ON SOME PROPERTIES OF SOLUTIONS OF $\Delta\psi + A(r^2)X\nabla\psi + C(r^2)\psi = 0$

V. MARIĆ

Dedicated to Charles Loewner on his 70th birthday

Introduction. The main task of Bergman's operator theory has been to establish and to study the mappings (in the large) of the algebra $\{f\}$ of functions of one or several complex variables onto the linear space $\{\psi\}$ of solutions of various classes of linear partial differential equations $L[\psi] = 0$. In the case of partial differential equations of two variables the mappings of the algebra of functions of one complex variable onto the solutions $\{\psi\}$ of a general class of equations is established by means of a comparatively simple integral operator in such a way that very many theorems concerning $\{f\}$ have their counterpart in $\{\psi\}$.

In the case of partial differential equations of three variables the mapping of the set of functions $\{f\}$ of two complex variables onto the space $\{G\}$ of harmonic function and onto the space $\{\psi\}$ of solutions of the equation

$$(1) \qquad \Delta\psi(x, y, z) + A(r^2)\, X\nabla\psi(x, y, z) + C(r^2)\,\psi(x, y, z) = 0$$

where $r^2 = x^2 + y^2 + z^2$, $X = (x, y, z)$, Δ is Laplace operator,

$$X\nabla = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and $A, C,$ are entire functions of r^2, in particular have been studied ([1], [4]).

In addition one can consider the mapping $\{G\} \to \{\psi\}$. Of course, this mapping could be obtained by combining $\{f\} \to \{G\}$ and $\{f\} \to \{\psi\}$. However, in this way one obtains a very complicated integral operator and since we are passing through functions of two complex variables which then have to be restricted to some special values, various relations could be lost. Therefore, it is of interest to study the direct mapping $\{G\} \to \{\psi\}$ which is the aim of the present paper.

Let

$$a_n(r) = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}\, C_n(r)$$

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where $C_n(r)$ are defined by means of the equations

$$(2)\quad \begin{aligned} &\frac{dC_1}{dr} - r\,B = 0,\ B = \frac{3}{2}A + \frac{1}{2}r\,Ar + \frac{1}{4}r^2A^2 - C\,, \qquad C_n(0) = 0 \\ &(2n-1)\frac{dC_n}{dr} + r\frac{d^2C_{n-1}}{dr^2} - (2n-3)\frac{dC_{n-1}}{dr} + r\,BC_{n-1} = 0\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 2, 3, \cdots \end{aligned}$$

Let further $G = G(X)$ be a harmonic function regular in a neighborhood of the origin. It was shown by Bergman [4, p. 68][1] that the integral operator

$$(3)\quad \boldsymbol{P}_3[G] = \exp\left(-\frac{1}{2}\int_0^r rA\,dr\right)\left\{G(X) + \sum_{n=1}^{\infty} a_n(r)\int_0^1 (1-t^2)^{n-1}t^2\,G(Xt^2)\,dt\right\}$$

transforms G into solution $\psi = \psi(X)$ of (1) i.e. that

$$(4)\qquad \psi = \boldsymbol{P}_3[G]\,.$$

The function G is called the associate function of the solution ψ, and the domain $\mathscr{A}(P_3, \psi)$ of validity of (4) is called the domain of association [3]. It is obvious from (4), that the solution ψ is uniquely determined and regular in every simply connected domain D containing the origin in which G is regular, and that $\mathscr{A}(P_3, \psi) \supset D$.

Here we show that the converse is also true, namely that for every solution ψ of (1) regular in a domain D there exists a uniquely determined associate function G regular in the same domain D, so that ψ can there be represented in the form (3). The domain of association $\mathscr{A}(P_3, \psi)$ coincides therefore, with the domain of regularity of ψ. We show further, that for a solution ψ which satisfies suitable conditions the associate function G is also harmonic in the domain.

