EXTREME EIGEN VALUES OF TOEPLITZ FORMS ASSOCIATED WITH JACOBI POLYNOMIALS

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Introduction. Let $t(\theta)$ be a real function in $L^1(T)$ where T is the real numbers modulo 1, and let

$$c(k) = \int_T t(\theta) e^{-2\pi i k\theta} d\theta \qquad k = 0, 1, \cdots ,$$

$$C_n = [c(j-k)]_{j,k=0,\cdots,n} .$$

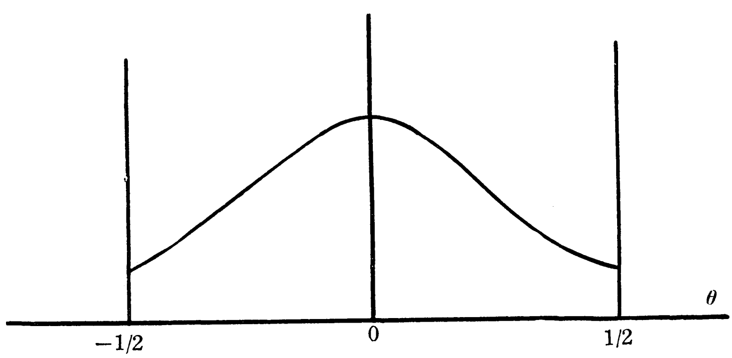
C_n is the Toeplitz matrix of index n associated with $t(\theta)$. C_n is clearly Hermitian and thus has real eigen values,

$$\lambda_{n,1} \geqq \lambda_{n,2} \geqq \cdots \geqq \lambda_{n,n+1} .$$

For some time studies have been made of the asymptotic behaviour of these eigen values as $n \to \infty$. Thus, for example, if $N(a, b; n)$ is, for n fixed, the number of $\lambda_{n,k}$'s which satisfy $a \leqq \lambda_{n,k} \leqq b$, and if $\nu(y)$ is the Lebesgue measure of the set $\{\theta \mid t(\theta) < y\}$ then

$$(1) \qquad \lim_{n\to\infty} n^{-1} N(a, b; n) = \nu(a) - \nu(b) ,$$

provided a and b are points of continuity of ν. This result was proved by Szegö, see [2; p. 64]. Detailed investigations have also been made of the behaviour of $\lambda_{n,k}$ as $n \to \infty$ while k is fixed, under various additional assumptions on $t(\theta)$. Suppose that $t(\theta)$ is continuous for $\theta \in T$, has a unique absolute maximum at $\theta = 0$, and that $t(\theta)$ is twice continuously differentiable in a neighborhood of $\theta = 0$ with $t''(0) < 0$.



It was shown in 1953 by Kac, Murdock, and Szegö that under these assumptions

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$$\lambda_{n,k} = t(0) - \frac{t''(0)}{8}k^2(n+1)^{-2} + o(n^{-2}) \tag{2}$$

as $n \to \infty$ for k fixed, $k = 1, 2, \cdots$. In 1958 Widom, [14], proved that if $t(\theta)$ is even and four times continously differentiable near $\theta = 0$ (in addition to the assumptions already made) then

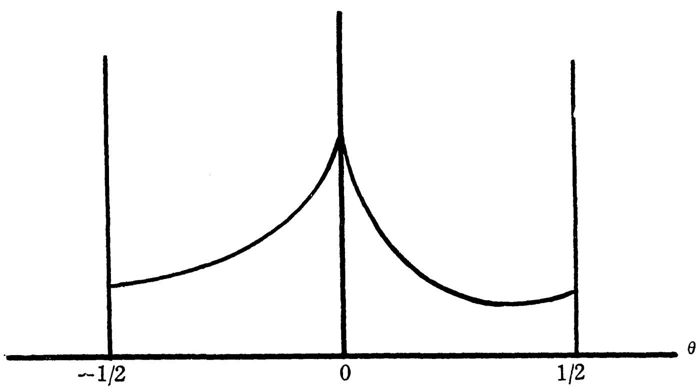
$$\lambda_{n,k} = t(0) - \frac{t''(0)}{8}k^2(n+1)^{-2}[1 + \alpha(n+1)^{-1}] + o(n^{-3}) \tag{3}$$

as $n \to \infty$, where

$$\alpha = \int_{-1/2}^{1/2} [csc^2 \pi\theta] \, log \left[2\pi^2 \left(\frac{t(0) - t(\theta)}{t''(0)} \right) ctn^2 \pi\theta \right] d\theta \,.$$

More recently Widom and Parter, see [9]–[11] and [15]–[17], have studied the behaviour of $\lambda_{n,k}$ under less restrictive assumptions on the nature of the maximum of $t(\theta)$. Suppose that $t(0)$ is again the unique maximum of $t(\theta)$, and that there exist constants $\sigma_1 > 0$, $\sigma_2 > 0$, and $\omega > 0$ such that

$$t(\theta) \backsim \begin{cases} t(0) - \sigma_1 |\theta|^\omega & \theta \longrightarrow 0+ \\ t(0) - \sigma_2 |\theta|^\omega & \theta \longrightarrow 0- \,. \end{cases}$$



Then

$$\lambda_{n,k} = t(0) - \mu_k n^{-\omega} + o(n^{-\omega}) \tag{4}$$

where $0 < \mu_1 \leqq \mu_2 \leqq \cdots$, $\lim_{k\to\infty} \mu_k = \infty$ are eigen values of a certain operator depending only on σ_1, σ_2, and ω. The formula (4) evidently includes (2) as a very special case.

Let $\alpha, \beta > 1$ be fixed and let

$$2^n n! P_n^{(\alpha\ \beta)}(x) = (-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}] \,,$$

where $D = d/dx$, be the Jacobi polynomial of order n, $n = 0, 1, 2, \cdots$. The Jacobi polynomials are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$w_{\alpha,\beta}(x) = w(x) = (1 - x)^{\alpha}(1 + x)^{\beta} ;$$

more precisely

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w(x) dx = \delta(n, m) h_n$$

where $\delta(n, m)$ is the Kronecker delta and where

$$(2n + \alpha + \beta + 1) n! \Gamma(n + \alpha + \beta + 1) h_n \\ = 2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1) .$$

Let $t(x)$ be a real function in $L^1(w)$ and let

$$c(j, k) = (h_j h_k)^{-1/2} \int_{-1}^{1} P_j^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) t(x) w(x) dx$$

for $j, k = 0, 1, \cdots$. If

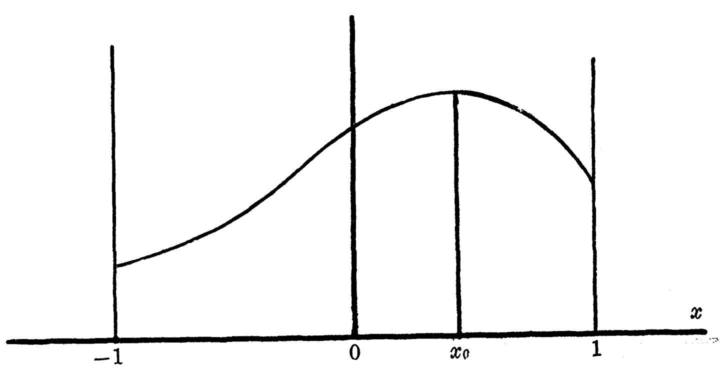
$$C_n = [c(j, k)] \qquad j, k = 0, \cdots, n$$

then C_n is a generalized Toeplitz matrix of index n associated with $t(x)$. Since C_n is real and symmetric its eigen values $\{\lambda_{n,k}\}_1^{n+1}$ are real. In part the studies carried out for ordinary Toeplitz matrices have also been carried out for various classes of generalized Toeplitz matrices, and in particular for the generalized Toeplitz matrices constructed using Jacobi polynomials. For example, if we again define $N(a, b; n)$ to be the numbers of $\lambda_{n,k}$'s which satisfy $a < \lambda_{n,k} \leqq b$ and if $\nu(y)$ is π^{-1} times the Lebesgue measure of the set $\{\theta \mid 0 \leqq \theta \leqq \pi, t(\cos\theta) < y\}$, then

$$\lim_{n\to\infty} \frac{1}{n} N(a, b; n) = \nu(a) - \nu(b) \tag{5}$$

whenever a and b are points of continuity of $\nu(y)$. See [2; p. 114].

