HOMOMORPHISMS OF NON-COMMUTATIVE $*$-ALGEBRAS

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1. Introduction. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras and ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. This paper is a study of the continuity properties of ν which depend only on the structure of $\mathfrak{A}$; $\mathfrak{B}$ is completely arbitrary. The algebras considered are non-commutative.

If ν is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then the function $|x| = \|\nu(x)\|$, $x \in \mathfrak{A}$, is a multiplicative semi-norm on $\mathfrak{A}$. Conversely, every multiplicative semi-norm on $\mathfrak{A}$ arises from a homomorphism in this way. Thus all results on continuity of homomorphisms can be stated in terms of multiplicative semi-norms.

Section 2 contains material concerning units in $\mathfrak{A}$ and $\mathfrak{B}$ and the relation between homomorphisms and multiplicative semi-norms.

Section 3 is devoted to the proof of the main technical device of the paper: If $\{g_n\}$ and $\{f_n\}$ are sequences in $\mathfrak{A}$ with $g_n g_m = 0, n \neq m$, and $f_n g_m = 0, n \neq m$, then, under any homomorphism ν of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$, the sequence $\{\|\nu(f_n g_n)\| / \|f_n\| \|g_n\|\}$ is bounded.

In § 4 the separating ideals for ν in $\mathfrak{A}$ and $\mathfrak{B}$ are defined and several of their properties are exhibited. The separating ideal $\mathscr{S}$ for ν in $\mathfrak{A}$ is the set of x in $\mathfrak{A}$ for which there is a sequence $\{x_n\}$ in $\mathfrak{A}$ with $x_n \to 0$ and $\nu(x_n) \to \nu(x)$. An application of the main boundedness theorem (Theorem 3.1) shows that if $\{x_n\}$ is a sequence in $\mathscr{S}$ with $x_n x_m = 0, n \neq m$, then $\nu(x_n)^3 = 0$ for all but a finite number of n.

In § 5 we restrict attention to the case in which ν is an isomorphism and $\mathfrak{A}$ is a B^* algebra. In this case $\mathscr{S}$ is the zero ideal. This fact enables us to show that there is a constant M such that $\|x\| \leqq M \|\nu(x)\|$, $x \in \mathfrak{A}$. This result is analogous to an important theorem of Kaplansky [4]: any multiplicative norm on the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space majorizes the supremum norm. A theorem due to Bonsall [2] implies the following similar result: if $|\cdot|$ is a multiplicative norm on the algebra $\mathfrak{A}$ of bounded operators on a Banach space, there is a constant β such that for $T \in \mathfrak{A}$, $\|T\| \leqq \beta |T|$, where $\|\cdot\|$ is the usual operator norm. Although our result is similar, our approach is quite different. Kaplansky's proof depends heavily on commutativity; Bonsall's on the existence of nonzero finite dimensional operators which, of course, are not necessarily present in an arbitrary B^* algebra. Notice that if $\mathfrak{A}$ is a Banach algebra with the property that for every

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isomorphism ν of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$ there is a constant M with $\|x\| \leqq M\|\nu(x)\|$, then every multiplicative norm on $\mathfrak{A}$ is complete if and only if every isomorphism of $\mathfrak{A}$ is continuous.

We also show in § 5 that if ν is an isomorphism of a B^* algebra $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$ then $Cl(\nu(\mathfrak{A}))$ is the direct sum of the range of ν and the separating ideal for ν in $\mathfrak{B}$. This is the desired generalization of a theorem due to Yood [11] which states that $Cl(\nu(\mathfrak{A})) = \nu(\mathfrak{A}) \oplus R$ when $\mathfrak{A}$ is a commutative B^* algebra. (R is the radical of $Cl\nu(\mathfrak{A})$). Yood's theorem is also true for certain regular commutative Banach algebras.

It is an open question whether or not there exists a discontinuous homomorphism of a B^* algebra. In § 6 a technique due to Bade and Curtis [1] is used to show that any homomorphism of a B^* algebra $\mathfrak{A}$ must be bounded on certain ideals in $\mathfrak{A}$.

2. Preliminaries. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras and ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. There is no loss of generality in assuming $\mathfrak{B} = Cl(\nu(\mathfrak{A}))$ since any restrictions on the algebras we consider will be placed on the domain. If $\mathfrak{A}$ has a unit e, then we may assume that $\mathfrak{B}$ has a unit e' and that $\nu(e) = e'$. Since for any Banach algebra with unit an equivalent norm may be found in which the unit has norm one and since renorming in this way does not affect continuity properties, we assume that if any algebra considered has a unit, then the unit has norm one.

The study of homomorphisms of a Banach algebra $\mathfrak{A}$ is equivalent to the study of multiplicative semi-norms on $\mathfrak{A}$ as was pointed out by Bade and Curtis [1].

DEFINITION 2.1. Let $\mathfrak{A}$ be a Banach algebra. A multiplicative semi-norm on $\mathfrak{A}$ is a function $|\cdot|$ on $\mathfrak{A}$ to $[0, \infty)$ satisfying

(i) $|x + y| \leqq |x| + |y|$, $x, y \in \mathfrak{A}$

(ii) $|xy| \leqq |x||y|$, $x, y \in \mathfrak{A}$

(iii) $|\alpha x| = |\alpha||x|$, $x \in \mathfrak{A}$, α scalar.

If $|x| = 0$ implies $x = 0$, then $|\cdot|$ is called a multiplicative norm on $\mathfrak{A}$.

