A REMARK ON ANALYTICITY OF FUNCTION ALGEBRAS

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1. Let A be a closed separating subalgebra of $C(X)$, X compact, with maximal ideal space $\mathfrak{M}_A$ and Šilov boundary ∂_A. Naturally A can also be viewed as a closed subalgebra of $C(\mathfrak{M}_A)$ or $C(\partial_A)$.

Call A *analytic on* X if the vanishing of $f \in A$ on a non-void open subset of X implies $f \equiv 0$, or simply *analytic* if this holds for $X = \mathfrak{M}_A$. Recently Kenneth Hoffman asked if the analyticity of A on ∂_A implied analyticity on $\mathfrak{M}_A$; the present note is devoted to a counter-example.[1] Evidently such an example, analytic on its Šilov boundary, must be an integral domain, so our algebra is a non-analytic integral domain.

The example was suggested by, and utilizes, an interpolation theorem of Rudin and Carleson [5, 9], recently generalized by Bishop [3], which in fact permits the construction of a variety of unfamiliar tractable subalgebras of familiar algebras; consequently we shall discuss the construction in more generality than is absolutely necessary. Finally we give a slightly more complicated example which is also dirichlet.

NOTATION. $M(X)$ will denote the space of (finite complex regular Borel) measures μ on X; for such a μ, μ is orthogonal to $A (\mu \perp A)$ if $\mu(f) = \int f d\mu = 0$, f in A. And μ_F will denote the usual restriction of μ to $F \subset X$, while $f \mid F$ will be the restriction of a function f, $A \mid F$ the set $\{f \mid F : f \in A\}$. An algebra A will always be assumed to contain the constants.

2. Our construction is based on the following fact.

(2.1) *Suppose F is a closed subset of X, and $\mu_F = 0$ for all μ in $M(X)$ orthogonal to A. Then*[2]

(2.1.1) $$A \mid F = C(F) \ [3]$$

(2.1.2) *if X is metric, F is a peak set of A, i.e., there is an f in*

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[1] After this note was completed, I found that analyticity of A on $\mathfrak{M}_A$ implies analyticity on ∂_A; this will appear in a subsequent paper.

[2] (2.11) is Bishop's generalization of the Rudin-Carleson result mentioned before, which applies to the special case in which A is the "disc algebra" and F a subset of measure zero of the unit circle. (2.12) will actually be avoided in the specific examples we construct.

A with $f(F) = 1$ *and* $|f| < 1$ *on* $X \backslash F$ [7, 4.8].

Now suppose we are given two uniformly closed algebras A_1, A_2, as subalgebras of $C(\mathfrak{M}_1)$, $C(\mathfrak{M}_2)$, where $\mathfrak{M}_i = \mathfrak{M}_{A_i}$ is metric, $i = 1, 2$. Further suppose $\partial_2 = \partial_{A_2}$ is homeomorphic to a (compact) subset F of ∂_1 satisfying the hypothesis of (2.1) with $A = A_1$, $X = \partial_1$, so that $A_1 | F = C(F)$. Identifying F and ∂_2 (via some homeomorphism) we may form a compact metric space $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ containing each $\mathfrak{M}_i$ as a subspace, with $\mathfrak{M}_1 \cap \mathfrak{M}_2 = F = \partial_2$. Now form the closed subalgebra A of $C(\mathfrak{M})$ consisting of those f with $f | \mathfrak{M}_i$ in A_i, $i = 1, 2$. (Since $\partial_2 \subset \partial_1$, A may also be viewed as a closed subalgebra of A_1.)

The consequences of (2.1) for A are the following facts.

(2.2) $$\mathfrak{M}_A = \mathfrak{M}$$

(2.3) $$\partial_A = \partial_1$$

(2.4) $k\mathfrak{M}_2 = \{f \in A : f(\mathfrak{M}_2) = 0\}$ *separates the points of* $\mathfrak{M} \backslash \mathfrak{M}_2$.

