COMMON FIXED POINTS FOR COMMUTING CONTRACTION MAPPINGS

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Kakutani [1] and Markov [2] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some nonempty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous (not necessarily linear) transformations. We shall show that it is true in a rather special, but non-trivial, case, thus giving some hope that further investigation of the general question will yield positive results. The main result of this paper is the following.

THEOREM. *Let B be a Banach space and let X be a nonempty compact convex subset of B. If $\mathscr{F}$ is a nonempty commutative family of contraction mappings of X into itself, then the family $\mathscr{F}$ has a common fixed point in X.*

Note 1. A mapping $f: X \to X$ is said to be a contraction mapping if $||f(x) - f(y)|| \leqq ||x - y||$ for all $x, y \in X$.

Note 2. If the norm for B is strictly convex, then the above theorem is almost trivial since in this case each contraction mapping has a fixed-point set which is nonempty, compact, and convex. In the general case, however, the fixed-point set of a contraction mapping is not convex. An example illustrating this fact is constructed as follows. Let B be the space of all ordered pairs (a, b) of real numbers, where if $x = (a, b)$, then $||x|| = \max\{|a|, |b|\}$. Define $X = \{x: ||x|| \leqq 1\}$ and $f: X \to X$ as follows: if $x = (a, b)$, then $f(x) = (|b|, b)$. It is easily shown that f is a contraction mapping and that $x = (1, 1)$ and $y = (1, -1)$ are fixed points for f. However, $1/2\,(x + y) = (1, 0)$ is not a fixed point for f.

In the proof of the theorem we shall make use of the following two lemmas.

LEMMA 1. *Let B be a Banach space and let M be a nonempty compact subset of B and let K be the closed convex hull of M. Let ρ be the diameter of M. If $\rho > 0$, then there exists an element $u \in K$ such that*

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$$\sup\{\|x-u\|: x \in M\} < \rho .$$

Proof. Since M is nonempty and compact, we may find $x_0, x_1 \in M$ such that $\|x_0 - x_1\| = \rho$. Let $M_0 \subset M$ be maximal so that $M_0 \supset \{x_0, x_1\}$ and $\|x - y\| = 0$ or ρ for all $x, y \in M_0$. Since M is compact and we are assuming $\rho > 0$, M_0 must be finite. Let us assume $M_0 = \{x_0, x_1, \cdots, x_n\}$. Now let us define

$$u = \sum_{k=0}^{n} \frac{1}{n+1} x_k \in K .$$

Since M is compact, we can find $y_0 \in M$ such that $\|y_0 - u\| = \sup\{\|x-u\|: x \in M\}$. Now

$$\|y_0 - u\| \leqq \sum_{k=0}^{n} \frac{1}{n+1} \|y_0 - x_k\| \leqq \rho$$

because $\|y_0 - x_k\| \leqq \rho$ for all $k = 0, 1, \cdots, n$. Therefore, if $\|y_0 - u\| = \rho$, then we must have $\|y_0 - x_k\| = \rho > 0$ for all $k = 0, 1, \cdots, n$, which means that $y_0 \in M_0$ by definition of M_0. But then we must have $y_0 = x_k$ for some $k = 0, 1, \cdots, n$, which is a contradiction. Therefore, $\|y_0 - u\| < \rho$.

LEMMA 2. *Let X_0 be a nonempty convex subset of a Banach space and let f be a contraction mapping of X_0 into itself. If there is a compact set $M \subset X_0$ such that $M = \{f(x): x \in M\}$ and M has at least two points, then there exists a nonempty closed convex set K_1 such that $f(x) \in K_1 \cap X_0$ for all $x \in K_1 \cap X_0$ and $M \cap K_1' \neq \phi$. (K_1' is the complement of K_1.)*

Proof. If we take K as the closed convex hull of M, then by Lemma 1 there exists an element $u \in K$ such that

$$\rho_1 = \sup\{\|x - u\|: x \in M\} < \rho ,$$

where ρ is the diameter of M. Since M has at least two points, we have $\rho > 0$, so that our use of Lemma 1 is valid.

For each $x \in M$ let us define $U(x) = \{y: \|y - x\| \leqq \rho_1\}$. Since $u \in U(x)$ for each $x \in M$, we have $K_1 = \bigcap_{x \in M} U(x) \neq \phi$. It is clear that K_1 is closed and convex. For any $x \in K_1 \cap X_0$ and any $z \in M$ we have $x \in U(z)$, i.e., $\|x - z\| \leqq \rho_1$. Since $M = \{f(y): y \in M\}$, there must exist $y \in M$ such that $z = f(y)$. Since f is a contraction mapping, we have

$$\|f(x) - z\| = \|f(x) - f(y)\| \leqq \|x - y\| \leqq \rho_1 ;$$

i.e., $f(x) \in U(z)$. Since this is true for any $z \in M$, we have $f(x) \in K_1 \cap X_0$. We have shown that $f(x) \in K_1 \cap X_0$ for all $x \in K_1 \cap X_0$.

Since M is compact, there exist $x_0, x_1 \in M$ such that $\|x_0 - x_1\| = \rho > \rho_1$. Thus, we see that x_1 does not belong to $U(x_0) \supset K_1$, i.e., $x_1 \in M \cap K_1' \neq \phi$.

Proof of the theorem. One may show by using Zorn's lemma that there exists a minimal nonempty compact convex set $X_0 \subset X$ such that X_0 is invariant under each $f \in \mathscr{F}$. If X_0 consists of a single point, then the theorem is proved. We shall now show that if X_0 consists of more than one point, then we obtain a contradiction.

We may use Zorn's lemma again to show that there exists a minimal nonempty compact (but not necessarily convex) set $M \subset X_0$ such that M is invariant under each $f \in \mathscr{F}$. We will now show that $M = \{f(x): x \in M\}$ for each $f \in \mathscr{F}$. Since each $f \in \mathscr{F}$ is continuous and M is compact, $f(M)$ must also be compact. For all $f \in \mathscr{F}$ we have $f(M) \subset M$. Let us assume that for some $g \in \mathscr{F}$ we have $g(M) = N \neq M$. Now for any $x \in N$ there exists $y \in M$ such that $x = g(y)$. Since all functions in $\mathscr{F}$ commute, we have for all $f \in \mathscr{F}$ $f(x) = f(g(y)) = g(f(y)) \in N$ because $f(y) \in M$. Thus, we have $f(N) \subset N \subset M$ for all $f \in \mathscr{F}$. But since N is a nonempty compact subset of X_0 which is invariant under each $f \in \mathscr{F}$ and since $N \subset M$ and $N \neq M$, we have contradicted the minimality of M. Consequently, our assumption that $M \neq N$ is false. We may assume that M has at least two points; otherwise, the theorem is proved.

We may now apply Lemma 2 to each $f \in \mathscr{F}$. Referring to the notation of Lemma 2, we see that the set $K_1 \cap X_0$ is invariant under each $f \in \mathscr{F}$. Since K_1 is closed, we see that $K_1 \cap X_0$ is a nonempty compact convex subset of X_0. Since $X_0 \cap K_1' \supset M \cap K_1' \neq \phi$, we see that $K_1 \cap X_0 \neq X_0$. Thus, we see that if X_0 has more than one point, then we obtain a contradiction to the minimality of X_0.

References

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2. A. Markov, *Quelques theoremes sur les ensembles Abeliens*, Doklady Akad. Nauk SSSR (N.S.) **10** (1936), 311-314.