THEOREM 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras and ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then the function $|x| = \|\nu(x)\|$, $x \in \mathfrak{A}$, is a multiplicative semi-norm on $\mathfrak{A}$. Conversely, if $\|\cdot\|_1$ is a multiplicative semi-norm on $\mathfrak{A}$, there is a Banach algebra $\mathfrak{B}$ and a homomorphism ν of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\|x\|_1 = \|\nu(x)\|$, $x \in \mathfrak{A}$.*

Proof. The first assertion is clear. To prove the second notice that $I = \{x \in \mathfrak{A} : \|x\|_1 = 0\}$ is a two-sided ideal in $\mathfrak{A}$ closed with respect to $\|\cdot\|_1$. Moreover, if x and y are congruent modulo I, then $\|x - y\|_1 = 0$

and since $| \|x\|_1 - \|y\|_1 | \leqq \|x - y\|_1 = 0$, $\|x\|_1 = \|y\|_1$. Thus $\mathfrak{A}/I$ is a normed algebra under the norm $\|x + I\| = \|x\|_1$. Let ν be the natural map of $\mathfrak{A}$ into the completion of $\mathfrak{A}/I$ in this norm. Then ν has the required properties.

3. The main boundedness theorems.

The main boundedness theorems are the principal device of the paper. The present form is due to P. C. Curtis, Jr. The corollary is found in [1].

THEOREM 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Suppose $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathfrak{A}$ satisfying*

(i) $g_m g_n = 0$, $n \neq m$

(ii) $f_m g_n = 0$ $(g_n f_m = 0)$, $m \neq n$.

Then

$$\sup_n \|\nu(f_n g_n)\|/\|f_n\| \|g_n\| < \infty \quad (\sup_n \|\nu(g_n f_n)\|/\|g_n\| \|f_n\| < \infty) .$$

Proof. We consider only the first part of the theorem as the proof of the second is completely analogous. Suppose the theorem is false. We shall show that a certain linear combination of the elements f_i must be mapped into an element of infinite norm.

If the theorem is false, we may select distinct elements u_{ij}, $i, j = 1, 2, \cdots$, from the sequence $\{g_n\}$ such that

$$\|\nu(v_{ij}u_{ij})\| \geqq 4^{i+j} \|v_{ij}\| \|u_{ij}\| \qquad i, j = 1, 2, \cdots ,$$

where v_{ij} is the element f_m corresponding to $g_m = u_{ij}$. Define

$$h_i = \sum_{k=1}^{\infty} u_{ik}/2^k \|u_{ik}\| \qquad i = 1, 2, \cdots .$$

Then $h_i \in \mathfrak{A}$ and $v_{lj}h_i = 0$ for $l \neq i$. If $l = i$, $v_{ij}h_i = v_{ij}u_{ij}/2^j \|u_{ij}\|$. Thus $\nu(h_i) \neq 0$, $i = 1, 2, \cdots$. For each i choose an integer $j(i)$ with $2^{j(i)} > \|\nu(h_i)\|$ and define $y = \sum_{k=1}^{\infty} v_{kj(k)}/2^k \|v_{kj(k)}\|$. It follows from (ii) that

$$yh_i = v_{ij(i)}u_{ij(i)}/2^{i+j(i)} \|v_{ij(i)}\| \|u_{ij(i)}\| , \qquad i = 1, 2, \cdots .$$

Then

$$\|\nu(y)\| \|\nu(h_i)\| \geqq \|\nu(yh_i)\| > 2^{i+j(i)} > 2^i \|\nu(h_i)\| .$$

Thus $\|\nu(y)\| > 2^i$ for every integer i.

COROLLARY 3.2. *If $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathfrak{A}$ satisfying*

(i) $g_n g_m = 0$, $n \neq m$

(ii) $f_n g_n = f_n$ $(g_n f_n = f_n)$, $n = 1, 2, \cdots$, *then* $\sup_n \|\nu(f_n)\|/\|g_n\|\,\|f_n\| < \infty$.

4. The separating ideals. In this section $\mathfrak{A}$ and $\mathfrak{B}$ denote arbitrary Banach algebras and ν a homomorphism of $\mathfrak{A}$ onto a dense subalgebra of $\mathfrak{B}$. Our objective is Theorem 4.9. The function Δ is a variant of the separating function defined by Rickart [7, p. 70]. The separating ideals were defined and used by Rickart [8] in this work on the uniqueness of norm problem. They have also been discussed by Yood [9].

DEFINITION 4.1. For $y \in \mathfrak{B}$, $\Delta(y) = \inf(\|x\| + \|y - \nu(x)\|)$ where the inf is taken over all $x \in \mathfrak{A}$.

PROPOSITION 4.2. The function Δ has the following properties

(i) $\Delta(y_1 + y_2) \leqq \Delta(y_1) + \Delta(y_2)$, $y_1, y_2 \in \mathfrak{B}$

(ii) $\Delta(\alpha y) = |\alpha|\,\Delta(y)$, $y \in \mathfrak{B}$, α scalar

(iii) $\Delta(y) \leqq \|y\|$, $y \in \mathfrak{B}$ and if $y = \nu(x)$ for some $x \in \mathfrak{A}$, $\Delta(y) = \Delta(\nu(x)) \leqq \|x\|$.

The proof is straightforward and is omitted.

