CONVERGENCE OF EXTENDED BERNSTEIN POLYNOMIALS IN THE COMPLEX PLANE

J. J. GERGEN, F. G. DRESSEL, AND W. H. PURCELL, JR.

1. Introduction. Let $f(x)$ be defined on $[0, 1]$. The following two theorems on the Bernstein polynomials corresponding to f,

$$B_n(x; f) = \sum_{\lambda=0}^{n} f\left(\frac{\lambda}{n}\right)\binom{n}{\lambda} x^\lambda (1-x)^{n-\lambda}, \qquad n = 1, 2, \cdots, \tag{1.1}$$

are well known.

THEOREM I. *If $f(x)$ is continuous on $[0, 1]$, then $B_n(x; f) \to f(x)$ as $n \to \infty$ uniformly on $[0, 1]$.*

THEOREM II. *If $f(z)$, $z = x + iy$, is analytic in the interior E of the ellipse with foci at $z = 0$ and $z = 1$, then $B_n(z; f) \to f(z)$ as $n \to \infty$ on E, this convergence being uniform on each closed subset of E.*

The first of these results is due to S. Bernstein [1], the second to L. V. Kantorovitch [6] (See also [4], [7]).

For $f(x)$ defined on $[0, \infty)$ the functions

$$P_k(x; f) = e^{-kx} \sum_{\lambda=0}^{\infty} \frac{(kx)^\lambda}{\lambda!} f\left(\frac{\lambda}{k}\right), \qquad 0 < k, \tag{1.2}$$

form a natural extension of the Bernstein polynomials, the terms of (1.2) corresponding to a Poisson distribution in much the same manner as the terms of (1.1) correspond to a binomial distribution. The functions (1.2) have been considered by Favard [5], Szász [9], and Butzer [3] for the real case. The results of Favard and Szász include the following analogue of Theorem I.

THEOREM III. *If $f(x)$ is continuous on $[0, \infty)$, and if $f(x) = O(x^A)$* [Szász], *or more generally, if $f(x) = O(e^{Ax})$* [Favard] *as $x \to \infty$, where A is a positive, real constant, then $P_k(x; f) \to f(x)$ as $k \to \infty$ for x on $[0, \infty)$, this convergence being uniform on each finite subinterval of $[0, \infty)$.*

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The order condition $f(x) = O(x^A)$ can be replaced by $O(e^{Ax})$ in Szasz' proof without difficulty through the application of the inequality

$$\sum_{|(\lambda/u)-x|\geqq\delta} \frac{(tux)^\lambda}{\lambda!} \leqq \frac{1}{\delta^2u^2}\sum_{\lambda=0}^{\infty}\frac{(\lambda-ux)^2(tux)^\lambda}{\lambda!}$$

$$= \frac{x}{\delta^2 u}[ux(t-1)^2 + t]e^{tux},$$

valid for $0 < u, x, \delta, t$, in Szász' treatment [9, p. 240] of S_4.

In this paper our objective is to obtain an analogue of Theorem II. Our principal results are stated in § 2 below. In our analysis we depend heavily upon the work [10] of Szász and Yeardley. Bohman [2] considers polynomials of the form $e^{-Nz}\sum_{\lambda=0}^{n}((Nz)^\lambda/\lambda!)f(\lambda/n)$, $N = N(n)$, in the complex plane, but there seems to be no existing treatment of the series (1.2) for the complex case.