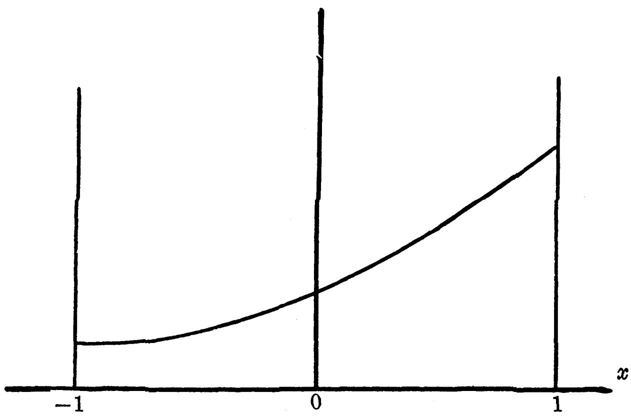
In [5] the author obtained formulas analgous to (2) and (3) but applying to generalized Toeplitz matrices constructed using the various classical orthogonal polynomials. Thus, for example, Let $t(x)$ be defined



and continuous for $-1 \leqq x \leqq 1$, and have a unique absolute maximum at x_0, $-1 < x_0 < 1$. Let $t(x)$ be continuously differentiable in a neighborhood of x_0 and let $t''(x_0) = -\sigma^2 < 0$. If C_n is the generalized Toeplitz matrix constructed from $t(x)$ using the Jacobi polynomials, then

$$(6) \qquad \lambda_{n,k} = t(x_0) - (1 - x_0^2)\sigma^2 k^2/8n^2 + o(n^{-2}) \qquad \text{as } n \to \infty .$$

Let $t(x)$ have a unique absolute maximum at $x = 1$, let $t(x)$ be continuously differentiable in a neighborhood of $x = 1$, and let $t'(1) = \sigma > 0$.

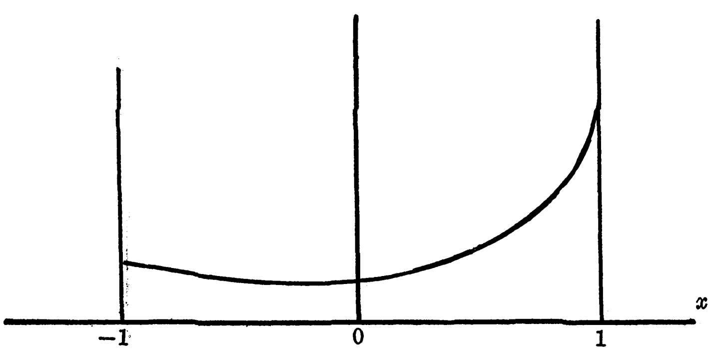


Then

$$(7) \qquad \lambda_{n,k} = t(1) - \frac{\sigma}{2}\left(\frac{z_{\alpha,k}}{n}\right)^2 + o(n^{-2})$$

where $0 < z_{\alpha,1} < z_{\alpha,2} < \cdots$, are the positive zeros of $J_\alpha(z)$. See [5], where a more precise result analagous to (3) is also given.

In the present paper we will obtain formulas analogous to (4) for generalized Toeplitz matrices constructed using Jacobi polynomials. For example let $t(x)$ be continuous for $-1 \leqq x \leqq 1$, let the unique absolute maximum be at $x = 1$, and let



$$t(x) \backsim t(1) - \sigma(1-x)^{\omega} \qquad x \to 1- ,$$

where σ and ω are positive. We then have

$$(8) \qquad \lambda_{n,k} = t(1) - \mu_k(k/n)^{2\omega} + o(n^{-2\omega})$$

as $n \to \infty$ where $0 < \mu_1 \leqq \mu_2 \leqq \cdots$, $\lim_{k\to\infty} \mu_k = \infty$, are the eigen values of an operator depending only upon σ and ω, and α but not otherwise upon $t(x)$ nor upon β. The case of a unique absolute maximum in the interior of $-1 \leqq x \leqq 1$ is also considered.

The program of demonstration of our results runs parallel to that employed in [17]. Sections 2–7 are devoted to developing an appropriate perturbation theory in Hilbert space. This theory is a rearticulation and partial generalization of the perturbation theory constructed by Widom. In sections 8–14 and 15–19 this theory is applied to the case where the maximum of $t(x)$ occurs at an end point of $-1 \leqq x \leqq 1$, and to the case where the maximum occurs at an interior point, respectively.

A large number of known properties of Jacobi polynomials, Jacobi functions of the second kind, Bessel functions, etc. are required in the course of this paper. Many of these results are collected in the Appendix.

2. A perturbation problem. Let $\boldsymbol{H}$ be a Hilbert space with elements f, g, h, etc. The inner product and norm in $\boldsymbol{H}$ are denoted by $(\,|\,)$ and $\|\ \|$. Let S and S_n be unbounded self-adjoint operators in $\boldsymbol{H}$ with spectral resolutions.

$$S = \int \lambda d\Phi(\lambda) ,$$

$$S_n = \int \lambda d\Phi_n(\lambda) .$$

If S is the closure of the strong limit of the S_n's as $n \to \infty$ then Rellich's theorem asserts that in the strong operator topology

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda)$$

for every λ, $-\infty < \lambda < \infty$, not in the point spectrum of S. See [13, p. 56].

Let F and F_n be bounded not necessarily self-adjoint transformations of $\boldsymbol{H}$, such that F is the strong limit of the F_n's as $n \to \infty$. In order to fix our attention suppose that the S_n's are bounded, but not necessarily S. Then for each n $S_{n,F} = F_n^* S_n F_n$ is a bounded self-adjoint transformation. Let its spectral resolution be

(1) $$S_{n,F} = \int \lambda d\Psi_n(\lambda) .$$

Formally let $S_F = F^*SF$,

(2) $$S_F = \int \lambda d\Psi(\lambda) .$$

The problem we wish to study is that of passing from the convergence of the S_n's to S and the F_n's to F to the convergence of the $\Psi_n(\lambda)$'s to $\Psi(\lambda)$. However there are several difficulties. First F^*SF is not in general self-adjoint or even densely defined. Secondly the $S_{n,F}$'s may not suitably converge to S_F. In §§ 3–6 we will show essentially that if $0 \leqq S_n$ $n = 1, 2, \cdots$, $0 \leqq S$, (that is if the S_n's and S are bounded from below) then these difficulties can be overcome.

Throughout we assume that the Hilbert space $\boldsymbol{H}$ is separable. while this is not at all necessary, it makes possible a simpler and more intuitive language.

3. Definition of S_F. We assume henceforth that:

i. $0 \leqq S$ is a self-adjoint operator on $\boldsymbol{H}$;

ii. F is a bounded operator on $\boldsymbol{H}$.

We define

$$\boldsymbol{S} = \{f \mid Ff \in \boldsymbol{D}(S^{1/2})\} .$$

Here $S^{1/2}$ is the unique positive square root of S and $\boldsymbol{D}(S^{1/2})$ is its domain. We do *not* assume that $\boldsymbol{S}$ is dense in $\boldsymbol{H}$ although this is the most interesting special case. Let $\boldsymbol{M}$ be the closure of $\boldsymbol{S}$ in $\boldsymbol{H}$. $\boldsymbol{M}$ is a closed subspace of $\boldsymbol{H}$ and inherits the structure of a Hilbert space from $\boldsymbol{H}$. Our goal is to construct a self-adjoint transformation S_F on the Hilbert space $\boldsymbol{M}$ with the properties:

(1) $$\boldsymbol{D}(S_F) \subset \boldsymbol{S} ;$$

(2) $$(S_F f \mid g) = (S^{1/2}Ff \mid S^{1/2}Fg)$$

for all $f \in \boldsymbol{D}(S_F)$ and for all $g \in \boldsymbol{S}$. The construction of S_F with these properties has long been known, see for example [13; p. 35], however it is included for the sake of completeness. We will need the following simple and well known fact which we record as a lemma.

LEMMA 3a. *Let A be a self-adjoint transformation on $\boldsymbol{H}$ and let $h_n \in \boldsymbol{D}(A)$ $n = 1, 2, \cdots$. If*

$$h_n \to h \qquad \textit{as} \quad n \to \infty$$

and

$$\| Ah_n \| = O(1) \qquad as\ n \to \infty ,$$

then $h \in \boldsymbol{D}(A)$ *and* $Ah_n \rightharpoonup Ah$.

Here "$\to$" indicates strong convergence and "$\rightharpoonup$" indicates weak convergence in $\boldsymbol{H}$. Lemma 3a is a special case of Lemma 4a which is proved in § 4.

For $f, g \in \boldsymbol{S}$ let us define

$$\begin{aligned} \langle f \mid g \rangle &= (S^{1/2}Ff \mid S^{1/2}Fg) + (f \mid g) , \\ |||f||| &= \langle f \mid f \rangle^{1/2} . \end{aligned} \tag{3}$$

LEMMA 3b. *With the definition of inner product and norm given by* (3) $\boldsymbol{S}$ *is a Hilbert space.*

Proof. It is evident that $\boldsymbol{S}$ is a pre-Hilbert space. We need only verify that $\boldsymbol{S}$ is complete. Suppose $f_n \in \boldsymbol{S}$ $n = 1, 2, \cdots$, $|||f_n - f_m||| \to 0$ as $n, m \to \infty$. Since $\|f_n - f_m\| \leqq |||f_n - f_m|||$ there exists $f \in \boldsymbol{H}$ such that $\|f - f_n\| \to 0$ as $n \to \infty$. Since $\| S^{1/2}F(f_n - f_m) \| \leqq |||f_n - f_m|||$ there exists $g \in \boldsymbol{H}$ such that $\| S^{1/2}Ff_n - g \| \to 0$ as $n \to \infty$. Applying Lemma 3a with $h_n = Ff_n$ and $A = S^{1/2}$ we see (since weak and strong limits coincide when both exist) that $Ff \in D(S^{1/2})$ and that $g = S^{1/2}Ff$. Thus $f \in \boldsymbol{S}$ and

$$|||f - f_n|||^2 = \| S^{1/2}F(f - f_n) \|^2 + \|f - f_n\|^2 \to 0 \qquad \text{as } n \to \infty .$$

LEMMA 3c. *There exists a linear transformation* W *of* $\boldsymbol{M}$ *into* $\boldsymbol{S}$ *such that* $(f|g) = < f| Wg >$ for all $f \in \boldsymbol{S}$, $g \in \boldsymbol{M}$ and:

i. $\| Wf \| \leqq ||| Wf ||| \leqq \|f\|$ *for all* $f \in \boldsymbol{M}$;
ii. $(Wf \mid g) = (f \mid Wg)$ *for all* $f, g \in \boldsymbol{M}$;
iii. $0 < (Wf \mid f)$ *for all* $f \in \boldsymbol{M}$.