These results permit us to show that many theorems about representations, summation methods of the series expansion and behavior on the boundary holding for solutions of partial differential equations of two variables [4, p. 22], can be also proved in the case of equation (1).

As an illustration of the derivation of this kind, we give here some representations of solutions ψ of (1).

1.1. In order to get the representations of solutions mentioned above, we need the following.

LEMMA. *For every singlevalued solution ψ regular in a simply connected domain D containing the origin, there exists a uniquely determined function G regular in the same domain so that ψ can*

[1] The exponential factor has been omitted there by misprint.

be represented there in the form (4).

Proof. Since the series in (3) converges uniformly for $r \leqq R$ where R is an arbitrary large (fixed) number,[2] we have, in fact, to prove the existence, uniqueness and regularity in a domain, of a solution of the integral equation

$$\psi^*(X) = G(X) + \int_0^1 f(r, t)\, G(Xt^2)\, dt \tag{5}$$

where

$$\psi^* = \psi \exp\left(\frac{1}{2}\int_0^r r\, A\, dr\right), \qquad f(r, t) = \sum_{n=1}^{\infty} a_n(r)\,(1 - t^2)^{n-1}\, t^2\,.$$

Introducing in (5) spherical coordinates r, θ, φ, and putting $t^2 r = t$ we get

$$\psi^*(r, \theta, \varphi) = G(r, \theta, \varphi) + \int_0^r K(r, t)\, G(t, \theta, \varphi)\, dt$$

where

$$K(r, t) = \sum_{n=1}^{\infty} a_n(r)\left(1 - \frac{t}{r}\right)^{n-1} \frac{\sqrt{t}}{2r^{3/2}}$$

—a Volterra type integral equation of the second kind with θ, φ as parameters.

Since the series $\Sigma\, a_n(r)$, $\Sigma\, a_n'(r)$ both converge for $r \leqq R$ uniformly[2] and since from (2) we conclude that $a_n(0) = a_n'(0) = 0$ it follows that $K(r, t) \to 0$, $r \to 0$ uniformly in t. We define $K(0, 0) = 0$ so that the kernel $K(r, t)$ is continuous in $0 \leqq t \leqq r \leqq R$. Therefore, for continuous ψ the existence and uniqueness of the function G follows and

$$G(r, \theta, \varphi) = \psi^*(r, \theta, \varphi) + \int_0^r L(r, t)\, \psi^*(t, \theta, \varphi)\, dt$$

where $L(r, t)$ is the resolvent kernel

$$L(r, t) = \sum_{\nu=1}^{\infty} (-1)^{\nu} K^{(\nu)}(r, t) \tag{6}$$

and $K^{(\nu)}(r, t)$ are iterated kernels of (5) [10, p. 243]. Since $K(r, t)$ is a regular function of r for $r \leqq R$, the same holds for both iterated and resolvent kernels because of the uniform convergence of the series (6). Hence, if ψ is regular in a simply connected domain D (which also holds for ψ^*), it follows that G is regular in the same domain, which completes the proof of the lemma.

[2] This can be shown by an obvious modification of the argument used in [**1**].

1.1.1. Suppose now that there exists a positive function $A(r, \theta, \varphi)$ bounded in every closed subdomain of the sphere $\mathscr{S}$ of the radius ρ, so that every regular solution $\psi(r, \theta, \varphi)$ satisfies the inequality

$$(6a) \qquad |\psi(r, \theta, \varphi)| \leqq A(r, \theta, \varphi) \max_{\rho} |\psi(\rho, \theta, \varphi)|, (r, \theta, \varphi) \in \mathscr{S} .$$

