ON AN INEQUALITY OF P. R. BEESACK

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In a recent paper [1], P. R. Beesack derived the inequality

$$|g(x, s)| \leqq \frac{\prod_{\nu=1}^{n} |x - a_\nu|}{(a_n - a_1)(n-1)!} \tag{1}$$

for the Green's function $g(x, s)$ of the differential system

$$\begin{gathered} y^{(n)} = 0, \qquad y(a_\nu) = 0, \qquad \nu = 1, 2, \cdots, n, \\ -\infty < a_1 < a_2 < \cdots < a_n < \infty. \end{gathered} \tag{2}$$

In addition to being interesting in its own right, this inequality is a useful tool in the study of the oscillatory behavior of nth order differential equations. It would therefore appear to be worth while to give a short proof of (1). The derivation of this inequality in [1] is rather complicated.

We denote by $[x_0, x_1, \cdots, x_k]$ the kth difference quotient of the function $g(x) = g(x, s)$, i.e., we set

$$[x_0, x_1] = \frac{g(x_0) - g(x_1)}{x_0 - x_1},$$

$$[x_0, x_1, \cdots, x_\nu] = \frac{[x_0, x_1, \cdots, x_{\nu-1}] - [x_1, x_2, \cdots, x_\nu]}{x_0 - x_\nu}, \qquad \nu = 2, \cdots.$$

This difference quotient can also be represented in the form

$$[x_0, \cdots, x_k] = \int \cdots \int g^{(k)}(t_0x_0 + t_1x_1 + \cdots + t_kx_k)dt_0dt_1 \cdots dt_{k-1}, \tag{3}$$

where the integration is to be extended over all the positive values of the t_ν for which

$$t_0 + t_1 + \cdots + t_k = 1. \tag{4}$$

This formula, which goes back to Hermite, is easily verified by induction (cf., e.g., [2]). It holds if $g(x)$ has continuous derivatives up to the order $k - 1$, and if $g^{(k)}$ is piecewise continuous.

Since, by its definition, $g(x, s)$ has continuous derivatives up to the order $n - 2$, while $g^{(n-1)}$ has the jump

$$g_+^{(n-1)}(s) - g_-^{(n-1)}(s) = -1 \tag{5}$$

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at $x = s$, we may apply (3) with $k = n - 1$. We shall do so twice, identifying the points $x_0, \cdots, x_{n-1}$ with $x, a_1, \cdots, a_{n-1}$ and $x, a_2, \cdots, a_n$, respectively. Since, because of $g(a_\nu, s) = 0$, $\nu = 1, \cdots, n$, we have

$$[x, a_1, \cdots, a_{n-1}] = \frac{g(x, s)}{\prod_{\nu=1}^{n-1} (x - a_\nu)}$$

and

$$[x, a_2, \cdots, a_n] = \frac{g(x, s)}{\prod_{\nu=2}^{n} (x - a_\nu)} ,$$

we obtain, upon subtracting these expressions from each other,

$$(6) \qquad \frac{(a_n - a_1) g(x, s)}{\prod_{\nu=1}^{n} (x - a_\nu)} = \int_D g^{(n-1)}(v) dt - \int_D g^{(n-1)}(u) dt ,$$

where, for brevity, $dt = dt_0 dt_1 \cdots dt_{n-2}$, D denotes the region defined by (4) (with $k = n - 1$ and $t_\nu > 0$, $\nu = 0, \cdots, n - 1$), and

$$(7) \qquad u = t_0 x + t_1 a_1 + \cdots + t_{n-1} a_{n-1},\ v = t_0 x + t_1 a_2 + \cdots + t_{n-1} a_n .$$

Both for $a_1 \leqq x < s$ and $s < x \leqq a_n$, $g(x, s)$ is a polynomial of degree $n - 1$. Accordingly, the function $g^{(n-1)}(x, s)$ is capable only of two constant values, say α and β, which according to (5) are related by $\alpha = \beta + 1$. If we denote by D_1 the subset of D in which $a_1 \leqq u < s$ (where u is defined in (7), we have

$$\int_D g^{(n-1)}(u) dt = \alpha \int_{D_1} dt + \beta \int_{D-D_1} dt = \alpha \int_{D_1} dt + (\alpha - 1) \int_{D-D_1} dt$$
$$= \alpha \int_D dt - \int_{D-D_1} dt .$$

Similarly,

$$\int_D g^{(n-1)}(v) dt = \alpha \int_D dt - \int_{D-D_2} dt ,$$

where D_2 is the subset of D in which $a_1 \leqq v < s$. Substituting these expressions in (6), we obtain

$$(8) \qquad \frac{(a_n - a_1) g(x, s)}{\prod_{\nu=1}^{n} (x - a_\nu)} = \int_{D-D_2} dt - \int_{D-D_1} dt .$$

The differential dt is positive, and we thus have

$$-\int_D dt \leqq -\int_{D-D_1} dt \leqq \int_{D-D_2} dt - \int_{D-D_1} dt \leqq \int_{D-D_2} dt \leqq \int_D dt .$$

Since

$$\int_D dt = \frac{1}{(n-1)!}$$

(as can be seen be applying (3) to the function x^{n-1} and setting $k = n - 1$), this shows that

$$\left| \int_{D-D_2} dt - \int_{D-D_1} dt \right| \leqq \frac{1}{(n-1)!}$$

In view of (8), this establishes the inequality (1).

REFERENCES

1. P. R. Beesack, *On the Green's function of an n-point boundary value problem,* Pacific J. Math., **12** (1962), 801-812.
2. N. E. Nörlund, *Leçons sur les séries d'interpolation,* Paris, Gauthier-Villars, 1926.

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