Proof. For $g \in \boldsymbol{M}$ fixed $(f \mid g)$ is a linear functional on $\boldsymbol{S}$ and since

$$|(f \mid g)| \leqq \|f\| \|g\| \leqq |||f||| \|g\|$$

$(f \mid g)$ is a bounded linear functional on $\boldsymbol{S}$. Therefore there exists a unique element $g^1 \in \boldsymbol{S}$ such that

$$(f \mid g) = \langle f \mid g^1 \rangle \qquad \text{for all } f \in \boldsymbol{S} .$$

Clearly the mapping $g \to g^1$ defines a linear transformation of $\boldsymbol{M}$ into $\boldsymbol{S}$, $g^1 = Wg$. It is evident that $||| Wg ||| \leqq \| g \|$ so that i. holds. Suppose that $f, g \in \boldsymbol{S}$. Then

$$(Wf \mid g) = \langle Wf \mid Wg \rangle = (f \mid Wg)$$

so that ii. is valid if $f, g \in \boldsymbol{S}$. By continuity it is also valid for $f, g \in \boldsymbol{M}$. Thus W is a self-adjoint transformation on $\boldsymbol{M}$. Since

$$(Wf \mid f) = \langle Wf \mid Wf \rangle > 0 \qquad f \in \boldsymbol{S}$$

and since $\boldsymbol{S}$ is dense in $\boldsymbol{M}$ we have $0 \leqq W$. To show that $0 < W$ we need only verify that $Wf = 0$ is impossible unless $f = 0$. If $Wf = 0$, then

$$(g \mid f) = \langle g \mid Wf \rangle = 0 \qquad \text{for all } g \in \boldsymbol{S},$$

but since $\boldsymbol{S}$ is dense in $\boldsymbol{M}$ this implies that $f = 0$.

THEOREM 3d. *There exists a self-adjoint operator S_F on $\boldsymbol{M}$ satisfying conditions* (1) *and* (2).

Proof. We define

$$S_F = W^{-1} - I.$$

It is evident from this definition that S_F is a self-adjoint operator, and that

$$\boldsymbol{D}(S_F) = \boldsymbol{D}(W^{-1}) = \boldsymbol{R}(W) \subset \boldsymbol{S},$$

where $\boldsymbol{R}(W)$ is the range of W. If $f \in \boldsymbol{D}(S_F)$ and $g \in \boldsymbol{S}$ then

$$\begin{aligned}(S_F f \mid g) &= (W^{-1} f \mid g) - (f \mid g) = \langle f \mid g \rangle - (f \mid g), \\ &= (S^{1/2} F f \mid S^{1/2} F g),\end{aligned}$$

and our proof is complete.

4. The resolvant relation.

Let A be a closed linear operator on $\boldsymbol{M}$. It is *not* assumed that $\boldsymbol{D}(A)$ is dense in $\boldsymbol{M}$. A subset $\boldsymbol{C} \subset \boldsymbol{D}(A)$ is said to be a core for A if the set $\{(f, g) \mid g = Af, f \in \boldsymbol{C}\}$ in $\boldsymbol{H} \times \boldsymbol{H}$ is dense in the set $\{(f, g) \mid g = Af, f \in \boldsymbol{D}(A)\}$. Let A_n and A be closed linear operators in $\boldsymbol{H}$ and let $\boldsymbol{C} = \{f \mid A_n f \to Af \text{ as } n \to \infty\}$. If $\boldsymbol{C} = \boldsymbol{D}(A)$ we say that A is the strong limit of the A_n's; if $\boldsymbol{C}$ is a core for A we say that A is the closure of the strong limit of the A_n's.

LEMMA 4a. *Let A_n and A be self-adjoint operators on $\boldsymbol{H}$ and let A be the closure of the strong limit of the A_n's. Then if*

$$f_n \rightharpoonup f, \qquad \| A_n f_n \| = O(1),$$

we have

$$f \in \boldsymbol{D}(A) \quad \textit{and} \quad A_n f_n \; A \rightharpoonup f.$$

Proof. We denote by $\mathfrak{p}$ the positive integers $\{1, 2, 3, \cdots\}$. A subsequence $\mathfrak{p}_1$ of $\mathfrak{p}$ is then a subset $\{n_1, n_2, n_3 \cdots\}$ of $\mathfrak{p}$ with $1 \leqq n_1 < n_2 < \cdots$. By "$a_n \to a$ as $n \to \infty$ in $\mathfrak{p}_1$" we mean that $\lim_{k\to\infty} a_{n_k} = a$. This notation enables us to dispense with awkward subscripts.

Let $\boldsymbol{C} = \{f \mid A_n f \to Af \text{ as } n \to \infty\}$. By assumption $\boldsymbol{C}$ is a core for A. Since $\| A_n f_n \| = O(1)$ given any subsequence $\mathfrak{p}_1$ of $\mathfrak{p}$ there exists a subsequence $\mathfrak{p}_2$ of $\mathfrak{p}_1$ such that $A_n f_n \rightharpoonup g$ as $n \to \infty$ in $\mathfrak{p}_2$ for some $g \in \boldsymbol{H}$. This is because bounded sets in $\boldsymbol{H}$ are weakly conditionally compact. In particular if $h \in \boldsymbol{C}$ then

$$(A_n f_n \mid h) \to (g \mid h) \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_2 .$$

On the other hand

$$(A_n f_n \mid h) = (f_n \mid A_n h)$$

for all (large) n and thus

$$(A_n f_n \mid h) \to (f \mid Ah) \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_2 ,$$

so that

$$(g \mid h) = (f \mid Ah) .$$

Given $k \in \boldsymbol{D}(A)$ and $\delta > 0$ there exists $h \in \boldsymbol{C}$ such that $\| k - h \| < \delta$, $\| Ak - Ah \| < \delta$. This implies that

$$(g \mid k) = (f \mid Ak) \qquad \text{for all } k \in \boldsymbol{D}(A) .$$

Consequently $f \in \boldsymbol{D}(A^*)$ and $A^* f = g$; but $A^* = A$. Since every subsequence $\mathfrak{p}_1$ contains a subsequence $\mathfrak{p}_2$ such that $A_n f_n \rightharpoonup Af$ as $n \to \infty$ in $\mathfrak{p}_2$ it follows that $A_n f_n \rightharpoonup Af$ as $n \to \infty$ in $\mathfrak{p}$.

In what follows we assume that:

iii. $O \leqq S_n$ is a self-adjoint transformation on $\boldsymbol{H}$ $n = 1, 2, \cdots$; F_n is a bounded transformation on $\boldsymbol{H}$ $n = 1, 2, \cdots$; $\boldsymbol{R}(F_n) \subset \boldsymbol{D}(S_n)$ $n = 1, 2, \cdots$. Assumption iii. implies that $S_{n,F} = F_n^* S_n F_n$ is a bounded operator on $\boldsymbol{H}$ for $n = 1, 2, \cdots$. We further assume that:

iv. F is a bounded operator on $\boldsymbol{H}$ and F is the strong limit of F_n as $n \to \infty$;

v. $S^{1/2}$ is the closure of the strong limit of $S_n^{1/2}$ as $n \to \infty$;

vi. $S^{1/2}F$ is the closure of the strong limit of $S_n^{1/2}F_n$ as $n \to \infty$.

We set

$$\boldsymbol{S}' = \{f \mid S_n^{1/2} F_n f \to S^{1/2} F f \text{ as } n \to \infty\} .$$

It is evident that $\boldsymbol{S}' \subset \boldsymbol{S}$.

