2-SIGNALIZERS OF FINITE GROUPS

JOHN G. THOMPSON

DEFINITION. Let π be a set of primes and $\mathfrak{G}$ a group. The subgroup $\mathfrak{A}$ of $\mathfrak{G}$ is a π-signalizer of $\mathfrak{G}$ if and only if $|\mathfrak{A}|$ and $|\mathfrak{G}:\boldsymbol{N}_{\mathfrak{G}}(\mathfrak{A})|$ are π'-numbers.[1]

Let $s_\pi(\mathfrak{G}) = \max|\mathfrak{A}|$, $\mathfrak{A}$ ranging over the π-signalizers of $\mathfrak{G}$.

THEOREM 1. *For each pair of integers m, n, there are only finitely many (isomorphism classes of) finite groups $\mathfrak{G}$ such that*

$$|\mathfrak{G}|_2 \leqq m \quad \textit{and} \quad s_2(\mathfrak{G}) \leqq n\ .$$

Proof. Let $|\mathfrak{G}|_2 = 2^r$ and proceed by induction on r, as the theorem holds for $r = 0$. We must bound $|\mathfrak{G}|$ by a function of r and $s_2(\mathfrak{G})$. If $\mathfrak{H}$ is a normal $2'$-subgroup of $\mathfrak{G}$, then $s_2(\mathfrak{G}/\mathfrak{H}) = s_2(\mathfrak{G})/|\mathfrak{H}|$, so we assume without loss of generality that 1 is the only normal $2'$-subgroup of $\mathfrak{G}$. Suppose $\mathfrak{G}$ contains a normal 2-subgroup $\mathfrak{R} \neq 1$. It suffices to bound $|\mathfrak{G}:\mathfrak{R}|$, so it suffices to bound $s_2(\mathfrak{G}/\mathfrak{R})$, by our induction hypothesis. Let $\mathfrak{A}/\mathfrak{R}$ be a 2-signalizer of $\mathfrak{G}/\mathfrak{R}$ and let $\mathfrak{T}/\mathfrak{R}$ be a S_2-subgroup of $\mathfrak{G}/\mathfrak{R}$ normalizing $\mathfrak{A}/\mathfrak{R}$. Let $\mathfrak{B} = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{R})$, so that $\mathfrak{B} = \boldsymbol{Z}(\mathfrak{R}) \times \mathfrak{C}$, where $\mathfrak{C}$ char $\mathfrak{B} \lhd \mathfrak{A}\mathfrak{T}$. Hence, $|\mathfrak{C}| \leqq s_2(\mathfrak{G})$, $\mathfrak{C}$ being a 2-signalizer of $\mathfrak{G}$. Hence, $|\mathfrak{A}| \leqq |\mathfrak{R}||\mathfrak{C}||\operatorname{Aut}\mathfrak{R}|_{2'} \leqq s_2(\mathfrak{G})2^{r^2}$. This gives a bound for $|\mathfrak{A}:\mathfrak{R}|$, so also for $|\mathfrak{G}:\mathfrak{R}|$.

We therefore assume without loss of generality that 1 is the only normal $2'$-subgroup of $\mathfrak{G}$ and 1 is the only normal 2-subgroup of $\mathfrak{G}$. Hence, 1 is the only normal abelian subgroup of $\mathfrak{G}$, so that $\mathfrak{N}$, the join of all minimal normal subgroups of $\mathfrak{G}$, is the direct product of subgroups $\mathfrak{N}_i$, each of which is a non abelian simple group, $1 \leqq i \leqq t$. Since $1 = \boldsymbol{C}_{\mathfrak{G}}(\mathfrak{N})$, it suffices to bound $|\mathfrak{N}|$. Clearly, each $\mathfrak{N}_i$ is of even order, since 1 is the only normal $2'$-subgroup of $\mathfrak{G}$. This yields $t \leqq r$, so it suffices to bound $|\mathfrak{N}_1|$. Let $\mathfrak{T}$ be a S_2-subgroup of $\mathfrak{G}$ and let J be an involution in $\boldsymbol{Z}(\mathfrak{T}) \cap \mathfrak{N}$, $J = J_1 \cdots J_t$, J_i in $\mathfrak{N}_i$, $1 \leqq i \leqq t$. Let $\mathfrak{A}/\langle J\rangle$ be a 2-signalizer of $\boldsymbol{C}_{\mathfrak{G}}(J)/\langle J\rangle$, so that $\mathfrak{A} = \langle J\rangle \times \mathfrak{B}$, where $\mathfrak{B}$ is a 2-signalizer of $\mathfrak{G}$. Hence, $s_2(\boldsymbol{C}_{\mathfrak{G}}(J)/\langle J\rangle) \leqq s_2(\mathfrak{G})$. By our induction hypothesis, $|\boldsymbol{C}_{\mathfrak{G}}(J)|$ is bounded by a function of r and $s_2(\mathfrak{G})$. Since $\boldsymbol{C}_{\mathfrak{N}}(J) = \boldsymbol{C}_{\mathfrak{N}_1}(J_1) \times \cdots \times \boldsymbol{C}_{\mathfrak{N}_t}(J_t)$, $|\boldsymbol{C}_{\mathfrak{N}_1}(J_1)|$ is bounded. If $J_1 = 1$, we have a bound for $|\mathfrak{N}_1|$; otherwise, $|\mathfrak{N}_1| \leqq \{|\boldsymbol{C}_{\mathfrak{N}_1}(J_1)|^2\}!$, by a well known theorem of Brauer and Fowler [1]. The proof is complete.