In particular (2.4) implies there are many functions in A vanishing on the (possibly void) open subset $\mathfrak{M} \backslash \mathfrak{M}_1 = \mathfrak{M}_2 \backslash \partial_2$ of $\mathfrak{M} = \mathfrak{M}_A$.

Note that since $A_1 | F = C(F)$, for any f in A_2, $f | \partial_2 = f | F$ has an extension to $\mathfrak{M}_1$ in A_1; consequently f itself has an extension to $\mathfrak{M}$ in A. Thus

(2.5) $$A | \mathfrak{M}_2 = A_2 ,$$

and A separates the points of $\mathfrak{M}_2$. On the other hand trivially

(2.6) f *in* A_1 *and* $f(F) = f(\partial_2) = 0$ *imply* f *has an extension* ($\equiv 0$ *on* $\mathfrak{M}_2$) *in* A.

Now the f in A_1 satisfying the hypothesis of (2.6) form an ideal kF of A_1, and of course the quotient algebra A_1/kF has the hull of kF as its maximal ideal space. But A_1/kF is naturally isomorphic to $A_1 | F = C(F)$, so that F is the maximal ideal space, hence the hull of kF. So (as is well known and easily proved) the Banach algebra kF has

(2.7) $$\partial_{kF} = \partial_1 \backslash F = \partial_1 \backslash \partial_2 , \quad \mathfrak{M}_{kF} = \mathfrak{M}_1 \backslash F .$$

Hence from the trivial relation (2.6), $k\mathfrak{M}_2 = \{f \in A : f(\mathfrak{M}_2) = 0\}$ separates the points of $\mathfrak{M}_1 \backslash F = \mathfrak{M} \backslash \mathfrak{M}_2$, yielding (2.4), and separates any element of $\mathfrak{M} \backslash \mathfrak{M}_2$ from one of $\mathfrak{M}_2$. Since A separates the points of $\mathfrak{M}_2$ by (2.5), A separates $\mathfrak{M}$, and $\mathfrak{M}$ is a subspace of $\mathfrak{M}_A$. Moreover by (2.6) kF and $k\mathfrak{M}_2$ are isomorphic, whence $\partial_{k\mathfrak{M}_2} = \partial_1 \backslash \partial_2$, so that

(2.8) $$\partial_1 \backslash \partial_2 \subset \partial_A .$$

The remainder of (2.2) now follows by a standard argument: if a multiplicative linear functional φ on A vanishes on $k\mathfrak{M}_2$, hence corresponds to an element of $\mathfrak{M}_{A/k\mathfrak{M}_2}$, then the isomorphism of $A/k\mathfrak{M}_2$ and $A | \mathfrak{M}_2 = A_2$ shows φ arises from a point in $\mathfrak{M}_2 \subset \mathfrak{M}$. But if φ does not vanish on $k\mathfrak{M}_2$ it provides a nonzero functional on this algebra, hence on kF, and (since $\mathfrak{M}_{kF} = \mathfrak{M}_1 \backslash F$) we have some x in $\mathfrak{M}_1$ for which $\varphi(f) = f(x)$, f in $k\mathfrak{M}_2$. Choosing f in $k\mathfrak{M}_2$ with $f(x) = \varphi(f) = 1$, we have fg in $k\mathfrak{M}_2$ for any g in A, so $\varphi(g) = \varphi(fg) = fg(x) = g(x)$.

For (2.3), we already have $\partial_A \subset \partial_1$ (since $f \in A$ assumes its maximum modulus on ∂_1 by the definition of A) and $\partial_1 \backslash \partial_2 \subset \partial_A$ by (2.8). Consequently (2.3) follows immediately if $F = \partial_2$ is nowhere dense in ∂_1 (as in the case of our examples to follow) since $\partial_1 = (\partial_1 \backslash \partial_2)^- \subset \partial_A$.