THEOREM 4b. *Under assumptions* i–vi *if there exists* $\delta > 0$ *such that* $\operatorname{dist}\{z, \sigma(S_F)\} \geqq \delta$, $\operatorname{dist}\{z, \sigma(S_{n,F})\} \geqq \delta$, $n = 1, 2, \cdots$, *then for all* $f \in \boldsymbol{M}$

$$\{S_{n,F} - zI\}^{-1}f \rightharpoonup \{S_F - zI\}^{-1}f \qquad as\ n \to \infty\ .$$

Proof. Take $f \in \boldsymbol{M}$. We will show that if $\mathfrak{p}_1$ is an arbitrary subsequence of $\mathfrak{p}$ then $\mathfrak{p}_1$ contains a subsequence $\mathfrak{p}_2$ such that

$$\{S_{n,F} - zI\}^{-1}f \rightharpoonup \{S_F - zI\}^{-1}f \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_2\ .$$

This will prove our result. Because dist $\{z, \sigma(S_{n,F})\} \geqq \delta$ it follows that $\| \{S_{n,F} - zI\}^{-1}f \| = 0(1)$ as $n \to \infty$. Therefore we can find a subsequence $\mathfrak{p}_2$ of $\mathfrak{p}_1$ such that if $g_n = \{S_{n,F} - zI\}^{-1}f$ then $g_n \rightharpoonup g$ as $n \to \infty$ in $\mathfrak{p}_2$ for some $g \in \boldsymbol{H}$. We must show that $g = \{S_F - zI\}^{-1}f$. Since F is the strong limit of F_n we have

$$F_n g_n \rightharpoonup Fg \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_2\ ,$$

and since $S_{n,F}g_n = f + zg_n$ we have

$$\begin{aligned}(S_n^{1/2}F_n g_n \mid S_n^{1/2}F_n g_n) &= (S_{n,F}\, g_n \mid g_n) \\ &= (f + zg_n \mid g_n) = O(1)\ .\end{aligned}$$

Therefore by Lemma 4a $Fg \in \boldsymbol{D}(S^{1/2})$ and $S_n^{1/2}F_n g_n \rightharpoonup S^{1/2}Fg$ as $n \to \infty$ in $\mathfrak{p}_2$. In particular $g \in \boldsymbol{S}$. Take $h \in \boldsymbol{S}'$; then by the above

$$\begin{aligned}\lim_{\mathfrak{p}_2} (S_{n,F}g_n \mid h) &= \lim_{\mathfrak{p}_2} (S_n^{1/2}F_n g_n \mid S_n^{1/2}F_n h) \\ &= (S^{1/2}Fg \mid S^{1/2}Fh)\ .\end{aligned}$$

On the other hand

$$\lim_{\mathfrak{p}_2} (S_{n,F}g_n \mid h) = \lim_{\mathfrak{p}_2} (f + zg_n \mid h) = (f + zg \mid h)\ .$$

Thus

$$(1) \qquad (S^{1/2}Fg \mid S^{1/2}Fh) = (f + zg \mid h)$$

for all $h \in \boldsymbol{S}'$. Since $\boldsymbol{S}'$ is by assumption a core for $S^{1/2}F$ (1) holds for all $h \in \boldsymbol{S}$, and thus for all $h \in \boldsymbol{D}(S_F)$. For such an h we have

$$(S^{1/2}Fg \mid S^{1/2}Fh) = (g \mid S_F h)$$

by Theorem 3d. Consequently we have shown that

$$(g \mid S_F h) = (f + zg \mid h)$$

or equivalently

$$(g \mid \{S_F - z^*I\}h) = (f \mid h)$$

for all $h \in \boldsymbol{D}(S_F)$. This implies that

$$\{S_F - z^*I\}^*g = f$$

and hence that

$$g = \{S_F - zI\}^{-1}f\ .$$

5. Spectral resolutions. Let

$$S_{n,F} = \int_{0-}^{\infty} \lambda d\Psi_n(\lambda)$$

be the spectral resolution of $S_{n,F}$ on $\boldsymbol{H}$ and

$$S_F = \int_{0-}^{\infty} \lambda d\Psi(\lambda)$$

be the spectral resolution of S_F on $\boldsymbol{M}$. We assume throughout that $\Psi_n(\lambda) = \Psi_n(\lambda+)$, $0 \leqq \lambda < \infty$, $n = 1, 2, \cdots$, that $\Psi_n(0-) = 0$, and similarly for $\Psi(\lambda)$.

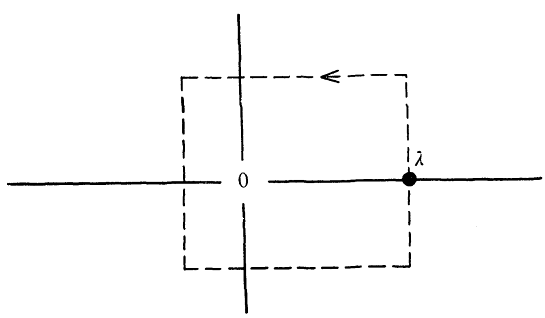
THEOREM 5a. *If $\lambda \geqq 0$ is not in the point spectrum of S_F and if $f \in \boldsymbol{M}$ then*

$$\Psi_n(\lambda)f \rightharpoonup \Psi(\lambda)f \qquad \text{as } n \to \infty\ .$$

Proof. Fix $f \in \boldsymbol{M}$, and let $h = \Psi(\lambda)f$, $h_n = \Psi_n(\lambda)f$. It is enough to show that if $\mathfrak{p}_1$ is any sequence such that $h_n \rightharpoonup h'$ as $n \to \infty$ in $\mathfrak{p}_1$, then $h' = h$. In order to identify h' we proceed as follows. We assert that if $g \in \boldsymbol{H}$ then

$$(1) \qquad \lim_{\mathfrak{p}} (2\pi i)^{-1}\int_C (\{S_{n,F} - zI\}^{-1}f \mid g)(\lambda - z)dz = (2\pi i)^{-1}\int_C (\{S_F - zI\}^{-1}f \mid g)(\lambda - z)dz\ ,$$

where C is the curve pictured below.



Indeed by Theorem 4b

$$\lim_{\mathfrak{p}} (\{S_{n,F} - zI\}^{-1}f \mid g) = (\{S_F - zI\}^{-1}f \mid g)$$

for all z on C except $z = \lambda$. Moreover starting from the inequality $\| (A - zI)^{-1} \| \leqq 1/\text{dist}\{z, \sigma(A)\}$ one can easily show that for z on C and some constant $k(C)$

$$|(\{S_{n,F} - zI\}^{-1}f \mid g)| \leqq k(C)|\lambda - z|^{-1}\| f \| \| g \| .$$

Applying the Lebesgue limit theorem we obtain (1).

A simple computation gives

$$(2) \qquad (2\pi i)^{-1}\int_C \frac{\lambda - z}{\mu - z} dz = \begin{cases} \mu - \lambda & \text{if } 0 \leqq \mu \leqq \lambda \\ 0 & \text{if } \mu > \lambda . \end{cases}$$

We have

$$\begin{aligned}(2\pi i)^{-1}&\int_C (\{S_F - zI\}^{-1}f \mid g)(\lambda - z)dz \\ &= (2\pi i)^{-1}\int_C (\lambda - z)dz\int_{0-}^{\infty} (\mu - z)^{-1}d_\mu(\Psi(\mu)f \mid g) .\end{aligned}$$

This iterated integral is absolutely convergent and therefore using Fubini's theorem and (2) we obtain

$$\begin{aligned}(3) \qquad (2\pi i)^{-1}\int_C (\{S_F - zI\}^{-1}f \mid g)(\lambda - z)dz &= \int_{0-}^{\lambda} (\mu - \lambda)d_\mu(\Psi(\mu)f \mid g) , \\ &= (\{S_F - \lambda I\}\Psi(\lambda)f \mid g) , \\ &= (\{S_F - \lambda I\}h \mid g) .\end{aligned}$$

Similarly

$$\begin{aligned}(4) \qquad (2\pi i)^{-1}\int_C (\{S_{n,F} - zI\}^{-1}f \mid g)(\lambda - z)dz &= (\{S_{n,F} - \lambda I\}\Psi_n(\lambda)f \mid g) , \\ &= (\{S_{n,F} - \lambda I\}h_n \mid g) .\end{aligned}$$

Using (1), (3), and (4) we see that

$$(5) \qquad (\{S_{n,F} - \lambda I\}h_n \mid g) \to (\{S_F - \lambda I\}h \mid g) \qquad \text{as } n \to \infty \text{ in } \mathfrak{p} .$$

Since $h_n = \Psi_n(\lambda)f$ it follows that

$$\begin{aligned}(S_n^{1/2}F_nh_n \mid S_n^{1/2}F_nh_n) &= (S_{n,F}h_n \mid h_n) \\ &= (S_{n,F}\Psi_n(\lambda)f \mid \Psi_n(\lambda)f) \\ &= \int_{0-}^{\lambda} \mu d_\mu\| \Psi_n(\mu)f \|^2 \\ &\leqq \lambda\| f \|^2 .\end{aligned}$$

We also have, since F is the strong limit of F_n, that

$$F_nh_n \rightharpoonup Fh' \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_1 .$$

Applying Lemma 4a we find that $h' \in S$ and that $S_n^{1/2}F_nh_n \rightharpoonup S^{1/2}Fh'$

as $n \to \infty$ in $\mathfrak{p}_1$. Suppose that $g \in \boldsymbol{S}$; then

$$(S_{n,F}h_n \mid g) = (S_n^{1/2}F_nh_n \mid S_n^{1/2}F_ng)$$

and thus

$$(6) \qquad (S_{n,F}h_n \mid g) \to (S^{1/2}Fh' \mid S^{1/2}Fg) \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_1 ;$$

also

$$(7) \qquad (S_Fh \mid g) = (S^{1/2}Fh \mid S^{1/2}Fg) .$$

Inserting (6) and (7) in (5) we find that

$$(8) \qquad (S^{1/2}Fh' \mid S^{1/2}Fg) - \lambda(h' \mid g) = (S^{1/2}Fh \mid S^{1/2}Fg) - \lambda(h \mid g)$$

for all $g \in \boldsymbol{S}'$. Using assumption vi. we see that (8) holds for all $g \in \boldsymbol{S}$ and therefore in particular for all $g \in \boldsymbol{D}(S_F)$. Appealing to Theorem 3d we obtain

$$(h' - h \mid S_Fg) = \lambda(h' - h \mid g)$$

for all $g \in \boldsymbol{D}(S_F)$. Since $h' - h \in \boldsymbol{S} \subset \boldsymbol{M}$ this implies that

$$S_F(h' - h) = \lambda(h' - h) .$$

However by assumption λ is *not* in the point spectrum of S_F so that $h' - h = 0$ and our proof is complete.