2. Principal results Corresponding to the positive number d, let $p(d)$ denote the parabolic set $\{z \,||\, z| < x + 2d^2\}$. We will say that a function $f(z)$ defined in $p(d)$ has property B in $p(d)$ if there corresponds to each b, $0 < b < d$, a positive number $B(b)$ such that for $z \in p(b)$

$$(2.1)\qquad |f(z)| \leqq B(b)\exp\left\{\frac{1}{2}x - |x|^{1/2}\left[b^2 - \frac{1}{2}(|z| - x)\right]^{1/2}\right\}.$$

A collection of functions $\{f_k(x)\}_{0<k}$, each defined in $p(d)$, will be said to have property B uniformly in $p(d)$ if there corresponds to each b, $0 < b < d$, a positive number $B(b)$, independent of k, such that (2.1) holds for each f_k. Our principal theorem is then

THEOREM IV. *Suppose that $f(z)$ is analytic and has property B in $p(d)$, where d is a positive number. Then the functions*

$$(2.2)\qquad P_k(z;f) = e^{-kz}\sum_{\lambda=0}^{\infty}\frac{(kz)^\lambda}{\lambda!}f\left(\frac{\lambda}{k}\right), \qquad 0 < k,$$

satisfy the following four conditions. (1) *$P_k(z;f)$ is an entire function of z for each k.* (2) *$P_k(z;f) \to f(z)$ as $k \to \infty$ in $p(d)$.* (3) *The convergence in* (2) *is uniform on each compact subset of $p(d)$.* (4) *The functions $\{P_k(z/\chi_k;f)\}_{0<k}$, where $\chi_k = \exp[1/(2k)]$, have property B uniformly in $p(d)$.*

We note the result of Pollard [8] and Szász and Yeardley [10] that, in order that a function $f(z)$ be analytic and have property B in $p(d)$, $0 < d$, it is necessary and sufficient that $f(z)$ possess a Laguerre series (of order 0),

$$f(z) \sim \sum_{n=0}^{\infty} a_n L_n(z),\ a_n = \int_0^{\infty} e^{-x} L_n(x) f(x) dx ,$$

which converges to it in $p(d)$. As a consequence of this result, the hypothesis in Theorem IV that $f(z)$ be analytic and have property B in $p(d)$ can be replaced by the hypothesis that $f(z)$ possess a Laguerre series which converges to it in $p(d)$. The result of Szász and Yeardley [10] is valid as well for general Laguerre series.

3. *Lemmas for Theorem IV.* It is convenient to develop the proof of Theorem IV in lemmas. Unless the contrary is stated we assume z arbitrary and $0 < k$.

LEMMA 1. *If $f(z)$ is a polynomial, then $P_k(z; f)$ is a polynomial of the same degree as f.*

Proof. We can suppose $f \equiv z^n$, where n is a nonnegative integer. We have

$$e^{-z} \sum_{\lambda=0}^{\infty} \frac{z^{\lambda}}{\lambda!} \lambda^n = e^{-z}(zD_z)^n e^z = \sum_{j=0}^{n} c_j^{(n)} z^j ,$$

where the $c_j^{(n)}$ are constants. We obtain then

$$P_k(z; f) = e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^{\lambda}}{\lambda!} \left(\frac{\lambda}{k}\right)^n = \frac{1}{k^n} \sum_{j=0}^{n} c_j^{(n)} (kz)^j$$

and the lemma follows.

We may observe that $c_n^{(n)} = 1$. It follows that $P_k(z; f) \to z^n$ as $k \to \infty$ for every z, the convergence being uniform on each compact set. The same result then holds for any polynomial.

LEMMA 2. *Denote by $G_k^{(n)}(z)$ the polynomial*

$$G_k^{(n)}(z) = P_k(z; L_n) , \qquad n = 0, 1, 2, \cdots,$$

where L_n is the nth Laguerre polynomial of order 0. *Then*

$$|G_k^{(n)}(z)| \leqq \exp(-kx + k\chi_k |z|) , \qquad n = 1, 2, \cdots, \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} G_k^{(n)}(z) w^n = \frac{1}{1-w} \exp\left\{-kz + kz \exp\left[\frac{-w}{k(1-w)}\right]\right\}, \quad |w| < 1. \tag{3.2}$$

Proof. The inequality (3.1) follows from the fact that [11, p. 162]

$$|L_n(x)| \leqq \exp(\tfrac{1}{2}x) , \qquad 0 \leqq x, n = 1, 2, \cdots . \tag{3.3}$$