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[1] The notation in this paper conforms with *Solvability of Groups of Odd Order*, W. Feit and J. Thompson, this Journal, 1963, and is for the most part self-explanatory.

REMARK. Defining $f(0, n) = n$, $n = 1, 2, \cdots$, and $f(m+1, n) = z^z$, where $z = f(m, 2^m n)^2$, $m = 0, 1, \cdots$, it follows readily that if $|\mathfrak{G}|_2 = 2^m$ and $s_2(\mathfrak{G}) = n$, then $|\mathfrak{G}| \leqq f(m, n)$. While the alternating groups show that $|\mathfrak{G}|$ is not bounded by a polynomial function of $|\mathfrak{G}|_2$ and $s_2(\mathfrak{G})$, the given f is hardly to the point.

THEOREM 2. *Suppose the S_2-subgroup $\mathfrak{T}$ of $\mathfrak{G}$ is abelian and $s_2(\mathfrak{G}) = 1$. Then $\mho^1(\mathfrak{T}) \lhd \mathfrak{G}$.*

Proof. We proceed by induction on $|\mathfrak{G}|$. Let $\mathfrak{H}$ be the largest normal 2-subgroup of $\mathfrak{G}$, and let $\mathfrak{C} = C_{\mathfrak{G}}(\mathfrak{H})$. Suppose $\mathfrak{C} \subset \mathfrak{G}$. Then by induction, $\mho^1(T) \lhd \mathfrak{C}$, so $\mho^1(\mathfrak{T})$ char $\mathfrak{C}$, as $\mho^1(\mathfrak{T})$ is the set of squares of 2-elements of $\mathfrak{C}$. Hence, $\mho^1(\mathfrak{T}) \lhd \mathfrak{G}$ and we are done. Suppose $\mathfrak{C} = \mathfrak{G}$, but $\mathfrak{H} \neq 1$. Let J be an involution of $\mathfrak{H}$. If $\langle J \rangle$ is a direct factor of $\mathfrak{T}$, then $\langle J \rangle$ is a direct factor of $\mathfrak{G}$ and we are done by induction. If $\langle J \rangle$ is not a direct factor of $\mathfrak{G}$, then $J \in \mho^1(\mathfrak{T})$, so $\mho^1(\mathfrak{T}/\langle J \rangle) = \mho^1(\mathfrak{T})/\langle J \rangle$, and we are again done by induction. Hence, we may assume $\mathfrak{H} = 1$.

Since 1 is the only normal $2'$-subgroup of $\mathfrak{G}$, the join $\mathfrak{N}$ of all minimal normal subgroups of $\mathfrak{G}$ is the direct product of subgroups $\mathfrak{N}_i$, each being a non abelian simple group, $1 \leqq i \leqq n$. Furthermore, $1 = C_{\mathfrak{G}}(\mathfrak{N})$.

Let J be an involution of $\mathfrak{T}$. Since $C_{\mathfrak{G}}(J) \subset \mathfrak{G}$, we have $\mho^1(\mathfrak{T}) \lhd C_{\mathfrak{G}}(J)$. Hence, $N_{\mathfrak{G}}(\mho^1(\mathfrak{T})) = \mathfrak{R}$ contains the centralizer of each of its involutions. Suppose J, K are involutions of $\mathfrak{R}$ which are not conjugate in $\mathfrak{G}$. If K_1 is any conjugate of K, there is an involution L centralizing J and K_1. Hence, $L \in C_{\mathfrak{G}}(J) \subseteqq \mathfrak{R}$, and $K_1 \in C_{\mathfrak{G}}(L) \subseteqq \mathfrak{R}$, and so $\mathfrak{R}$ contains the normal closure of K in $\mathfrak{G}$. Hence, $\mathfrak{R}$ contains the normal closure of each of its involutions, which implies that $\mathfrak{N} \subseteqq \mathfrak{R}$. Hence, $[\mathfrak{N}, \mho^1(\mathfrak{T})] \subseteqq \mho^1(\mathfrak{T}) \cap \mathfrak{N}$. As every c.f. of $\mathfrak{N}$ is nonsolvable, $\mho^1(\mathfrak{T}) \cap \mathfrak{N} = 1$. Since $1 = C_{\mathfrak{G}}(\mathfrak{N})$, so also $1 = \mho^1(\mathfrak{T})$, and we are done. We may therefore assume that any two involutions of $\mathfrak{R}$ are conjugate in $\mathfrak{G}$. This implies that $\mathfrak{T}$ is homocyclic. If $\mathfrak{T}$ is elementary, we are done. Otherwise, $C_{\mathfrak{G}}(\mho^1(\mathfrak{T}))$ has a normal 2-complement, which must be 1, since $s_2(\mathfrak{G}) = 1$. This in turn implies that $\mathfrak{T} = C_{\mathfrak{G}}(\mho^1(\mathfrak{T})) \lhd \mathfrak{R}$. As $\mathfrak{R}$ contains the centralizer of each of its involutions, $\mathfrak{T}$ is a T.I. set in $\mathfrak{G}$. By a fundamental result of Suzuki [2], $\mathfrak{T}$ is elementary. The proof is complete.

Conjecture. If $\mathfrak{G}$ is a simple group, then every 2-signalizer of $\mathfrak{G}$ is abelian.

REFERENCES

1. R. Brauer and K. Fowler, *On groups of even order*, Annals of Math., **62** no. 3, (1955).
2. M. Suzuki, *Finite groups whose Sylow 2-subgroups are independent*, (to appear).

UNIVERSITY OF CHICAGO