DEFINITION 4.3. The separating ideal for ν in $\mathfrak{B}$, denoted $\mathscr{S}'$ is the set of y in $\mathfrak{B}$ for which $\Delta(y) = 0$. The separating ideal $\mathscr{S}$ for ν in $\mathfrak{A}$ is the set of x in $\mathfrak{A}$ for which $\Delta(\nu(x)) = 0$.

THEOREM 4.4. *$\mathscr{S}$ is a closed two-sided ideal in $\mathfrak{A}$; $\mathscr{S}'$ is a closed two-sided ideal in $\mathfrak{B}$.*

Proof. Parts (i) and (ii) of the proposition show that $\mathscr{S}(\mathscr{S}')$ is a linear subspace of $\mathfrak{A}(\mathfrak{B})$. If $\{y_n\}$ is a sequence in $\mathscr{S}'$ and $y_n \to y_0$, then by the triangle inequality for Δ and part (iii) of the proposition

$$\Delta(y_0) \leqq \Delta(y_0 - y_n) \leqq \|y_0 - y_n\| \to 0 .$$

Thus $\mathscr{S}'$ is closed. A similar argument using the last part of (iii) shows that $\mathscr{S}$ is closed in $\mathfrak{A}$.

To complete the proof notice that $y \in \mathscr{S}'$ if and only if there is a sequence $\{x_n\}$ in $\mathfrak{A}$ with $x_n \to 0$ and $\nu(x_n) \to y$. Suppose $y \in \mathscr{S}'$ and $w = \nu(z) \in \mathfrak{B}$. Let $\{x_n\}$ be a sequence in $\mathfrak{A}$ with $x_n \to 0$ and $\nu(x_n) \to y$. Then $x_n z \to 0$ and $\nu(x_n z) = \nu(x_n)\nu(z) \to yw$, which implies $yw \in \mathscr{S}'$. Similarly, $wy \in \mathscr{S}'$. If w is an arbitrary element of $\mathfrak{B}$, then $w = \lim \nu(z_n)$ for some sequence $\{z_n\}$ in $\mathfrak{A}$. For each n, $y\nu(z_n) \in \mathscr{S}'$ and $\nu(z_n)y \in \mathscr{S}'$. But $y\nu(z_n) \to yw$ and $\nu(z_n)y \to wy$. Since $\mathscr{S}'$ is closed, yw and wy belong to $\mathscr{S}'$. The argument also shows that $\mathscr{S}$ is a two-sided ideal in $\mathfrak{A}$.

PROPOSITION 4.5. The homomorphism ν is continuous if and only if $\mathscr{S}' = (0)$.

Proof. If $\mathscr{S}' = (0)$, continuity is immediate from the closed graph theorem. The converse is obvious.

The next theorem shows that if both separating ideals are factored out the resulting map is continuous.

THEOREM 4.6. *The map ν' of $A/\mathscr{S}$ into $\mathfrak{B}/\mathscr{S}'$ given by*

$$\nu'(a + \mathscr{S}) = \nu(a) + \mathscr{S}'$$

is continuous isomorphism of $\mathfrak{A}/\mathscr{S}$ onto a dense subalgebra of $\mathfrak{B}/\mathscr{S}'$.

Proof. Since $\mathscr{S}$ and $\mathscr{S}'$ are closed two-sided ideals, $\mathfrak{A}/\mathscr{S}$ and $\mathfrak{B}/\mathscr{S}'$ are Banach algebras under the usual quotient norm. It is easily verified that ν' is a well-defined isomorphism whose range is dense in $\mathfrak{B}/\mathscr{S}'$. To show ν' is continuous it suffices to show $J = (0)$ where J is the separating ideal for ν' in $\mathfrak{B}/\mathscr{S}'$. We shall show that $\Delta(b + \mathscr{S}') = \Delta(b)$, $b \in \mathfrak{B}$.

Let φ and π denote the natural maps of $\mathfrak{A} \rightarrow \mathfrak{A}/\mathscr{S}$ and $\mathfrak{B} \rightarrow \mathfrak{B}/\mathscr{S}'$ respectively. Since both φ and π are norm decreasing, we have for $b \in \mathfrak{B}$

$$\begin{aligned}\Delta(b + \mathscr{S}') &= \inf (\|\varphi(a)\| + \|\pi(b - \nu(a))\| \\ &\leqq \inf (\|a\| + \|b - \nu(a)\| = \Delta(b) .\end{aligned}$$

To prove the reverse inequality let $\varepsilon > 0$ and choose $a \in \mathfrak{A}$ with

$$\|\varphi(a)\| + \|\pi(b - \nu(a))\| < \Delta(b + \mathfrak{S}') + \varepsilon/3 .$$

Then choose $s_1 \in \mathscr{S}$, $s_2 \in \mathscr{S}'$, such that $\|a + s_1\| \leqq \|\varphi(a)\| + \varepsilon/3$ and $\|b - \nu(a) + s_2\| < \|\pi(b - \nu(a))\| + \varepsilon/3$. Since $\nu(s_1) + s_2 \in \mathscr{S}'$, we have

$$\begin{aligned}\Delta(b) &\leqq \Delta(b + \nu(s_1) + s_2) \\ &\leqq \|a + s_1\| + \|b - \nu(a) + s_2\| < \Delta(b + \mathscr{S}') + \varepsilon .\end{aligned}$$

For any Banach algebra $\mathfrak{A}$ the spectrum of x, denoted $\sigma_{\mathfrak{A}}(x)$, is the set of complex numbers λ such that $\lambda^{-1}x$ has no quasi-inverse in $\mathfrak{A}$. The spectral radius of x, denoted $r_{\mathfrak{A}}(x)$, is $\sup |\lambda|$ where the sup is taken over all $\lambda \in \sigma_{\mathfrak{A}}(x)$. When no confusion will result, we omit the subscript $\mathfrak{A}$ and write $\sigma(x)$ or $r(x)$. It is well known that $r(x)$ is the limit as $n \rightarrow \infty$ of $\|x^n\|^{1/n}$ [7, p. 10].