For the general case we need only show x in ∂_2 lies in ∂_A, and for this part of the argument we shall restrict our attention to ∂_1 and regard A and A_1 as subalgebras of $C(\partial_1)$, A_2 as one of $C(\partial_2)$. By (2.12) (with $X = \partial_1$, $F = \partial_2$ and A_1 our algebra) we have an element f of A_1 peaking on F, so $f(F) = 1$, $|f| < 1$ on $\partial_1 \backslash F$; and of course $f \in A$. For our x in ∂_2 and any open neighborhood U of x in ∂_1 we know there is a g_2 in A_2 assuming its maximum modulus over ∂_2 — 1 say—only within $\partial_2 \cap U$, and by (2.5) g_2 has an extension g in A. Moreover for some $\varepsilon > 0$, $|g_2| < 1 - \varepsilon$ on $\partial_2 \backslash U$, so $|g| < 1 - \varepsilon$ on some open subset V of ∂_1 containing $\partial_2 \backslash U$. Since ∂_2 is contained in the open subset $U \cup V$ of ∂_1, $\sup |f(\partial_1 \backslash (U \cup V))| < 1$, so $|f^n g| < 1 - \varepsilon$ on $\partial_1 \backslash (U \cup V)$ for some n, while $|f^n g| \leqq |g| < 1 - \varepsilon$ on V. Thus $|f^n g| < 1 - \varepsilon$ on $\partial_1 \backslash U$; since $f^n g = g$ on ∂_2 the element $f^n g$ of A assumes its maximum modulus 1 only within U, whence $x \in \partial_A$ and $\partial_2 \subset \partial_A$ as desired.

2.2 REMARK. (2.2)–(2.4) apply to a more general construction; for with $F \subset \partial_1$ having $\mu_F = 0$ for all μ in $M(\partial_1)$ orthogonal to A_1 as before, and ρ any (not one-to-one) continuous map of F onto ∂_2 we can set

$$A = \{f \in A_1 : f \mid F \in A_2 \circ \rho\}$$

and again arrive at the same conclusions. Here, of course, in forming $\mathfrak{M}$ there is some identification of points in F, while ∂_A is ∂_1 with just such identifications. (An appropriate modification of (4.1) below can also be obtained in this setting.)

3. We can now write down our example. Let A_1 be the disc algebra of all functions continuous in the disc $D = \{z : |z| \leqq 1\}$ and analytic on $|z| < 1$. Let A_2 be Rudin's algebra [10] of all functions continuous on the Riemann sphere S and analytic off a compact perfect 0-dimensional subset E of the plane with $E \cap U$ void or of positive plane measure for each open U. Then[3] $E = \partial_2$ and $\mathfrak{M}_2 = S$ [2].

[3] This follows from the argument of [**10**, p. 826]. For if U is open in S and $E \cap U \neq \phi$ is open and closed in E then—with $E \cap U$ in place of E—[**10**] shows there are non-constant f in $C(S)$ analytic off $E \cap U$, hence elements of A assuming their maximum modulus only within $E \cap U$.

Now pick a Cantor set F of measure 0 on the unit circle $T^1 = \partial_1$ so $\mu_F = 0$ for each μ in $M(T^1)$ orthogonal to A_1 by the F. and M. Riesz theorem [8]. $E = \partial_2$ and F are homeomorphic so we may identify these sets as before, in effect tacking S onto D along F. Our algebra A on the resulting space $\mathfrak{M} = D \cup S$ consists of all functions continuous on an open subset of $\partial_A = \partial_1 = T^1$ must vanish on $\mathfrak{M}$ and analytic off T^1.

Now $S \backslash E = \mathfrak{M}_2 \backslash F$ is a non-void open subset of $\mathfrak{M}_A = \mathfrak{M}$ on which nonzero elements of A *do* vanish by (2.4); but an f in A which vanishes on all of T^1, being analytic on the interior of D, whence $f \equiv 0$.

4. We conclude with a modification of our example in which our nonanalytic integral domain is also a dirichlet algebra on its Šilov boundary [8]. In order to see the example is dirichlet, we require the following additional information, which holds in the context of § 2.