Let further the boundary values $\psi(\rho, \theta, \varphi)$ of ψ be expanded in a series of Legendre associate functions (this holds under very general hypotheses which can be found in [9]). Then we have [1, p. 429]

$$(7) \qquad \psi(r, \theta, \varphi) = \sum_{n=0}^{\infty} r^n T_n(r) \Big[A_{n,0} P_{n,0}(\cos\theta) + \sum_{m=1}^{n} (A_{n,m} \cos m\varphi + B_{n,m} \operatorname{sim} m\varphi) P_{n,m}(\cos\theta) \Big]$$

where

$$T_n(r) = \int_{-1}^{1} E(r, t) (1 - t^2)^n \, dt, \; E(r, t) = \exp\Big(-\frac{1}{2} r \, A \, dr \Big) \sum_{n=0}^{\infty} C_n(r) t^{2n} ,$$

$C_n(r)$, $n = 1, 2, \cdots$, have been defined by (2) and $C_0(r) \overset{\text{def}}{=} 1$, and the series in (7) converges uniformly in every compact subset of $\mathscr{S}$. By interchanging the order of summation and integration in (7) we get

$$\psi(r, \theta, \varphi) = \int_{-1}^{1} E(r, t) \sum_{n=0}^{\infty} r^n (1 - t^2)^n \Big[A_{n,0} P_{n,0}(\cos\theta) , + \sum_{m=1}^{n} (A_{n,m} \cos m\varphi + B_{n,m} \sin m\varphi), P_{n,m}(\cos\theta) \Big]$$

or

$$\psi(X) = \int_{-1}^{1} E(r, t) \, H(X(1 - t^2)) \, dt$$

where H is a harmonic function. The above formula is identical to (3).

The above procedure is similar to the one Gilbert has used in [7, 8] when considering singularities of three-dimensional harmonic functions.

One can summarize the results in § 1.1 as follows: *If the solution* $\psi(X)$ *of* (1), *singlevalued and regular in a simply connected domain* D *containing the origin, satisfies the condition* (6a), *then its associate function* $G(X)$ *in the representation* (4) *is harmonic in* D.

2.1. Let $\{f_\nu(X)\}$ be a sequence of functions regular in a given domain. We introduce now, quite formally, a special transform of a given series expansion of the function

$$(8) \qquad f(X) = s\{f_\nu(X)\} \equiv \sum \alpha_\nu f_\nu(X)$$

for which we suppose to be convergent absolutely and uniformly in a domain B_1. Let further a sequence $\{h_n(X)\}$ be defined by the absolutely and uniformly convergent series

$$h_n(X) = \sum_{\nu=n}^{\infty} \beta_\nu^{(n)} f_\nu(X) .$$

If the sequence $\{\gamma_\nu\}$ is such that the series

$$\Sigma \gamma_\nu h_\nu(X) \tag{9}$$

converges absolutely and uniformly to $f(X)$ in $B_2 \supset B_1$, we shall call (9) the T-transform of the series (8) and write

$$f(X) = T\{s(f_\nu(X))\} .$$

T-transform obviously provides an analytic continuation of $s\{f_\nu(X)\}$ into B_2 (cf. [6], [5]).

Let $G(X)$ be a harmonic function regular in a simply connected domain D. In order to obtain the representations of ψ mentioned in the introduction we introduce here the T-transform of the series expansion of G in spherical harmonics

$$G(X) = \sum_{\kappa=1}^{\infty} \sum_{\lambda=1}^{2\kappa+1} b_{\kappa,\lambda}\, g_{\kappa,\lambda}(X) \tag{10}$$

which converges uniformly and absolutely in a sphere $\mathscr{S} \subset D$. For the sake of brevity we shall denote the right hand series in (10) by $s\{g_{\kappa,\lambda}(X)\}$. We denote further, by $\phi_{m,n}(X)$ the harmonic function whose expansion in spherical harmonics begins with the nth such polynomials of the degree m i.e.

$$\phi_{m\, n}(X) = \sum_{\lambda=n}^{2m+1} a_{m,\lambda}^{(m,n)}\, g_{m,\lambda}(X) + \sum_{\kappa=m+1}^{\infty} \sum_{\lambda=1}^{2\kappa+1} a_{\kappa,\lambda}^{(m,n)}\, g_{\kappa,\lambda}(X) . \tag{11}$$