PROPOSITION 4.7. If $\mathfrak{A}$ has an identity, then $\mathscr{S}$ is a proper ideal; if $\mathfrak{B}$ has an identity, then $\mathscr{S}'$ is a proper ideal.

Proof. First notice that for $x \in \mathfrak{A}$, $\sigma_{\mathfrak{B}}(\nu(x))$ is contained in $\sigma_{\mathfrak{A}}(x)$, For, if $\lambda^{-1}x$ has a quasi-inverse $y \in \mathfrak{A}$, then clearly $\nu(y)$ is a quasi-inverse in $\mathfrak{B}$ of $\lambda^{-1}(\nu(x))$. Thus $r_{\mathfrak{B}}(\nu(x)) \leqq r_{\mathfrak{A}}(x) \leqq \|x\|$, $x \in \mathfrak{A}$.

If c is any element in the centre of $\mathfrak{B}$, then by the remark above,

$$r_{\mathfrak{B}}(c) \leqq r_{\mathfrak{B}}(c - \nu(x)) + r_{\mathfrak{B}}(\nu(x)) \leqq \|c - \nu(x)\| + \|x\| .$$

Thus for c in the centre of $\mathfrak{B}$ we have $r_{\mathfrak{B}}(c) \leqq \Delta(c)$.

Suppose that $\mathfrak{A}$ has an identity e and $e \in \mathscr{S}$. Since $\nu(e)$ is in the centre of $\mathfrak{B}$, $r_{\mathfrak{B}}(\nu(e)) \leqq \Delta(\nu(e)) = 0$. But $r_{\mathfrak{B}}(\nu(e)) = 1$. Hence $\mathscr{S}$ is a proper ideal. The same argument proves the statement about $\mathscr{S}'$.

Proposition 4.7 is used by Yood [9, Th. 3.5] to show that $\mathscr{S}'$ is contained in the Brown-McCoy radical of $\mathfrak{B}$. It is also used in the following theorem which is due to Yood [10, Th. 3.10].

THEOREM 4.8. *Let p be an indempotent in $\mathfrak{A}$. If $\nu(p) \neq 0$, then $p \notin \mathscr{S}$.*

Proof. Suppose p is an indempotent in $\mathfrak{A}$ and $\nu(p) \neq 0$. Let $\mathfrak{A}' = p\mathfrak{A}p$ and $\mathfrak{B}' = \nu(p)\mathfrak{B}\nu(p)$. Then $\mathfrak{A}'$ and $\mathfrak{B}'$ are Banach algebras and $\nu(\mathfrak{A}')$ is dense in $\mathfrak{B}'$. Let ν' be the restriction of ν to $\mathfrak{A}'$. Then ν' is a homomorphism of $\mathfrak{A}'$ onto a dense subalgebra of $\mathfrak{B}'$.

Suppose $p \in \mathfrak{S}$. Let $\{x_n\}$ be a sequence in $\mathfrak{A}$ with $x_n \to 0$ and $\nu(x_n) \to \nu(p)$. Then $px_np \to 0$ and $\nu(px_np) \to \nu(p)$. Thus p belongs to the separating ideal for ν' in $\mathfrak{A}'$. But p is the identity in $\mathfrak{A}'$. This contradicts Proposition 4.7.

REMARK. If $\mathfrak{A}$ is a W^* or AW^* algebra, every closed two-sided ideal in $\mathfrak{A}$ is the closure of the two-sided ideal generated by its projections. (See [3] and [5]). Thus if p is a projection in $\mathfrak{A}$ which belongs to $\mathscr{S}$, then by the theorem p belongs to the kernel of ν. Hence $\mathscr{S}$ is contained in the closure of the kernel of ν. The reverse inclusion clearly holds. It follows immediately that if ν is an isomorphism of a W^* or AW^* algebra, then $\mathscr{S}$ is the zero ideal. We shall prove this later (Theorem 5.1) for any B^* algebra but it will require more work. The next theorem is the crucial step.

THEOREM 4.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, $\mathscr{S}$ the separating ideal for ν in $\mathfrak{A}$. If $\{g_n\}$ is a sequence in $\mathscr{S}$ with $g_ng_m = 0, n \neq m$, then $\nu(g_k)^3 = 0$ for all but a finite number of k.*

Proof. Suppose on the contrary that $\nu(g_k)^3 \neq 0$ for infinitely many k. By a suitable renumbering we may assume $\nu(g_k)^3 \neq 0$ for $k = 1, 2, \cdots$.