For the Laguerre polynomials L_n we have [11, p. 100]

$$\sum_{n=0}^{\infty} L_n(z)w^n = \frac{1}{1-w}\exp\left(\frac{-zw}{1-w}\right), \qquad |w|<1,$$

from which we obtain

$$\begin{aligned} e^{-kz}\sum_{\lambda=0}^{\infty}\frac{(kz)^\lambda}{\lambda!}\sum_{n=0}^{\infty} L_n\left(\frac{\lambda}{k}\right)w^n &= \frac{e^{-kz}}{1-w}\sum_{\lambda=0}^{\infty}\frac{(kz)^\lambda}{\lambda!}\exp\left[\frac{-\lambda w}{k(1-w)}\right] \\ &= \frac{1}{1-w}\exp\left\{-kz + kz\exp\left[\frac{-w}{k(1-w)}\right]\right\}. \end{aligned}$$

For z, k, w, fixed, $|w|<1$, the double series on the left here is absolutely convergent. Interchanging the order of summation in this series we get (3.2).

LEMMA 3. *Let*

$$H_k(z, w) = \mathscr{R}\left\{-kz + kz\exp\left[\frac{-w}{k(1-w)}\right]\right\}.$$

Then

$$(3.4) \qquad H_k(z, w) \leqq \chi_k r(|z| - rx)/(1-r^2), \qquad |w| = r < 1.$$

This is a principal lemma for the proof of Theorem IV. We show that

$$(3.5) \qquad H_k(z, w) \leqq \alpha r(|z| - rx)/(1-r^2), \qquad |w| = r < 1,$$

where $\alpha = \alpha(r, k) = \exp\{r/[k(1+r)]\}$. This inequality is slightly stronger than (3.4). The proof is based on the representation (3.6), the use of which was suggested by the referee and results in a simpler proof than that originally submitted by the authors for (3.4).

Proof. The inequality (3.5) is trivial for $z=0$ or $w=0$. We assume then $|z|$, $|w|$, k fixed with $z \neq 0$, $0<r<1$. We write

$$\begin{aligned} &z = |z|e^{i\phi}, \qquad \rho = r/(1-r^2), \qquad e^{i\theta} = w(1-\bar{w})/[r(1-w)], \\ &a = 1/k, \qquad \Phi = \phi - a\rho\sin\theta. \end{aligned}$$

We have then

$$(3.6) \qquad w/(1-w) = \rho(r + e^{i\theta}),$$

and we find that (3.5) holds provided

$$(3.7) \qquad T(\theta, \phi) = (a\alpha r\rho - 1)\cos\phi + e^{-a\rho(r+\cos\theta)}\cos\Phi \leqq a\alpha\rho$$

for $|\theta|$, $|\phi| \leqq \pi$. Since T is symmetric in the origin in the (θ, ϕ) –

plane, it is enough to show that (3.7) holds for (θ, ϕ) in the rectangle R: $0 \leqq \theta \leqq \pi$, $|\phi| \leqq \pi$.

Suppose first that $1 \leqq a\alpha r\rho$. Since $e^t \leqq 1 + te^t$, $0 \leqq t$, we then have

$$T \leqq a\alpha r\rho - 1 + \alpha \leqq a\alpha r\rho + a\alpha r/(1 + r) = a\alpha\rho,$$

which is (3.7) for this case.