6. The perturbation theorem. In this section and also in § 7 we make the following convention. Suppose that $\boldsymbol{P}$ is a subspace of $\boldsymbol{H}$. If E is a projection of $\boldsymbol{P}$ onto a subspace $\boldsymbol{Q}$ of $\boldsymbol{P}$ then E may also be regarded as projection of $\boldsymbol{H}$, namely the projection of $\boldsymbol{H}$ onto $\boldsymbol{Q}$.

THEOREM 6a. *Under assumptions* i–vi *we have for every* $f \in \boldsymbol{H}$

$$\Psi_n(\lambda)f \to \Psi(\lambda)f \qquad \textit{as } n \to \infty ,$$

for every λ *not in the point spectrum of* S_F.

Proof. It follows from Theorem 5a that

$$(1) \qquad \Psi_n(\lambda)f \rightharpoonup \Psi(\lambda)f \qquad \text{as } n \to \infty ,$$

for all $f \in \boldsymbol{M}$. Suppose next that $g \perp \boldsymbol{M}$. Since $\| \Psi_n(\lambda)g \| = 0(1)$, given any subsequence $\mathfrak{p}_1$ there is a subsequence $\mathfrak{p}_2$ of $\mathfrak{p}_1$ such that

$$\Psi_n(\lambda)g \rightharpoonup h \qquad \text{as } n \to \infty \text{ in } \mathfrak{p}_2$$

for some $h \in \boldsymbol{H}$. If $f \in \boldsymbol{M}$ then

$$\lim_{\mathfrak{p}_2} (\Psi_n(\lambda)g \mid f) = (h \mid f) .$$

Since $(\Psi_n(\lambda)f \mid g) = (f \mid \Psi_n(\lambda)g)$ we have using (1)

$$\lim_{\mathfrak{p}_2} (\Psi_n(\lambda)g \mid f) = (g \mid \Psi(\lambda)f) = 0 .$$

Thus $(h \mid f) = 0$; i.e. $h \perp \boldsymbol{M}$. Now

$$\begin{aligned} \| S_n^{1/2} F_n \Psi_n(\lambda) g \|^2 &= (S_{n,F} \Psi_n(\lambda) g \mid g), \\ &= \int_{0-}^{\lambda} \mu d_\mu \| \Psi_n(\mu) g \|^2 , \\ &\leqq \lambda \| g \|^2 . \end{aligned}$$

Therefore by Lemma 4a $Fh \in \boldsymbol{D}(S^{1/2})$; that is, $h \in \boldsymbol{S} \subset \boldsymbol{M}$. But $h \perp \boldsymbol{M}$ so that $h = 0$. We have thus shown that

$$(2) \qquad \Psi_n(\lambda) g \rightharpoonup 0 \qquad \text{as } n \to \infty \text{ if } g \perp \boldsymbol{M}$$

The relations (1) and (2) together prove that

$$(3) \qquad \Psi_n(\lambda) f \rightharpoonup \Psi(\lambda) f \qquad \text{as } n \to \infty ,$$

for all $f \in \boldsymbol{H}$. Since weak convergence of projections implies strong convergence our proof is complete.

7. Convergence in dimension. In this section we will show how starting from the *conclusion* of Theorem 6a and one further assumption it is possible to prove that the dimensions of the spectral projections converge. Suppose that $0 \leqq R_n$ $n = 1, 2, \cdots$ are bounded self-adjoint operators defined on subspaces $\boldsymbol{N}_n$ of a Hilbert space $\boldsymbol{H}$. Let $0 \leqq R$ be a self-adjoint operator on a subspace $\boldsymbol{N}$ of $\boldsymbol{H}$. Let

$$\begin{aligned} R_n &= \int_{0-}^{\infty} \lambda dE_n(\lambda) , \\ R &= \int_{0-}^{\infty} \lambda dE(\lambda) , \end{aligned}$$

be the spectral resolutions of R_n on $\boldsymbol{N}_n$ and of R on $\boldsymbol{N}$. We list two conditions.

a. $E_n(\lambda) \to E(\lambda)$ as $n \to \infty$ for all $\lambda > 0$, $\lambda \notin \sigma_p(R)$, the point spectrum of R. Here "$\to$" is in $\boldsymbol{H}$.

b. there is a number $m > 0$ such that if $f_n \in \boldsymbol{N}_n$, $\| f_n \| = 1$, and $(R_n f_n \mid f_n) \leqq m_1 < m$ for $n \in \mathfrak{p}_1$, then $\mathfrak{p}_1$ contains a subsequence $\mathfrak{p}_2$ such that $f_n \rightharpoonup f \neq 0$ as $n \to \infty$ in $\mathfrak{p}_2$. Here "$\rightharpoonup$" is in $\boldsymbol{H}$.

THEOREM 7a. *Under assumptions* a. *and* b. *we have*

$$(1) \qquad \dim E(\lambda) < \infty \qquad 0 \leqq \lambda < m ,$$

and

$$\lim_{n\to\infty} \dim E_n(\lambda) = \dim E(\lambda) \tag{2}$$

for $0 \leqq \lambda < m$, $\lambda \notin \sigma_p(R)$.

Proof. We first note that assumption a. alone implies that if $0 \leqq \lambda < \infty$, $\lambda \notin \sigma_p(R)$, then

$$\lim_{n\to\infty} \dim E_n(\lambda) \geqq \dim E(\lambda) \, . \tag{3}$$

In (3) we admit "$\infty \geqq \infty$". Suppose $\dim E(\lambda) \geqq k$. Then there exist orthonormal vectors $g_1, g_2, \cdots, g_k$ in $E(\lambda)\boldsymbol{H}$. By assumption a. we have

$$\lim_{n\to\infty} E_n(\lambda) g_j = E(\lambda) g_j = g_j \qquad j = 1, \cdots, k \, ,$$

from which it follows that for all sufficiently large n $\{E_n(\lambda)g_j\}_1^k$, which belong to $E_n(\lambda)\boldsymbol{N}_n$, are linearly independant.

From this point on we use assumptions a. *and* b. We suppose that $\lambda \notin \sigma_p(R)$ and that $0 \leqq \lambda < m$. If $\dim E(\lambda) = \infty$ then we can find an infinite orthonormal set of vectors $\{g_j\}_1^\infty$ in $E(\lambda)\boldsymbol{H}$. Using a. we see that there exists a subsequence $\mathfrak{p}_1 = \{0 < n_1 < n_2 < \cdots\}$ such that

$$\| E_{n_k}(\lambda) g_k - g_k \| \to 0 \qquad \text{as } k \to \infty \, .$$

If we set $f_{n_k} = E_{n_k}(\lambda)g_k / \| E_{n_k}(\lambda) g_k \|$ then f_n is defined for $n \in \mathfrak{p}_1$. We have $f_n \in \boldsymbol{N}_n$, $\| f_n \| = 1$, and $(R_n f_n \mid f_n) \leqq \lambda$ for $n \in \mathfrak{p}_1$. Therefore by b. there is a subsequence $\mathfrak{p}_2$ of $\mathfrak{p}_1$, such that $f_n \rightharpoonup f$ as $n \to \infty$ in $\mathfrak{p}_2$, and $f \neq 0$. But then $g_n \rightharpoonup f \neq 0$ as $n \to \infty$ in $\mathfrak{p}_2$. However it is obvious that $g_n \rightharpoonup 0$ as $n \to \infty$. Thus $\dim E(\lambda) = \infty$ leads to a contradiction and (1) is true.

We assert that (2) is true. Set $k = \dim E(\lambda)$. If (2) is not true then in view of (3) there is a subsequence $\mathfrak{p}_1$ such that $\dim E_n(\lambda) > k$ for $n \in \mathfrak{p}_1$. Let $g_1, \cdots, g_k$ be an orthonormal basis for $E(\lambda)\boldsymbol{H}$. For each $n \in \mathfrak{p}_1$ we can choose $f_n \in E_n(\lambda)\boldsymbol{N}_n$ such that $\| f_n \| = 1$, $f_n \perp g_1, \cdots, g_k$. We have $(R_n f_n \mid f_n) \leqq \lambda$ and therefore by b. there is a subsequence $\mathfrak{p}_2$ of $\mathfrak{p}_1$ such that $f_n \rightharpoonup f \neq 0$ as $n \to \infty$ in $\mathfrak{p}_2$. Now $f_n = E_n(\lambda) f_n$ and by a. $E_n(\lambda) f_n \rightharpoonup E(\lambda) f$ as $n \to \infty$ in $\mathfrak{p}_2$. Therefore $f = E(\lambda) f$ and $f \in E(\lambda) H$. Since $f \perp g_1, \cdots, g_k$ f must be 0. This is a contradiction and our assertion follows.