Since $g_n \in \mathscr{S}$, there exists for each n a sequence $\{z_{nj}\}$ in $\mathfrak{A}$ with $\lim_j z_{nj} = 0$ and $\lim_j \nu(z_{nj}) = \nu(g_n)$. For each n, $\lim_j z_{nj}g_n = 0$ and $\lim_j \nu(z_{nj}g_n^2) = \nu(g_n)^3 \neq 0$, and hence for $n = 1, 2, \cdots$,

$$\| \nu(z_{nj}g_n^2) \| / \| z_{nj}g_n \| \to \infty \quad \text{as } j \to \infty .$$

For each n pick $j(n)$ such that

$$\| \nu(z_{nj(n)}g_n^2) \| / \| z_{nj(n)}g_n \| > n \| g_n \| .$$

Let $f_n = z_{nj(n)}g_n$, $n = 1, 2, \cdots$. The sequences $\{f_n\}$, $\{g_n\}$ satisfy the following conditions.

(i) $g_n g_m = 0$, $n \neq m$

(ii) $f_m g_n = 0$, $n \neq m$

(iii) $\| \nu(f_n g_n) \| / \| f_n \| \| g_n \| > n$.

But this contradicts the main boundedness theorem.

5. Isomorphisms of B^* algebras. In this section we restrict attention to the case in which $\mathfrak{A}$ is a B^* algebra and ν is an isomorphism of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$. By a B^* algebra we mean a Banach $*$-algebra $\mathfrak{A}$ with $\| x \|^2 = \| xx^* \|$, $x \in \mathfrak{A}$.

THEOREM 5.1. *If ν is an isomorphism of a B^* algebra $\mathfrak{A}$, then $\mathscr{S} = (0)$.*

Proof. Suppose $\mathscr{S} \neq (0)$. Since a closed two-sided ideal in a B^* algebra is a $*$-ideal [7, p. 249], we may assume that $\mathscr{S}$ contains non-zero self-adjoint elements. Notice that $\mathscr{S}$ cannot contain a sequence of orthogonal self-adjoint elements. For if $\{g_n\}$ is such a sequence, then by Theorem 4.9 $\nu(g_n)^3 = 0$ for all but a finite number of n. Since ν is an isomorphism, $g_n^3 = 0$ and thus $r(g_n) = 0$ for all but a finite number of n. But in a B^* algebra $r(x) = \| x \|$ for x normal [7, p. 240]. Hence $g_n = 0$ for all but a finite number of n.

Let $x \in \mathscr{S}$, $x = x^*$, $x \neq 0$. We show that $\sigma(x)$ must be finite. The closure of all polynomials in x (without constant term) is a commutative B^* algebra $\mathfrak{C}$. Since $\mathscr{S}$ is a closed two-sided ideal, $\mathfrak{C} \subseteqq \mathscr{S}$. $\mathfrak{C}$ may be regarded as the algebra of continuous functions vanishing at infinity on the locally compact Hausdorff space $\sigma(x) \sim \{0\}$, [6, p. 232]. If $\sigma(x)$ is infinite, a sequence $\{\lambda_n\}$ of its points may be separated by a sequence of disjoint open sets $\{U_n\}$. Using local compactness we choose V_n open with $\bar{V}_n$ compact and $\lambda_n \in V_n \subseteqq \bar{V}_n \subseteqq U_n$, $n = 1, 2, \cdots$. For each n let f_n be continuous, $0 \leqq f_n \leqq 1$, $f_n(\bar{V}_n) = 1$, and $f_n(\sim U_n) = 0$. Then $f_n f_m = 0$, $n \neq m$. But for each n, $f_n \in \mathfrak{C}$ and hence in $\mathscr{S}$, f_n is self-adjoint, and $f_n \neq 0$. This contradicts the fact that $\mathscr{S}$ cannot contain a sequence of orthogonal self-adjoint elements. Thus $\sigma(x)$

must be finite.

Let $\lambda \in \sigma(x)$. Since $\sigma(x)$ is finite, the function f which is one at λ and vanishes on $\sigma(x) \sim \{\lambda\}$ is continuous and so belongs to $\mathscr{S}$. But $f^2 = f$ and $f \neq 0$. This contradicts Theorem 4.8. Hence $\mathscr{S} = (0)$.

THEOREM 5.2. *Let $C(X)$ be the algebra of all real or complex valued continuous functions vanishing at infinity on a locally compact Hausdorff space X. If $|\cdot|$ is a multiplicative norm on $C(X)$, then for $f \in C(X)$, $||f|| \leqq |f|$ where $||\cdot||$ is the usual* sup *norm.*

Proof. See Kaplansky [4].

LEMMA 5.3. *If ψ is a continuous isomorphism of a B^* algebra $\mathfrak{A}$, the range of ψ is closed.*

Proof. Suppose $||\psi(x)|| \leqq M||x||$, $x \in \mathfrak{A}$. Consider the self-adjoint element xx^*. Applying Kaplansky's theorem to the commutative subalgebra generated by xx^* we have $||xx^*|| \leqq ||\psi(xx^*)||$. Combining this with the B^* condition we have

$$||x^*||^2 = ||xx^*|| \leqq ||\psi(xx^*)|| \leqq ||\psi(x)||\,||\psi(x^*)|| \leqq M||x^*||\,||\psi(x)|| \,.$$

Thus for $x \in \mathfrak{A}$, $||x^*|| = ||x|| \leqq M||\psi(x)||$.

Now if $b_n = \psi(x_n) \in \text{range } \psi$ and $b_n \to b_0$, then $||x_n - x_m|| \leqq M||\psi(x_n) - \psi(x_m)||$. Hence $\{x_n\}$ is Cauchy in $\mathfrak{A}$ and so for some $x_0 \in \mathfrak{A}$, $x_n \to x_0$. By continuity $\psi(x_n) \to \psi(x_0) = b_0$.