Suppose then that $a\alpha r\rho < 1$. Let (θ, ϕ) denote a maximal point of T on R. We consider three possible cases

$$\theta = 0 , \qquad \theta = \pi , \qquad 0 < \theta < \pi .$$

If $\theta = 0$, then

$$T = (a\alpha r\rho - 1 + e^{-ar/(1-r)}) \cos \phi .$$

If the coefficient of $\cos \phi$ here is nonnegative, we have immediately

$$T \leqq a\alpha r\rho \leqq a\alpha\rho .$$

If this coefficient is negative, we have

$$\begin{aligned} T &\leqq e^{ar/(1-r)}(e^{ar/(1-r)} - 1) - a\alpha r\rho \\ &\leqq ar/(1 - r) - a\alpha r\rho \leqq a\alpha\rho . \end{aligned}$$

If $\theta = \pi$, then

$$T = (a\alpha r\rho - 1 + \alpha) \cos \phi \leqq a\alpha\rho .$$

Accordingly, to complete the proof it remains to consider the case $0 < \theta < \pi$.

At (θ, ϕ) both first partial derivatives of T vanish. Accordingly we obtain

$$\begin{aligned} &\sin (\theta + \Phi) = \sin \theta \cos \Phi + \cos \theta \sin \Phi = 0 , \\ &(a\alpha r\rho - 1) \sin \phi + e^{-a\rho(r+\cos\theta)} \sin \Phi = 0 . \end{aligned} \tag{3.8}$$

From these relations we then get

$$\begin{aligned} T \sin \theta &= (a\alpha r\rho - 1) \sin \theta \cos \phi + e^{-a\rho(r+\cos\theta)} \sin \theta \cos \Phi \\ &= (a\alpha r\rho - 1) \sin \theta \cos \phi - e^{-a\rho(r+\cos\theta)} \cos \theta \sin \Phi \\ &= (a\alpha r\rho - 1) \sin (\theta + \phi) . \end{aligned}$$

Now from (3.8) $\theta + \Phi = n\pi$, where $n = 0, \pm 1, \cdots$. Thus $\theta + \phi = \theta + \Phi + a\rho \sin \theta = n\pi + a\rho \sin \theta$, and

$$T \sin \theta = (a\alpha r\rho - 1) \sin (n\pi + a\rho \sin \theta) . \tag{3.9}$$

From (3.9) we get, since $a\alpha r\rho < 1$ and $0 < \theta < \pi$,

$$(3.10) \qquad T \sin\theta \leqq (1 - a\alpha r\rho)a\rho \sin\theta \leqq a\rho \sin\theta .$$

The inequality (3.10) gives $T \leqq \alpha a \rho$, which completes the proof.

LEMMA 4. *Let* α, β, γ *be positive constants such that* $\alpha \leqq \beta$. *Put* $u(t) = 4\alpha^2/t + t\beta^2/(4+t)$. *Then*

$$I(\alpha, \beta, \gamma) = \int_0^\infty \frac{1}{1-e^{-t}} \frac{1}{t^{3/2}} \exp\left[-u(t) - \frac{4\gamma^2}{t}\right] dt \leqq M_1(\gamma) \exp(\alpha^2 - 2\alpha\beta) ,$$

where

$$M_1(\gamma) = e[2 + \sqrt{\pi}/(16\alpha^3)]/(e-1) .$$

This lemma and the next two are closely related to results obtained by Szász and Yeardley [10]. Our proofs are somewhat different from theirs. The precise bound M_3 appearing in Lemma 6 does not occur in their article.

Proof. If $\alpha = \beta$, then $u(t) = \alpha^2 + 16\alpha^2/[t(4+t)] > \alpha^2 = 2\alpha\beta - \beta^2$ for $0 < t$. If $\alpha < \beta$, then $u(t)$ has the minimum value $2\alpha\beta - \alpha^2$ on this interval. Thus

$$I \leqq \exp(\alpha^2 - 2\alpha\beta) \int_0^\infty \frac{1}{1-e^{-t}} \frac{1}{t^{3/2}} \exp\left(-\frac{4\gamma^2}{t}\right) dt .$$

For $0 < t \leqq 1$ we have $t(1 - 1/e) \leqq 1 - e^{-t}$, and for $1 \leqq t$ we have $1 - 1/e \leqq 1 - e^{-t}$. This gives

$$I \leqq [e/(e-1)] \exp(\alpha^2 - 2\alpha\beta)$$
$$\times \left[\int_0^1 t^{-5/2} \exp(-4\alpha^2/t) dt + \int_1^\infty t^{-3/2} \exp(-4\alpha^2/t) dt\right] .$$