8. Maximum at the end point. As we announced in the introduction §§ 8–14 are devoted to the case in which $t(x)$ has a unique absolute maximum at $x = 1$. We assume that $t(x)$ is continuous for $-1 \leqq x \leqq 1$ and that

$$(1) \qquad t(x) < t(1) \qquad -1 \leqq x < 1 .$$

We further assume that

$$(2) \qquad t(1) - t(x) \backsim (1 - x)^{\omega} L(1 - x) \qquad \text{as } x \to 1- .$$

Here $\omega > 0$, and $L(y)$, defined for $0 < y \leqq 2$, is positive, continuous, and slowly oscillating as $y \to 0+$. We recall that $L(y)$ "slowly oscillating" means that for every $\varepsilon > 0$, $L(y)y^{\varepsilon}$ is increasing and $L(y)y^{-\varepsilon}$ is decreasing for $0 < y < a(\varepsilon)$ if $a(\varepsilon)$ is sufficiently small.

In what follows it will be necessary for us to work with four Hilbert spaces. The first Hilbert space is $\boldsymbol{L}$, the elements of which are complex functions $f(k)$ defined for $k = 0, 1, \cdots$, with inner product

$$(f \mid g)_{L} = \sum_{k=0}^{\infty} f(k) g(k)^* .$$

The second Hilbert space is $\boldsymbol{L}^{\wedge}$ the elements of which are complex measurable functions on $-1 \leqq x \leqq 1$ with inner product defined by

$$(f \mid g)_{L^{\wedge}} = \int_{-1}^{1} f(x) g(x)^* w_{\alpha,\beta}(x) dx .$$

Here $w_{\alpha,\beta}(x) = w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, $\alpha > -1$, $\beta > -1$. The mapping ϕ from $\boldsymbol{L}$ to $\boldsymbol{L}^{\wedge}$ defined by

$$\phi f \cdot (x) = \sum_{k=0}^{\infty} f(k) h_k^{-1/2} P_k^{(\alpha,\beta)}(x)$$

(the partial sums of this series converge in the metric of $\boldsymbol{L}^{\wedge}$) and its and its inverse ϕ^{-1} from $\boldsymbol{L}^{\wedge}$ to $\boldsymbol{L}$ defined by

$$\phi^{-1} f \cdot (k) = \int_{-1}^{1} f(x) h_k^{-1/2} P_k^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) dx ,$$

are unitary mappings. Both $\boldsymbol{H}$ and $\boldsymbol{H}^{\wedge}$ have as elements complex measurable functions on $[0, \infty)$ with inner products

$$(f \mid g)_{H^{\wedge}} = \int_{0}^{\infty} f(u) g(u)^* u du ,$$

$$(f \mid g)_{H} = \int_{0}^{\infty} f(z) g(z)^* z dz .$$

The mapping ψ from $\boldsymbol{H}$ to $\boldsymbol{H}^{\wedge}$ defined by

$$\psi f \cdot (u) = \int_{0}^{\infty} f(z) J_{\alpha}(zu) z dz$$

(the partial integrals converge in the metric of $\boldsymbol{H}^{\wedge}$) and its inverse

$$\psi^{-1} f \cdot (z) = \int_{0}^{\infty} f(u) J_{\alpha}(uz) u du$$

are also unitary mappings. See in this connection [1; p. 73] and the references given there.

Let us set

$$\theta_n^{(\alpha,\beta)}(u) = \Bigl(1 - \cos\frac{u}{n}\Bigr)^{\alpha/2}\Bigl(1 + \cos\frac{u}{n}\Bigr)^{\beta/2}\Bigl(\sin\frac{u}{n}\Bigr)^{1/2} u^{-1/2}n^{-1/2}$$

for $0 < u \leqq \pi n$. If $u > n\pi$ then $\theta_n^{(\alpha\,\beta)}(u)$ is defined to be 0. For each $n = 1, 2, \cdots$ we define a mapping from $\boldsymbol{L}^\frown$ to $\boldsymbol{H}^\frown$ by the formula

$$\chi_n f\cdot(u) = f\Bigl(\cos\frac{u}{n}\Bigr)\theta_n^{(\alpha,\beta)}(u) \qquad 0 \leqq u \leqq n\pi .$$

Note that $\chi_n f\cdot(u)$ is 0 for $u > n\pi$. A simple computation shows that the mapping χ_n is isometric and into. We further define

$$\chi_n^* f\cdot(x) = f(n \operatorname{arc}\cos x)(n \operatorname{arc}\cos x)^{1/2}(1 - x^2)^{-1/4}w_{\alpha,\beta}(x)^{-1/2}n^{1/2} .$$

The mapping χ_n^* is a partial isometry of $\boldsymbol{H}^\frown$ onto $\boldsymbol{L}^\frown$. Specifically χ_n^* is isometric on $\chi_n\boldsymbol{L}^\frown$ and zero on $(\chi_n\boldsymbol{L}^\frown)^\perp$, the orthogonal complement of $\chi_n\boldsymbol{L}^\frown$ in $\boldsymbol{H}^\frown$. Note that $\chi_n^*\chi_n = I$ on $\boldsymbol{L}^\frown$ and $\chi_n\chi_n^* = I$ on $\chi_n\boldsymbol{L}^\frown$ and 0 on $(\chi_n\boldsymbol{L}^\frown)^\perp$.

We next introduce various operators on these Hilbert spaces.

a. E_n is defined on $\boldsymbol{L}$ by the formula

$$E_n f\cdot(k) = \begin{cases} f(k) & \text{if } 0 \leqq k \leqq n \\ 0 & \text{if } k > n . \end{cases}$$

The following operators are defined by "transferring" E_n:

$$E_n^\frown \text{ on } \boldsymbol{L}^\frown \text{ defined by } E_n^\frown = \phi E_n \phi^{-1} ;$$
$$F_n^\frown \text{ on } \boldsymbol{H}^\frown \text{ defined by } F_n^\frown = \chi_n E_n^\frown \chi_n^* .$$

b. $T^\frown$ is defined on $\boldsymbol{L}^\frown$ by

$$T^\frown f\cdot(x) = [t(1) - t(x)]f(x) .$$

We set:

$$T \text{ on } \boldsymbol{L} \quad \text{defined by} \quad T = \phi^{-1}T^\frown\phi ;$$
$$T_n^\frown \text{ on } \boldsymbol{H}^\frown \text{ defined by } T_n^\frown = \chi_n T^\frown \chi_n^* ;$$
$$S_n^\frown \text{ on } \boldsymbol{H}^\frown \text{ defined by } S_n^\frown = 2^\omega n^{2\omega} L(n^{-2})^{-1} T_n^\frown .$$

c. $S^\frown$ is defined on $\boldsymbol{H}^\frown$ by

$$S^\frown f\cdot(u) = u^{2\omega} f(u) .$$

d. F is defined on $\boldsymbol{H}$ by

$$Ff\cdot(z) = \begin{cases} f(z) & 0 \leqq z \leqq 1 \\ 0 & z > 1 . \end{cases}$$

We set

$$F^{\wedge} \text{ on } \boldsymbol{H} \text{ defined by } F^{\wedge} = \psi F \psi^{-1} .$$

If $\lambda_{n,1} \geqq \cdots \geqq \lambda_{n,n+1}$ are the eigen values of C_n, see §1, then

$$t(1) - \lambda_{n,1} \leqq \cdots \leqq t(1) - \lambda_{n,n+1}$$

are the eigen values of the following operators:

$$E_n T E_n \Big|_{E_n \boldsymbol{L}} ,$$

$$E_n^{\wedge} T^{\wedge} E_n^{\wedge} \Big|_{E_n^{\wedge} \boldsymbol{L}^{\wedge}} ,$$

$$F_n^{\wedge} T_n^{\wedge} F_n^{\wedge} \Big|_{F_n^{\wedge} \boldsymbol{H}^{\wedge}} ,$$

where these symbols are to be read "$E_n T E_n$ restricted to $E_n \boldsymbol{L}$", etc. The eigen values of

$$F_n^{\wedge} S_n^{\wedge} F_n^{\wedge} \Big|_{F_n^{\wedge} \boldsymbol{H}^{\wedge}}$$

are in increasing order $\{(t(1) - \lambda_{n,k}) 2^{\omega} n^{2\omega} L(n^{-2})^{-1}\}_{k=1}^{n+1}$. In the following sections we will show that $F_n^{\wedge} S_n^{\wedge} F_n^{\wedge}$ "converges" to $S^{\wedge}{}_{F^{\wedge}}$ as $n \rightarrow \infty$, and thus, using the results of sections 2–7, that if

$$0 < \mu_1 \leqq \mu_2 \leqq \cdots , \ \lim_{k \rightarrow \infty} \mu_k = +\infty$$

are the eigenvalues of

$$S^{\wedge}{}_{F^{\wedge}} \Big|_{F^{\wedge} \boldsymbol{H}^{\wedge}}$$

then

$$\lim_{n \rightarrow \infty} (t(1) - \lambda_{n,k}) 2^{\omega} n^{2\omega} L(n^{-2})^{-1} = \mu_k \qquad k = 1, 2, \cdots ,$$

or equivalently

$$\lambda_{n,k} = t(1) - \mu_k 2^{-\omega} n^{-2\omega} L(n^{-2}) + o[n^{-2\omega} L(n^{-2})] .$$