THEOREM 5.4. *If ν is an isomorphism of a B^* algebra $\mathfrak{A}$, then there exists a constant M such that $||x|| \leqq M||\nu(x)||$, $x \in \mathfrak{A}$.*

Proof. Since $\mathscr{S} = (0)$, the map ν' is a continuous isomorphism of $\mathfrak{A}$ onto a dense subalgebra of $\mathfrak{B}/\mathscr{S}'$ and $\nu'(a) = \nu(a) + \mathscr{S}' = \pi(\nu(a))$ where π is the natural map of $\mathfrak{B}$ onto $\mathfrak{B}/\mathscr{S}'$. By the lemma range ν' is closed and hence ν' is onto. By the open mapping theorem ν' has a continuous inverse and there exists a constant M such that

$$||x|| \leqq M||\nu'(x)|| = M||\pi\nu(x)|| \leqq M||\nu(x)|| \,,$$

since π is norm decreasing.

THEOREM 5.5. *If ν is an isomorphism of a B^* algebra $\mathfrak{A}$, then $\mathfrak{B} = Cl\nu(\mathfrak{A}) = \nu(\mathfrak{A}) \oplus \mathscr{S}'$.*

Proof. Let $b_0 \in \mathfrak{B}$, $b_0 = \lim b_n$ where $b_n = \nu(x_n)$. By the preceding theorem $||x_n - x_m|| \leqq M||\nu(x_n) - \nu(x_m)||$. Thus $\{x_n\}$ is Cauchy in $\mathfrak{A}$.

Let $x_n \to x_0$ in $\mathfrak{A}$. Then we have $x_n - x_0 \to 0$ and $\nu(x_n - x_0) \to b_0 - \nu(x_0)$. This shows that $b - \nu(x_0) \in \mathscr{S}'$. Thus every element of $\mathfrak{B}$ is the sum of an element in the range of ν and an element of $\mathscr{S}'$. Moreover, the decomposition is unique since $\mathscr{S} = (0)$.

6. Finite singularity sets. It is an open question whether or not there is a discontinuous homomorphism of a B^* algebra. Theorem 6.1 shows that if every isomorphism of a B^* algebra is continuous, then every homomorphism of a B^* algebra is continuous.

THEOREM 6.1. *If there is a discontinuous homomorphism of a B^* algebra, there is a discontinuous isomorphism of a B^* algebra.*

Proof. Suppose ν is a discontinuous homomorphism of a B^* algebra $\mathfrak{A}$. Let $\mathscr{S}$ be the separating ideal for ν in $\mathfrak{A}$. $\mathscr{S}$ is a $*$-ideal and $\mathfrak{A}/\mathscr{S}$ is a B^* algebra. The same arguments used in Proposition 4.2 and Theorem 4.4 show that $\nu(\mathscr{S})$ is a closed two-sided ideal in $\nu(\mathfrak{A})$. Thus $\nu(\mathfrak{A})/\nu(\mathscr{S})$ is a normed algebra under the usual quotient norm. Let $\mathfrak{B}$ be the completion of $\nu(\mathfrak{A})/\nu(\mathscr{S})$ in this norm. Let φ be the map of $\mathfrak{A}/\mathscr{S}$ into $\mathfrak{B}$ defined as follows. $\varphi(a) + \mathscr{S} = \nu(a) + \nu(\mathscr{S})$.

If $a \in \mathscr{S}$, then $\nu(a) \in \nu(\mathscr{S})$ and so $\varphi(0) = 0$. Thus φ is well-defined. It is clearly a homomorphism. If $\varphi(a + \mathscr{S}) = 0$, then $\nu(a) \in \nu(\mathscr{S})$ and $a \in \mathscr{S}$. Hence φ is an isomorphism. The range of φ is clearly dense in $\mathfrak{B}$.

Let $\mathfrak{A}$ be a B^* algebra and ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. We assume that $\mathfrak{A}$ has a unit e and that $\mathfrak{B} = Cl\nu(\mathfrak{A})$. The remainder of the paper is devoted to proving that ν is bounded on certain ideals in $\mathfrak{A}$. The method is essentially the same as that used by Bade and Curtis [1].

Let $\mathfrak{C}$ be any commutative B^* subalgebra of $\mathfrak{A}$ which contains e. $\mathfrak{C}$ is isometrically isomorphic to $C(\Omega)$, the algebra of all continuous functions on a compact Hausdorff space Ω [6, p. 232]. For $f \in \mathfrak{C}$ the carrier of f, denoted car f, is the closure of the set of $\omega \in \Omega$ for which $f(\omega) \neq 0$.

LEMMA 6.2. *Let V be any open subset of Ω and $IR(V)$ ($IL(V)$) be the right (left) ideal in $\mathfrak{A}$ generated by $\{g \ni \mathfrak{C}\colon car\, g \subseteqq V\}$. Then $IR(V) = \{ga\colon g \in \mathfrak{C}, car\, g \subseteqq V,$ and $a \in \mathfrak{A}\}$ and $IL(V) = \{ag\colon g \in \mathfrak{C},$ $car\, g \subseteqq V,$ and $a \in \mathfrak{A}.\}$*

Proof. Clearly $IR(V)$ consists of finite sums of elements of the form ga where $g \in \mathfrak{C}$, car $g \subseteqq V$ and $a \in \mathfrak{A}$. It is enough to show that any such combination belongs to the set on the right. Let $y = g_1a_1 + g_2a_2$ with $a_i \in \mathfrak{A}$, $g_i \in \mathfrak{C}$, and car $g_i \subseteqq V$, $i = 1, 2$. By normality of

Ω choose $g \in \mathfrak{C}$ such that g is one on the union of car g_1 and car g_2 and g vanishes on a neighborhood of $\Omega \sim V$. Then car $g \subseteqq V$, $gg_i = g_i$, $i = 1, 2$. Thus $gy = y$. The proof for $IL(V)$ is the same.