We form then the T-transform of (10) as follows

$$T\{s(g_{m,n}(X))\} = \sum_{m=1}^{\infty} \sum_{n=1}^{2m+1} c_{m,n}\, \phi_{m,n}(X) . \tag{12}$$

Formally, from (10), (11) and (12) by comparison of coefficient we can calculate the numbers $c_{m,n}$:

$$c_{m,n} = D_{m,n}^{*} / D_{m,n}$$

where

$$D_{m\, n} = a_{1,1}^{(1,1)}\, a_{1,2}^{(1,2)}\, a_{1,3}^{(1\, 3)}\, a_{2,1}^{(2\, 1)} \cdots a_{m,n}^{(m,n)}$$

$m = 1, 2, 3, \cdots, n = 1, 2, \cdots 2m+1$ so that there are $m^2 - 1 + n$ factors in $D_{m,n}$ and $D_{m,n}^{*}$ is the determinant

$$\begin{vmatrix} a_{1,1}^{(1,1)}, & 0, & 0, \cdots, 0, b_{1,1} \\ a_{1,2}^{(1,1)}, & a_{1,2}^{(1,2)}, & 0, \cdots, 0, b_{1,2} \\ \vdots & & \\ a_{m,n}^{(1,1)}, & a_{m,n}^{(1,2)}, & b_{m,n} \end{vmatrix}$$

of the $m^2 - 1 + n$ order.

We shall prove that the T transforms of $s\{g_{\kappa,\lambda}(X)\}$ converges uniformly and absolutely to the function $G(X)$ in the whole domain D, in the case when $\{\phi_{m,n}(X)\}$ is an orthogonal system in D in the sense that

$$\int_D \left\{ \frac{\partial\phi_\nu}{\partial x}\frac{\partial\phi_\mu}{\partial x} + \frac{\partial\phi_\nu}{\partial y}\frac{\partial\phi_\mu}{\partial y} + \frac{\partial\phi_\nu}{\partial z}\frac{\partial\phi_\mu}{\partial z} \right\} dv = \delta_{\nu,\mu}$$

where $\delta_{\nu,\mu}$ are Kroneker symbols.

That there exists such a set that one can conclude by considering the following variational problem: To find a function $G(X)$ which minimizes the integral

$$\int_D \{G_x^2 + G_y^2 + G_z^2\}\, dv$$

and satisfies the $(m-1)^2 + n$ conditions

$$G(0, 0, 0) = 0, \frac{\partial^{n-1}G(0, 0, 0)}{\partial x^\lambda \partial y^\mu \partial z^\nu} = 0\,, \qquad \lambda + \mu + \nu = m - 1, \nu \leqq 1$$

$$\frac{\partial^m G(0, 0, 0)}{\partial x^m} = 0, \frac{\partial^m G(0, 0, 0)}{\partial x^{m-1}\partial y} = 0, \cdots ; \frac{\partial^m G(0, 0, 0)}{\partial x^r \partial y^s \partial z^t} = 1\,,$$

where there are n conditions in the last row and if we order them according to decreasing order of derivatives in x, y, z then the last (i.e. the nth) is that in which we differentiate $\{m - [(n-1)/2]\}$ times in x, $\{m - [m - (n-1)/2] - (1 - (-1)^n)/2\}$ times in y and $(1 - (-1)^n)/2$ times in z, which is easily verified by induction.

This variational problem can be solved, mutatis mutandis, by the method used by Bergman for the corresponding problem for analytic functions of one complex variable [2. p. 10].

Suppose now,

$$\sum |c_\nu| < \infty\,.$$

Then it follows from the theorem of Riesz-Fischer for the system $\{\phi_{m,n}\}$ that

$$G(X) = T\{s(g_{m,n}(X)\}$$

and the right hand series converges uniformly and absolutely in D, providing the analytic continuation of G into D.

