and $\|T_1^t\| < \|T_1\|$. Then $|\sigma(T_1)| > |\sigma(T_p)|$, $p > 1$.*

Proof. Using the fact that the spectral radius of an operator is less than or equal to its norm, and the special case where $q = 1$, of the inequality

$$\|T_p\| \leqq \|T_q\|^{(q+p(1-q))/(2-q)p}\|(T^t)_q\|^{(p-q)/(2-q)p}\ ,$$

p between q and q', (which in turn is a special case of a more general inequality, (2), p. 729 in [2]), we see that

$$\text{(A)} \qquad |\sigma(T_p)| \leqq \| T_p \| \leqq \| T_1 \|^{1/p} \| T_1^t \|^{1-1/p} .$$

Since by hypothesis $\| T_1^t \| < \| T_1 \|$, it follows immediately that $|\sigma(T_p)| < \| T_1 \|$. But since by since by hypothesis $T^t/\| T_1 \|$ is regular, we have by our corollary that $\| T_1 \| = |\sigma(T_1)|$, and our theorem is proved.

One might wonder if the result of Theorem 1 is perhaps attributable to the "lopsided" nature of the matrix; that is, the property that the supremum of the l_1 norms of the column vectors is greater than that of the row vectors. The following theorem demonstrates that is not the case.

THEOREM 2. *Suppose that both $T/\| T_1^t \|$ and $T^t/\| T_1 \|$ are regular and that $\| T_1^t \| < \| T_1 \|$. Then $A = \bar{T}^t + T$ is a hermitian symmetric matrix such that $|\sigma(A_p)| < |\sigma(A_1)|$, $1 < p < \infty$.*

Proof. The assumptions of regularity guarantee that

$$\lim_{j\to\infty} \sum_{i=1}^{\infty} t_{ij} = \| T_1 \| \text{ and } \lim_{j\to\infty} \sum_{i=1}^{\infty} t_{ij}^t = \lim_{j\to\infty} \sum_{i=1}^{\infty} \overline{t_{ij}^t} = \| T_1^t \| .$$

Thus we see that

$$\| T_1 \| + \| \bar{T}_1^t \| \geqq \| T_1 + \bar{T}_1^t \| \geqq \limsup_{j\to\infty} \left| \sum_{i=1}^{\infty} (t_{ij} + t_{ij}^t) \right| = \| T_1 \| + \| T_1^t \| ,$$

whence $\| T_1 + \bar{T}_1^t \| = \| T_1 \| + \| T_1^t \|$.

Now

$$\begin{aligned} \| T_p + \bar{T}_p^t \| &\leqq \| T_p \| + \| \bar{T}_p^t \| = \| T_p \| + \| T_p^t \| \\ &\leqq \| T_1 \|^{1/p} \| T_1^t \|^{1-(1/p)} + \| T_1^t \|^{1/p} \| T_1 \|^{1-(1/p)} , \end{aligned}$$

the last inequality being a result of (A) above. We shall now show that the right hand member of this inequality is less than $\| T_1 \| + \| T_1^t \|$.

From the hypothesis that $\| T_1^t \| < \| T_1 \|$, we can conclude that $\| T_1 \|^{1/p} - \| T_1^t \|^{1/p} > 0$ and $\| T_1^t \|^{1-(1/p)} - \| T_1 \|^{1-(1/p)} < 0$ for $1 < p < \infty$. It is now an immediate consequence that

$$\begin{aligned} 0 &> (\| T_1 \|^{1/p} - \| T_1^t \|^{1/p})(\| T_1^t \|^{1-(1/p)} - \| T_1 \|^{1-(1/p)}) \\ &= -\| T_1 \| - \| T_1^t \| + \| T_1 \|^{1/p} \| T_1^t \|^{1-(1/p)} + \| T_1 \|^{1-(1/p)} \| T_1^t \|^{1/p} , \end{aligned}$$

whence

$$\| T_1 \|^{1/p} \| T_1^t \|^{1-(1/p)} + \| T_1 \|^{1-(1/p)} \| T_1^t \|^{1/p} < \| T_1 \| + \| T_1^t \| .$$

Using these inequalities together with the fact that

$$|\sigma(T_p + \bar{T}_p^t)| \leqq \| T_p + \bar{T}_p^t \| ,$$

we see that

$$|\sigma(T_p + \bar{T}_p^t)| < \| T_1 + \bar{T}_1^t \| .$$

It is obvious that the operator

$$\frac{T_1 + \bar{T}_1^t}{\| T_1 + T_1^t \|}$$

is regular and thus

$$|\sigma(T_1 + \bar{T}_1^t)| = \| T_1 + \bar{T}_1^t \| .$$

This with the last inequality implies the desired conclusion,

$$|\sigma(T_p + \bar{T}_p^t)| < |\sigma(T_1 + \bar{T}_1^t)| .$$

THEOREM 3. *Suppose $T = (t_{ij})$ defines an element of $[l_1]$, t_{ij} is positive for all i and j, and the infimum of the column sums of T is greater than $\| T_p \|$. Then $|\sigma(T_p)| < |\sigma(T_1)|$.*

Proof. Let $T^n = (t_{ij}^{(n)})$, $n > 1$. By hypothesis $\inf_j \sum_{i=1}^{\infty} t_{ij} = K > \| T_p \|$. If $\inf_j \sum_{i=1}^{\infty} t_{ij}^{(n)} \geqq K^n$, then

$$\begin{aligned}\inf_j \sum_{i=1}^{\infty} t_{ij}^{(n+1)} &= \inf_j \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}^{(n)} t_{kj} = \inf_j \left(\sum_{k=1}^{\infty} t_{kj} \sum_{i=1}^{\infty} t_{ik}^{(n)} \right) \\ &\geqq \inf_j \sum_{k=1}^{\infty} t_{kj} K^n = K^{n+1} .\end{aligned}$$

Thus by induction we have $\inf_j \sum_{i=1}^{\infty} t_{ij}^{(n)} \geqq K^n$ for all n. It follows that $\| T_1^n \| \geqq K^n$ for all n, whence

$$|\sigma(T_1)| \geqq K > \| T_p \| \geqq |\sigma(T_p)| ,$$

and our theorem is proved.

Final Remarks. Matrices satisfying the hypotheses of the above theorems are easily constructable. The matrix $T = (t_{ij})$,

$$t_{ij} = \begin{cases} j/(i-1)i & \text{if } i > j \\ 0 & \text{if } i \leqq j , \end{cases}$$

cited in [1], satisfies the hypotheses of each of theorems (where in particular $p = 2$ in Theorem 3).

That the set of matrices satisfying the hypotheses of any one of these theorems forms a semigroup is a simple matter of computation.

BIBLIOGRAPHY

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