THEOREM 6.3. *There exist finite sets F_R and F_L in Ω and a constant M such that $\|\nu(u)\| \leqq M\|u\|$, $u \in IR(\Omega \sim F_R)$ and $\|\nu(v)\| \leqq M\|v\|$, $v \in IL(\Omega \sim F_L)$.*

Proof. We shall show that there exists a finite set F_R in Ω and a constant M_1 such that $\|\nu(u)\| \leqq M_1\|u\|$ $u \in IR(\Omega \sim F_R)$. It can be shown in an analogous way that there exists a finite set F_L in Ω and a constant M_2 such that $\|\nu(v)\| \leqq M_2\|v\|$, $v \in IL(\Omega \sim F_L)$. Then we take $M = \max(M_1, M_2)$.

Suppose we have shown that there exists a finite set F in Ω and a constant K such that

$$(*) \qquad \|\nu(g^2a)\| \leqq K\|g\|\,\|ga\|, a \in \mathfrak{A}, g \in \mathfrak{C}, \operatorname{car} g \subseteqq \Omega \sim F .$$

Let $a \in \mathfrak{A}, f \in \mathfrak{C}$, and car $f \subseteqq \Omega \sim F$. Now f may be written as $u_1 - u_2 + i(u_3 - u_4)$ where car $u_i \subseteqq \Omega \sim F$, u_i is positive, and $\|u_i\| \leqq \|f\|$, $i = 1, 2, 3, 4$. Since u_i is positive, $u_i = h_i^2$ where $h_i \in \mathfrak{C}$, car $h_i \subseteqq \Omega \sim F$, and $\|h_i\|^2 = \|u_i\|$, $i = 1, 2, 3, 4$. Then we have

$$\nu(fa) = \sum_{i=1}^{4} \alpha_i \nu(u_i a) = \sum_{i=1}^{4} \alpha_i \nu(h_i^2 a)$$

where the α_i are the obvious scalars. Then

$$\|\nu(fa)\| \leqq K\|a\| \sum_{i=1}^{4} \|h_i\|^2 \leqq 4K\|f\|\,\|a\| .$$

Now suppose $u \in IR(\Omega \sim F)$. By the lemma $u = ga$ where $a \in \mathfrak{A}$, $g \in \mathfrak{C}$, and car $g \subseteqq \Omega \sim F$. By normality choose $h \in \mathfrak{C}$ such that h is one on a neighborhood of car g, h vanishes on a neighborhood of F, and $0 \leqq h \leqq 1$. Then car $h \subseteqq \Omega \sim F$ and $hu = hga = ga = u$. Applying the above inequality to h and u and using the fact that $\|u\| = 1$ we have

$$\|\nu(u)\| = \|\nu(hu)\| \leqq M\|h\|\,\|u\| = M\|u\| .$$

Thus ν is bounded on $IR(\Omega \sim F)$. To prove the theorem it suffices to prove (*). The proof will be broken up into a number of lemmas.

Let $\mathfrak{G}$ be the family of open subsets E of Ω with the property

$$\sup \|\nu(g^2a)\|/\|g\|\,\|ga\| \leqq M_E < \infty$$

where the sup is taken over all $a \in \mathfrak{A}$ and all $g \in \mathfrak{C}$ with car $g \subseteqq E$. We shall show that $\mathfrak{G}$ contains a maximal open set.

LEMMA 6.4. *If* $\{E_n\}$ *is a sequence of disjoint open sets in* Ω, *then for sufficiently large* n, $E_n \in \mathfrak{G}$.

Proof. Suppose the lemma is false. Then there is a disjoint sequence of open sets $\{E_m\}$ and sequences $\{a_m\}$ in $\mathfrak{A}$, $\{g_m\}$ in $\mathfrak{C}$ with $\text{car}\, g_m \subseteqq E_m$ and $\| \nu(g_m^2 a_m) \| > m \| g_m \| \| g_m a_m \|$. Since the E_m are disjoint, $g_m g_n = 0$, $n \neq m$. Set $f_n = g_n a_n$, $n = 1, 2, \cdots$. Then $g_m f_n = 0$, $n \neq m$. This contradicts the main boundedness theorem.

LEMMA 6.5. *If* $E_1, E_2 \in \mathfrak{G}$ *and* G *is an open set with* $\bar{G} \subseteqq E_2$, *then* $E_1 \cup G \in \mathfrak{G}$.

Proof. Since Ω is normal, we can choose $u_1 \in \mathfrak{C}$ such that $0 \leqq u_1 \leqq 1$, u_1 is one on a neighborhood of $\Omega \sim E_2$ and zero on a neighborhood of $\bar{G}$. Let $u_2 = 1 - u_1$. Since u_1 and u_2 are nonnegative, each has a square root in $\mathfrak{C}$ and $\text{car}\, \sqrt{u_i} \subseteqq \text{car}\, u_i$, $i = 1, 2$.