2.2. In this section we shall introduce a special set $\psi_{\kappa,\lambda}$ of solutions of (1) which has many properties similar to those of the solutions $g_{\kappa,\lambda}(X)$ of $\Delta u = 0$. Some of these properties are illustrated by the following.

THEOREM. *There exist a set $\psi_{\kappa,\lambda}(X)$ of solutions of* (1) *such that for every solution ψ regular in a simply connected domain D containing the origin, with harmonic associate G*

1. *the representation*

$$\psi(X) = \sum_{\kappa=1}^{\infty} \sum_{\lambda=1}^{2\kappa+1} b_{\kappa,\lambda}\, \psi_{\kappa,\lambda}(X) \tag{13}$$

holds in every sphere $\mathscr{S} \subset D$;

2. *the representation*

$$\psi(X) = T\{s(\psi_{\kappa,\lambda}(X))\} \tag{14}$$

holds in the domain D.

Proof 1. We use the representation (10) for the harmonic function $G(X)$ holding in the sphere $\mathscr{S} \subset D$, and we take for $\{\psi_{\kappa,\lambda}\}$ the sequence $\boldsymbol{P}_3[g_{\kappa,\lambda}(X)]$, $\kappa = 0, 1, \cdots, \lambda = 1, 2, \cdots, 2\kappa + 1$. Since $g_{\kappa,\lambda}(X)$ are homogeneous polynomials of the degree κ, we get from (3)

$$\psi_{\kappa,\lambda} = \exp\left(-\frac{1}{2}\int_0^r rA\, dr\right) g_{\kappa,\lambda}(X)\left\{1 + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_n(r)\, B\left(\kappa + \frac{3}{2}, n\right)\right\} \tag{15}$$

If we put in (3) the expression (10) for $G(X)$ and interchange the order of summation and integration (which is valid since the right hand series in (10) converges uniformly) we get

$$\psi(X) = \exp\left(-\frac{1}{2}\int_0^r rA\, dr\right)$$
$$\cdot \left\{G(X) + \sum_{n=1}^{\infty}\sum_{\kappa=1}^{\infty}\sum_{\lambda=1}^{2\kappa+1} \alpha_n(r)\, B\left(\kappa + \frac{3}{2}, n\right) g_{\kappa,\lambda}(X) b_{\kappa,\lambda}\right\}.$$

By interchanging the order of two summation in the above formula and using (15) we get the representation (13). That this procedure is valid one can easily conclude from the fact that the series (10) converges uniformly and absolutely and that $B(\kappa + (3/2), n) < 1/n$ for all κ.

2. In order to prove 2 we proceed as above in 1 applying (3) to

$$G(X) = T\{s(g_{\kappa,\lambda}(X))\} .$$

Thus we get by the same reasoning as above

$$\psi(X) = \exp\left(-\frac{1}{2}\int_0^r rA\,dr\right)\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{2m+1} c_{m,n}\phi_{m,n}(X) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\nu=1}^{2m+1} a_n(r)c_{m,\nu}\tilde{\phi}^{(n)}_{m,\nu}(X)\right\} \tag{16}$$

where we put

$$\tilde{\phi}^{(n)}_{m,\nu}(X) = \sum_{\mu=\nu}^{2m+1} a^{(m,\nu)}_{m,\mu} g_{m,\mu}(X) B\left(m + \frac{3}{2}, n\right) + \sum_{\lambda=m+1}^{\infty}\sum_{\mu=1}^{2\lambda+1} a^{(m,\nu)}_{\lambda,\mu} g_{\lambda,\mu}(X) B\left(\lambda + \frac{3}{2}, n\right) .$$

By interchanging the order of two summation in (16) arguing as in 1 we get the representation (14). Since the T-transform of (10) provides analytic continuation of $G(X)$ into its whole domain of regularity, the formula (14) is valid in the whole domain of regularity of ψ according to the lemma.

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University of Novi Sad (Yugoslavia) and
Stanford University