9. Convergence of $(S_n^{\wedge})^{1/2}$ to $(S^{\wedge})^{1/2}$. It follows from §8 that for every $f \in \boldsymbol{H}^{\wedge}$ we have

$$T_n^{\wedge} f \cdot (u) = t_n(u) f(u) \qquad 0 \leqq u < \infty ,$$

where

$$t_n(u) = \begin{cases} t(1) - t\left(\cos \dfrac{u}{n}\right) & 0 \leqq u \leqq n\pi \\ 0 & n\pi < u \, . \end{cases}$$

Consequently

$$S_n^{\wedge} f(u) = s_n(u) f(u)$$

where

$$s_n(u) = 2^{\omega} n^{2\omega} [L(n^{-2})]^{-1} t_n(u) \, .$$

LEMMA 9a. *Under the assumption of* § 8 *we have*

$$(1) \qquad \lim_{n\to\infty} s_n(u) = u^{2\omega} \qquad 0 \leqq u < \infty \, ,$$

and for any $\varepsilon > 0$ *there is a constant* $M(\varepsilon)$ *such that for* $n = 1, 2, \cdots$

$$(2) \qquad 0 \leqq s_n(u) \leqq M(\varepsilon)\{u^{\varepsilon} + u^{-\varepsilon}\} u^{2\omega} \, , \qquad 0 \leqq u < \infty \, .$$

Proof. By assumption

$$(3) \qquad t(1) - t(x) = (1 - x)^{\omega} L(1 - x) \Lambda(x) \qquad \text{as } x \to 1-$$

where $L(y)$, continuous and positive for $0 < y \leqq 2$, is slowly oscillating as $y \to 0+$, and where $\Lambda(1-) = 1$. It is well known and easily verified that this implies that if $0 < y_1$, $0 < y_2$, and $0 < a_1 \leqq y_1/y_2 \leqq a_2$ then

$$(4) \qquad L(y_1)/L(y_2) \to 1 \qquad \text{as } y_1 \text{ and } y_2 \to 0 \, .$$

We have

$$s_n(u) = \left(2n \sin \frac{u}{2n}\right)^{2\omega} \left[L\left(2 \sin^2 \frac{u}{2n}\right) \Big/ L(n^{-2})\right] \Lambda\left(\cos \frac{u}{n}\right)$$

for $0 \leqq u \leqq n\pi$, and (1) is an immediate consequence of this formula.

From the fact that $L(y)$ is slowly oscillating as $y \to 0+$ it is easily verified that for each $\varepsilon > 0$ there is a constant $A(\varepsilon)$ such that if $0 < y_1 \leqq 2$, $0 < y_2 \leqq 2$ then

$$(5) \qquad L(y_1)/L(y_2) \leqq A(\varepsilon)[(y_1/y_2)^{\varepsilon} + (y_1/y_2)^{-\varepsilon}] \, .$$

It follows from (3) that if M is sufficiently large then

$$0 \leqq t(1) - t(x) \leqq M(1 - x)^{\omega} L(1 - x) \qquad -1 \leqq x \leqq 1 \, .$$

Consequently if $0 \leqq u \leqq n\pi$ we have

$$0 \leqq s_n(u) \leqq M\Big(2n \sin \frac{u}{2n}\Big)^{2\omega}\Big[L\Big(2 \sin^2 \frac{u}{2n}\Big)\Big/L(n^{-2})\Big],$$

$$0 \leqq s_n(u) \leqq A(\varepsilon)Mu^{2\omega}\Big[\Big(2n^2 \sin^2 \frac{u}{2n}\Big)^{\varepsilon} + \Big(2n^2 \sin^2 \frac{u}{2n}\Big)^{-\varepsilon}\Big],$$

from which (2) follows.

THEOREM 9b. *$(S^\wedge)^{1/2}$ is the closure of the strong limit of $(S_n^\wedge)^{1/2}$ as $n \to \infty$.*

Proof. Let $f \in \boldsymbol{D}[(S^\wedge)^{1/2}]$ and $\varepsilon > 0$ be given. Let $f_\delta(u) = e^{-\delta u}f(u)$. It is evident that if δ is sufficiently small then

$$\| f - f_\delta \|_{H^\wedge} = \Big\{\int_0^\infty | f(u) |^2 | 1 - e^{-\delta u} |^2 u du\Big\}^{1/2} < \varepsilon ,$$

and

$$\| (S^\wedge)^{1/2}(f - f_\delta) \|_{H^\wedge} = \Big\{\int_0^\infty | f(u) |^2 | 1 - e^{-\delta u} |^2 u^{2\omega+1} du\Big\}^{1/2} < \varepsilon .$$

Moreover using (1) and (2) and the Lebesgue limit theorem it is evident that

$$(S_n^\wedge)^{1/2} f_\delta \to (S^\wedge)^{1/2} f_\delta \qquad \text{in } \boldsymbol{H}^\wedge \text{ as } n \to \infty .$$

10. Convergence of $F_n^\wedge$ to $F^\wedge$.

THEOREM 10a. *If $F_n^\wedge$ and $F^\wedge$ are defined as in § 8 then $F_n^\wedge$ converges strongly to $F^\wedge$ as $n \to \infty$.*

Proof. In order to shorten our formulas let us set

$$R(k, n, u) = h_k^{-1/2} P_k^{(\alpha\ \beta)}\Big(\cos \frac{u}{n}\Big)\Big\{w\Big(\cos \frac{u}{n}\Big)\Big\}^{1/2}\Big\{\sin \frac{u}{n}\Big\}^{1/2} u^{-1/2} .$$

Starting from the definition of $F_n^\wedge$ as $\chi_n E_n^\wedge \chi_n^*$ it is easy to verify that for all $f \in H^\wedge$

$$(1) \qquad F_n^\wedge f \cdot (u) = \frac{1}{n} \sum_{k=0}^{n} R(k, n, u) a(k, n)$$

if $0 \leqq u \leqq n\pi$ and $F_n^\wedge f \cdot (u) = 0$ if $u > n\pi$, where

$$a(k, n) = \int_0^{n\pi} f(\zeta) R(k, n, \zeta) \zeta d\zeta .$$

Let us now assume that $f(u)$ is continuous for $0 \leqq u < \infty$ and vanishes except for $0 < a_1 \leqq u \leqq a_2 < \infty$. We will show that under this assumption $F_n^{\wedge}f\cdot(u) \to F^{\wedge}f\cdot(u)$ uniformly in any subset $0 < b_1 \leqq u \leqq b_2 \leqq \infty$. We first note that

$$(2) \qquad \lim_{k\to\infty} (2^{\alpha+\beta}h_k^{-1})/k = 1 .$$

This follows immediately from the formula of § 1 defining h_k. Formula (5) of the Appendix asserts that

$$(3) \qquad \lim_{k\to\infty} k^{-\alpha}P_k^{(\alpha,\beta)}\left(\cos\frac{z}{k}\right) = (z/2)^{-\alpha}J_\alpha(z)$$

uniformly for z in any compact subset of the complex plane. It is easily deduced from this that there exists a constant M such that

$$(4) \qquad |R(k, n, \zeta)| \leqq M\left(\frac{k+1}{n}\right)^{\alpha+(1/2)}$$

if

$$0 < a_1 \leqq \zeta \leqq a_2 < \infty , \qquad 0 \leqq k \leqq n , \qquad n = 1, 2, \cdots .$$

Let us set

$$\sum_1 (\delta, n, u) = \frac{1}{n} \sum_{0\leqq k<n\delta} R(k, n, u)a(k, n) ,$$

$$\sum_2 (\delta, n, u) = \frac{1}{n} \sum_{n\delta\leqq k\leqq n} R(k, n, u)a(k, n) .$$