Let $a \in \mathfrak{A}$, $g \in \mathfrak{C}$, and $\text{car}\, g \subseteqq E_1 \cup G$. Then $\text{car}\, (g\sqrt{u_i}) \subseteqq E_i$, $i = 1, 2$. Since $E_i \in \mathfrak{G}$,

$$\begin{aligned} \| \nu(g^2 a) \| &\leqq \| \nu(g^2 u_1 a) \| + \| \nu(g^2 u_2 a) \| \\ &\leqq M_1 \| g\sqrt{u_1} \| \| (g\sqrt{u_1})a \| + M_2 \| g\sqrt{u_2} \| \| (g\sqrt{u_2})a \| \\ &\leqq \{M_1 \| u_1 \| + M_2 \| u_2 \|\} \| g \| \| ga \| \end{aligned}$$

LEMMA 6.6. *If* $E_1, E_2 \in \mathfrak{G}$ *and* G *is open with* $\bar{G} \subseteqq E_1 \cup E_2$, *then* $G \in \mathfrak{G}$.

Proof. The closed set $F = \bar{G} \cap (\Omega \sim E_1) \subseteqq E_2$. By normality choose U open with $F \subseteqq U \subseteqq \bar{U} \subseteqq E_2$ Then $G \subseteqq E_1 \cup U$ which belongs to $\mathfrak{G}$ by Lemma 6.5.

LEMMA 6.7. $\mathfrak{G}$ *is closed under finite unions.*

Proof. Let $E_1, E_2 \in \mathfrak{G}$ and suppose $E_1 \cup E_2 \notin G$. If F is closed and $F \subseteqq E_1 \cup E_2$, then $G = (E_1 \cup E_2) \sim F \notin \mathfrak{G}$. For choose U, V open with

$$F \subseteqq U \subseteqq \bar{U} \subseteqq V \subseteqq \bar{V} \subseteqq E_1 \cup E_2 .$$

Then $V \in \mathfrak{G}$ by Lemma 6.6. If $G \in \mathfrak{G}$, then by Lemma 6.5, $E_1 \cup E_2 \subseteqq G \cup U \in \mathfrak{G}$.

Since $E_1 \cup E_2 \notin \mathfrak{G}$, we can choose $a_1 \in \mathfrak{A}$, $g_1 \in \mathfrak{C}$, such that $\text{car}\, g_1 \subseteqq E_1 \cup E_2$ and $\| \nu(g_1^2 a_1) \| > \| g_1 \| \| g_1 a_1 \|$. Pick U_1 open such that $\text{car}\, g_1 \subseteqq U_1 \subseteqq \bar{U}_1 \subseteqq E_1 \cup E_2$. Then $G_2 = (E_1 \cup E_2) \sim \bar{U}_1 \notin \mathfrak{G}$. Hence we can choose $a_2 \in \mathfrak{A}$, $g_2 \in \mathfrak{C}$ with $\text{car}\, g_2 \subseteqq G_2$ and $\| \nu(g_2^2 a_2) \| > 2 \| g_2 \| \| g_2 a_2 \|$. Continuing inductively we obtain sequences $\{a_n\}$ in $\mathfrak{A}$, $\{g_n\}$ in $\mathfrak{C}$ with

(i) $g_n g_m = 0, n \neq m$

(ii) $g_n(g_m a_m) = 0, n \neq m$

(iii) $\| \nu(g_n^2 a_n) \| > n \| g_n \| \| g_n a_n \|, n = 1, 2, \cdots$.

This contradicts the main boundedness theorem.

LEMMA 6.8. *$\mathfrak{G}$ is closed under arbitrary unions.*

Proof. Let $E_0 = \bigcup E_\alpha$, where $E_\alpha \in \mathfrak{G}$. Suppose $E_0 \notin \mathfrak{G}$. If F is closed and $F \subseteqq E_0$, then $E_0 \sim F \notin \mathfrak{G}$. For by compactness F is covered by the union of a finite number of the E_α. This union E_1 belongs to $\mathfrak{G}$ and thus if $E_0 \sim F \in \mathfrak{G}$, $E_0 = (E_0 \sim F) \cup E_1 \in \mathfrak{G}$.

Repeating the construction of the last proof we obtain a contradiction.

LEMMA 6.9. *There exists a finite set F in Ω and a constant K such that*

$$\| \nu(g^2 a) \| \leqq K \| g \| \| ga \|, a \in \mathfrak{A}, g \in \mathfrak{C}, \operatorname{car} g \subseteqq \Omega \sim F .$$

Proof. Since $\mathfrak{G}$ is closed under arbitrary unions, the union of all sets in $\mathfrak{G}$ is a maximal open set G in $\mathfrak{G}$. If $F = \Omega \sim G$ were infinite, a sequence of its elements could be separated by a sequence $\{E_n\}$ of disjoint open sets. For large n, $E_n \in \mathfrak{G}$. Thus G would contain a point of its complement. Hence F must be finite and the lemma is proved.

Let Z be the center of $\mathfrak{A}$. Z is a commutative B^* subalgebra of $\mathfrak{A}$ containing the identity and we have the following corollary.

COROLLARY 6.10. *Let $\mathfrak{C} = Z$. Then for any open subset V of Ω, $I(V) = \{ga: g \in Z, car\, g \subseteqq V,$ and $a \in \mathfrak{A}\}$ is a two-sided ideal in $\mathfrak{A}$. There is a finite set F and a constant M such that $\| \nu(u) \| \leqq M \| u \|$ for $u \in I(\Omega \sim F)$.*

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