Let A, A_1, A_2 again be as in § 2. Let $A_i^\perp$ denote the measures on ∂_i orthogonal to A_i, and $A^\perp$ those on $\partial_A = \partial_1$ orthogonal to A. (Since $\partial_2 \subset \partial_1$, we shall view $A_2^\perp$ as consisting of measures on ∂_1.) Then

$$A^\perp = A_1^\perp + A_2^\perp . \tag{4.1}$$

(4.1) is a consequence of an argument of Browder and Wermer [4]. To obtain it, consider the weak* closed subspaces $A^\perp$, $A_i^\perp$ of the dual $M(\partial_1)$ of $C(\partial_1)$. Clearly $A_i^\perp \subset A^\perp$, so $A_1^\perp + A_2^\perp \subset A^\perp$. On the other hand any f in $C(\partial_1)$ orthogonal to $A_1^\perp + A_2^\perp$ has $f \mid \partial_i$ in $A_i \mid \partial_i$, so $f \mid \partial_i$ has an extension g_i in A_i, $i = 1, 2$; and evidently g_1 and g_2 combine to yield an extension g of f, $g \in A$. So $f \in A \mid \partial_1$, which shows $A_1^\perp + A_2^\perp$ is weak* dense in $A^\perp$.

So it suffices to prove $A_1^\perp + A_2^\perp$ is weak* closed in $M(\partial_1)$. But by hypothesis $\mu_{\partial_2} = 0$ for all μ in $A_1^\perp$, so μ in $A_1^\perp$ and ν in $A_2^\perp$ are mutually singular, and $\| \mu + \nu \| = \| \mu \| + \| \nu \|$. Consequently the argument of Browder and Wermer [4] applies to complete the proof of (4.1).

Now let Z^2 be the lattice points in the plane, α an irrational real number, and H the half-space of Z^2 of all (m, n) with

$$m\alpha + n \geqq 0 .$$

Let A_1 be the closed algebra of continuous functions on the torus T^2 spanned by the characters of T^2 corresponding to the elements of the semigroup H; alternatively A_1 consists of those f in $C(T^2)$ with Fourier coefficients vanishing off H. A description of $\mathfrak{M}_1$ can be found in [1]; but here we only need the fact that $\partial_1 = T^2$ [1], and that A_1 is a dirichlet algebra on T^2.

Let F be the subset $T^1 \times \{1\}$ of T^2. Then from an extension of the F. and M. Riesz theorem obtained recently by K. de Leeuw and the author [6] we have[4] (i) $\mu_F = 0$ for all μ in $M(T^2)$ orthogonal to A_1 [6, Th. 3.1], while (ii) any f in A_1 which vanishes on an open subset of T^2 vanishes identically [6, Th. 4.1]. From (i) we can apply our construction, identifying F with the boundary of the disc D, taking A_2 as the disc algebra. The resulting algebra A again contains nonzero elements vanishing on an open subset of $\mathfrak{M}_A$—the interior of D— and again is analytic on $\partial_A = T^2$ by (ii).

And A is dirichlet on T^2 by (4.1): for if λ is any real measure in $M(T^2)$ orthogonal to A, so that $\lambda = \mu_1 + \mu_2$, μ_i in $A_i^\perp$, then $\mu_2 = \lambda_F$, $\mu_1 = \lambda_{F'}$, by (i). Consequently μ_i is a real measure on ∂_i orthogonal to A_i, hence zero since A_i is dirichlet on ∂_i.

Finally, note that A has a simple description as a subalgebra of $C(T^2)$: viewing T^1 as the reals mod 2π, A consists of all f with

$$\int_0^{2\pi}\int_0^{2\pi} f(\theta, \varphi)e^{-i(m\theta+n\varphi)}d\theta d\varphi = 0\,, \qquad m\alpha + n < 0\,,$$

$$\int_0^{2\pi} f(0, \varphi)e^{-in\varphi}d\varphi = 0\,, \qquad n < 0\,.$$

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[4] Here the map ψ of [**6**] taking Z^2 into R is $(m, n) \to m\alpha + n$.