Now

$$\int_0^1 t^{-5/2} \exp(-4\alpha^2/t) dt \leqq \int_0^\infty t^{-5/2} \exp(-4\alpha^2/t) dt = \sqrt{\pi}/(16\gamma^3) ,$$
$$\int_1^\infty t^{-3/2} \exp(-4\gamma^2/t) dt \leqq \int_1^\infty t^{-3/2} dt = 2 ,$$

and the lemma follows.

LEMMA 5. *If* $0 < b < c$, *and*

$$J(b, c, z) = \int_0^\infty \frac{1}{1-e^{-t}} \frac{1}{t^{3/2}} \exp\left[-\frac{4c^2}{t} + \frac{2e^{-t/2}}{1-e^{-t}} (|z| - xe^{t/2})\right] dt ,$$

then

$$J(b, c, z) \leqq M_2(b, c) \exp\left\{x - 2|x|^{1/2}\left[b^2 - \frac{1}{2}(|z| - x)\right]^{1/2}\right\}$$

for $z \in p(b)$, *where*

$$M_2(b, c) = e^{4b^2} M_1((c^2 - b^2)^{3/2}) .$$

Proof. Suppose $z \in p(b)$, so that $0 < b^2 + x$. From the inequalities $e^{-t/2}/(1 - e^{-t}) \leqq 1/t$, $e^{-t/2}(1 - e^{-t/2})/(1 - e^{-t}) \leqq 2/(4 + t)$, valid for $0 < t$, we then obtain for $0 < t$

$$\begin{aligned}\frac{2e^{-t/2}}{1 - e^{-t}}(|z| - xe^{-t/2}) &= \frac{2e^{-t/2}}{1 - e^{-t}}[|z| - x + x(1 - e^{-t/2})] \\ &\leqq \frac{2e^{-t/2}}{1 - e^{-t}}[|z| - x + (x + b^2)(1 - e^{-t/2})] \\ &\leqq 2(|z| - x)/t + 4(x + b^2)/(4 + t) \\ &= 2(|z| - x)/t + x + b^2 - t(x + b^2)/(4 + t) .\end{aligned}$$

Thus

$$J \leqq e^{x+b^2}\int_0^\infty \frac{1}{1 - e^{-t}} \frac{1}{t^{3/2}} \exp\left\{\frac{-4(c^2 - b^2)}{t} - \frac{4}{t}\left[b^2 - \frac{1}{2}(|z| - x)\right] - \frac{t(x + b^2)}{4 + t}\right\} dt .$$

Since $b^2 - \frac{1}{2}(|z| - x) \leqq x + b^2$, Lemma 4 is applicable. Applying this lemma we then get for $z \in p(b)$

$$\begin{aligned}J &\leqq e^{x+b^2} M_1((c^2 - b^2)^{3/2}) \\ &\quad \cdot \exp\left\{b^2 - \frac{1}{2}(|z| - x) - 2(x + b^2)^{1/2}\left[b^2 - \frac{1}{2}(|z| - x)\right]^{1/2}\right\} .\end{aligned}$$

Now $|x|^{1/2} - b \leqq (x + b^2)^{1/2}$ for $z \in p(b)$, and the lemma follows readily.

LEMMA 6. *Suppose* $0 < b < c$. *Then*

$$\begin{aligned}&\sum_{n=0}^\infty |G_k^{(n)}(z/\chi_k)|^2 \exp(-4c\sqrt{n}) \\ &\qquad \leqq M_3(b, c) \exp\left\{x - 2|x|^{1/2}\left[b^2 - \frac{1}{2}(|z| - x)\right]^{1/2}\right\}\end{aligned}$$

for $z \in p(b)$, *where*

$$M_3(b, c) = (2c\sqrt{\pi}) M_2(b, c) .$$

Proof. Let C_r, $0 < r < 1$, denote the circle of radius r about the origin in the w-plane. Making use of Lemmas 2 and 3 and a classical integral formula we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} |G_k^{(n)}(z)|^2 r^{2n} &= \frac{1}{2\pi r}\int_{C_r}\frac{1}{|1-w|^2}\left|\exp\left\{-kz+kz\exp\left[\frac{-w}{k(1-w)}\right]\right\}\right|^2 |dw| \\ &= \frac{1}{2\pi r}\int_{C_r}\frac{1}{|1-w|^2}\exp[2H_k(z;w)]\,|dw| \\ &\leqq \frac{1}{2\pi r}\int_{C_r}\frac{1}{|1-w|^2}\exp\{2\chi_k r(z-rx)/(1-r^2)\}\,|dw| \\ &= \frac{1}{1-r^2}\exp[2\chi_k r(|z|-rx)/(1-r^2)]\,.\end{aligned}$$

Thus, if $0 < t$, then

$$\begin{aligned}&\sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 e^{-nt} \\ &\qquad \leqq [1/(1-e^{-t})]\exp\{2e^{-t/2}(|z|-xe^{-t/2})/(1-e^{-t})\}\,.\end{aligned}$$

On the other hand,

$$\exp(-4c\sqrt{n}) = (2c/\sqrt{\pi})\int_0^{\infty} t^{-3/2}\exp(-nt-4c^2/t)dt\,.$$

Hence, applying Lemma 5, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 \exp(-4c\sqrt{n}) \\ &= (2c/\sqrt{\pi})\sum_{n=0}^{\infty} |G_k^{(n)}(z/\chi_k)|^2 \int_0^{\infty} t^{-3/2}\exp(-nt-4c^2/t)dt \\ &= (2c/\sqrt{\pi})\int_0^{\infty} t^{-3/2}\exp\left(-(4c^2/t)\left[\sum_{n=0}^{\infty}|G_k(z/\chi_k)|^2\exp(-nt)\right]dt\right) \\ &\leqq (2c/\sqrt{\pi})\int_0^{\infty}\frac{t^{-3/2}}{1-e^{-t}}\exp\left[\frac{-4c^2}{t}+\frac{2e^{-t/2}}{1-e^{-t}}(|z|-xe^{-t/2})\right]dt \\ &\leqq (2c/\sqrt{\pi})M_2(b,c)\exp\left\{x-2|x|^{1/2}\left[b^2-\frac{1}{2}(|z|-x)\right]^{1/2}\right\}\end{aligned}$$

for $z \in p(b)$. This is the required inequality.

4. *Proof of Theorem* IV. Assume the hypotheses of Theorem IV hold. We note first that under these hypotheses $f(x)$ satisfies

$$(4.1)\qquad |f(x)| \leqq Ae^{x/2}\,,\qquad 0 \leqq x\,,$$

for some positive constant A. It is seen then that the series in (2.2) converges for z, k arbitrary, $0 < k$. Thus conclusion (1) of Theorem IV holds.

Next, by the theorem of Pollard, and Szász and Yeardley noted in § 2 above, the hypotheses of Theorem IV imply that f can be represented in $p(d)$ by a convergent Laguerre series:

$$f(z) = \sum_{n=0}^{\infty} a_n L_n(z),\ z \in p(d)\ ; \qquad a_n = \int_0^{\infty} e^{-x} L_n(x) f(x) dx\ . \tag{4.2}$$

From the convergence in $p(d)$ of the series (4.2) it follows that, if ε is an arbitrary positive number, then

$$|a_n| \leqq A_\varepsilon \exp[2n(-d+\varepsilon)]\ , \qquad n = 1, 2, \cdots, \tag{4.3}$$

for a suitably chosen positive constant A_ε. From (4.3) we obtain

$$\sum_{n=0}^{\infty} |a_n| < \infty,\ M(c; f) = \sum_{n=0}^{\infty} |a_n|^2 \exp(4c\sqrt{n}) < \infty \tag{4.4}$$

the latter provided $0 < c < d$.

Now consider $P_k(z; f)$. We have formally

$$\begin{aligned} P_k(z; f) &= e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} \sum_{n=0}^{\infty} a_n L_n(\lambda/k) \\ &= \sum_{n=0}^{\infty} a_n \left[e^{-kz} \sum_{\lambda=0}^{\infty} \frac{(kz)^\lambda}{\lambda!} L_n\left(\frac{\lambda}{k}\right) \right] \\ &= \sum_{n=0}^{\infty} a_n G_k^{(n)}(z)\ . \end{aligned} \tag{4.5}$$

Making use of (3.3) and the first inequality in (4.4) we see that the series in the first line of (4.5) converges absolutely for z, k arbitrary, $0 < k$. This justifies the formal manipulation in (4.5) and we accordingly have

$$P_k(z; f) = \sum_{n=0}^{\infty} a_n G_k^{(n)}(z) \tag{4.6}$$

for z, k arbitrary, $0 < k$. From (4.6) we get

$$|P_k(z; f)|^2 \leqq \sum_{n=0}^{\infty} |a_n|^2 \exp(4c\sqrt{n}) \sum_{n=0}^{\infty} |G_k^{(n)}(z)|^2 \exp(-4c\sqrt{n})\ .$$

Thus, by Lemma 6, if $0 < b < c < d$, then

$$|P_k(z/\chi_k; f)|^2 \leqq M(c; f) \cdot M_3(b, c) \cdot \exp\left\{x - 2|x|^{1/2}\left[b^2 - \frac{1}{2}(|z| - x)\right]^{1/2}\right\}$$

for $z \in p(b)$. For a fixed b, $0 < b < d$, on taking $c = \frac{1}{2}(b + d)$, say, we find then that conclusion (4) holds with

$$B(b) = [M(c; f) M_3(b, c)]^{1/2}\ , \qquad c = \frac{1}{2}(b + d)\ .$$

It remains to consider conclusions (2) and (3). It is enough to show that, if S is a compact subset of $p(d)$, then $P_k(z; f) \to f(z)$, $k \to \infty$, uniformly on S. For $0 < b$, $0 < x_0$ let

$$U(b, x_0) = \{z \mid |z| < x + 2b^2, x < x_0\} .$$

Choose b_1, b_2, b_3; x_1, x_2, x_3 such that $0 < b_1 < b_2 < b_3 < d$, $0 < x_1 < x_2 < x_3$, and $S \subset U(b_1, x_1)$. Making use of conclusion (4), we infer that there exists a constant M^* such that

$$|P_k(z/\chi_k; f)| \leqq M^*, z \in U(b_3, x_3) .$$

Choose $k_0 = \max\{[4 \cdot ln(b_3/b_2)]^{-1}, [2 \cdot ln(x_3/x_2)]^{-1}\}$. Then for $k_0 < k$ and $z \in U(b_2, x_2)$ we have $z\chi_k \in U(b_3, x_3)$. Thus

$$(4.7) \qquad |P_k(z; f)| = |P_k(z\chi_k/\chi_k; f)| \leqq M^*, k_0 < k, z \in U(b_2, x_2) .$$

Recalling (4.1), we have also, by Theorem III,

$$P_k(x; f) \to f(x), k \to \infty, 0 < x < x_2 .$$

By an application of Vitali's theorem, $\{P_k(z; f)\}_{k_0 < k}$ converges uniformly on $U(b_1, x_1)$ to a function $F(z)$, analytic on $U(b_1, x_1)$. Since $f(z)$ is analytic on $U(b_1, x_1)$ and $F(x) = f(x)$, $0 < x < x_1$, it follows that $F(z) = f(z)$ throughout $U(b_1, x_1)$, and the proof of complete.

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