Then $F_n^{\wedge}f\cdot(u) = \sum_1 (\delta, n, u) + \sum_2 (\delta, n, u)$. Using (2), the inequality (4), and the corresponding inequality for $R(k, n, u)$ when $0 < b_1 \leqq u \leqq b_2 < \infty$ we find that

$$\left|\sum_1 (\delta, n, u)\right| \leqq Mn^{-2-2\alpha} \sum_{0\leqq k<n\delta} (k+1)^{2\alpha+1} \leqq M\delta^{2\alpha+2}$$

for

$$b_1 \leqq u \leqq b_2 , \qquad 0 < \delta < 1 , \qquad n = 1, 2, \cdots .$$

It follows from (3) that

$$(5) \qquad \lim_{n\to\infty} \left[R(k, n, \zeta) - \left(\frac{k}{n}\right)^{1/2} J_\alpha(k\zeta n^{-1})\right] = 0$$

uniformly for

$$n\delta \leqq k \leqq n , \qquad a_1 \leqq \zeta \leqq a_2 .$$

Consequently

$$\lim_{n\to\infty} [a(k, n) - g(kn^{-1})(k/n)^{1/2}] = 0$$

where

$$g(kn^{-1}) = \int_0^\infty f(\zeta)J_\alpha(k\zeta n^{-1})\zeta d\zeta ,$$

uniformly in k for $n\delta \leqq k \leqq n$. Here of course $g = \psi^{-1}f$. Using (5) again we have

$$\lim_{n\to\infty}\left[R(k, n, u) - \left(\frac{k}{n}\right)^{1/2} J_\alpha(kun^{-1})\right] = 0$$

uniformly for

$$n\delta \leqq k \leqq n \qquad b_1 \leqq u \leqq b_2 .$$

It now follows that

$$(6) \qquad \lim_{n\to\infty}\left|\sum_2(\delta, n, u) - \frac{1}{n}\sum_{n\delta\leqq k\leqq n} g(kn^{-1})J_\alpha(kn^{-1}u)\frac{k}{n}\right| = 0$$

uniformly for $b_1 \leqq u \leqq b_2$. We assert that

$$(7) \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{n\delta\leqq k\leqq n} g(kn^{-1})J_\alpha(kn^{-1}u)\frac{k}{n} = \int_\delta^1 g(z)J_\alpha(uz)zdz$$

uniformly for $b_1 \leqq u \leqq b_2$. Indeed the sum on the left is a Riemann sum for the integral on the right, so that (7) is certainly true for each $u > 0$. To show that it holds uniformly for $b_1 \leqq u \leqq b_2$ it is sufficient to note that

$$\left|\frac{d}{du}\frac{1}{n}\sum_{n\delta\leqq k\leqq n} g(k^{-1}n)J_\alpha(kn^{-1}u)\frac{k}{n}\right| \leqq M$$

uniformly for $b_1 \leqq u \leqq b_2$ $n = 1, 2, \cdots$ so that the sequence of functions on the left in (7) is equicontinuous. Given $\varepsilon > 0$ let us choose $\delta > 0$ so small that

$$\left|\sum_1(\delta, n, u)\right| < \varepsilon/2 , \qquad \left|\int_0^\delta g(z)J_\alpha(uz)zdz\right| < \varepsilon/2 ,$$

for $b_1 \leqq u \leqq b_2$. It then follows on collecting our estimates that $|F_n^\wedge f\cdot(u) - F^\wedge f(u)| < \varepsilon$ for $b_1 \leqq u \leqq b_2$ and for all sufficiently large n.

Let $C^\wedge$ be the set of functions $f \in H^\wedge$ which are continuous and have support in $a_1 \leqq u \leqq a_2$ for some $0 < a_1 < a_2 < \infty$. Using what we have proved above and the fact that $\| F_n^\wedge \| = 1$ $n = 1, 2, \cdots$, we see that if $f \in C^\wedge$ then $F_n^\wedge f \rightharpoonup F^\wedge f$ as $n \to \infty$. Since $C^\wedge$ is dense in $H^\wedge$ we see, again using the fact that $\| F_n^\wedge \| = 1$ $n = 1, 2, \cdots$, that $F_n^\wedge \rightharpoonup F$ as $n \to \infty$. However weak convergence for projections implies strong convergence so that $F_n^\wedge \to F^\wedge$ as $n \to \infty$.

11. Convergence of $(S_n^\wedge)^{1/2}F_n^\wedge$ to $(S^\wedge)^{1/2}F$ Part I. It remains to prove that $(S^\wedge)^{1/2}F^\wedge$ is the closure of the strong limit of $(S_n^\wedge)^{1/2}F_n^\wedge$. The considerations here are considerably more involved than those of §§ 9 and 10 and will occupy §§ 11–13.

Let $\boldsymbol{D}$ be the set of functions $h(z)$ in $\boldsymbol{H}$ which can be written in the form $h(z) = z^\alpha h_1(z)$ where $h_1(z)$ defined for $-\infty < z < \infty$ is even, infinitely differentiable and has compact support. We set $\boldsymbol{D}^\wedge = \psi \boldsymbol{D}$.

LEMMA 11a. *If* $f \in \boldsymbol{D}^\wedge$ *then* $f(u) = u^\alpha f_1(u)$ *where* $f_1(u)$ *is the restriction to* $0 < u < \infty$ *of an even continuous function satisfying* $f_1(u) = 0(u^{-r})$ *as* $u \to +\infty$ *for every* r.

Proof. Suppose that $f \in \boldsymbol{D}^\wedge$ then, with an evident notation,

$$f_1(u) = \int_0^\infty h_1(z)\mathfrak{J}_\alpha(zu)z^{2\alpha+1}dz$$

where $\mathfrak{J}_\alpha(z) = z^{-\alpha}J_\alpha(z)$ is an even continuous function satisfying $|\mathfrak{J}_\alpha(z)| \leqq A(1 + |z|^q)$ for $0 \leqq z \leqq \infty$. Here $q = \max[0, -\alpha - (1/2)]$. If we set

$$\Delta k \cdot (z) = k''(z) + \frac{2\alpha + 1}{z} k'(z) ,$$

then

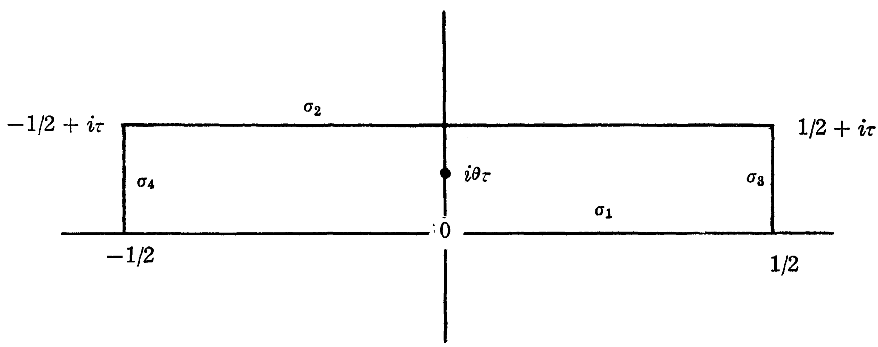
$$\Delta\mathfrak{J}_\alpha(zu) = -u^2\mathfrak{J}_\alpha(zu) .$$

Consequently

$$\begin{aligned}(-u^2)^m f_1(u) &= \int_0^\infty h_1(z)\{(-u^2)^m\mathfrak{J}_\alpha(uz)\}z^{2\alpha+1}dz ,\\ &= \int_0^\infty h_1(z)\{\Delta^m\mathfrak{J}_\alpha(uz)\}z^{2\alpha+1}dz ,\\ &= \int_0^\infty \{\Delta^m h_1(z)\}\mathfrak{J}_\alpha(uz)z^{2\alpha+1}dz ,\end{aligned}$$

where in the last step we have integrated by parts repeatedly. It is easy to deduce our assertion from this last formula.

Consider the rectangle

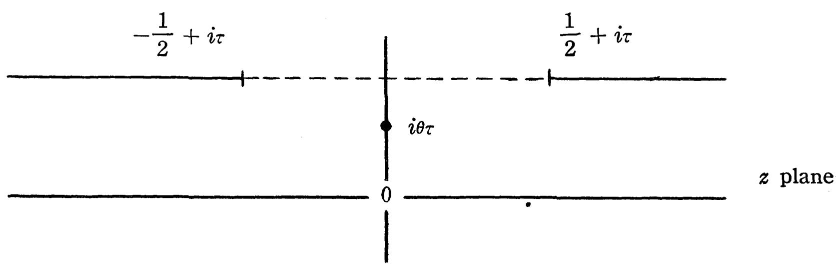


Let $\gamma_k(\theta)$ be the harmonic measure of the side σ_k with respect to the point $i\theta\tau$. Later in this section we will need estimates of the $\gamma_k(\theta)$'s.

LEMMA 11b. *With the above notations we have the inequalities*:

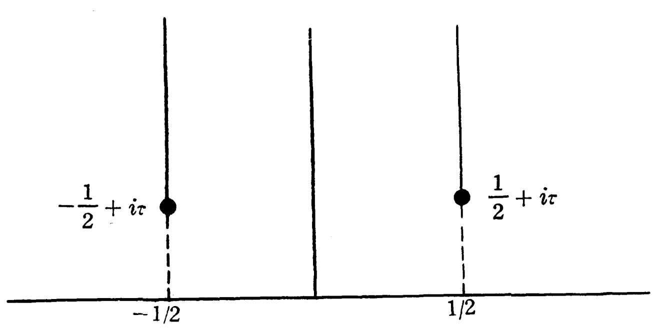
$$\gamma_1(\theta) \geqq 1 - \theta - 2\theta\tau \cosh \tau\pi \ ,$$
$$\gamma_2(\theta) \leqq \theta$$
$$\gamma_3(\theta) \leqq \tau\theta \cosh \tau\pi \ ,$$
$$\gamma_4(\theta) \leqq \tau\theta \cosh \tau\pi \ .$$

Proof. For the definition of harmonic measure and its basic properties see [8]. By the principle of domain extension $\gamma_2(\theta)$ is less than or equal to the harmonic measure of the line segment connection $-(1/2) + i\tau$ to $(1/2) + i\tau$ in the strip bounded by the lines $Imz = \tau$ and $Imz = 0$

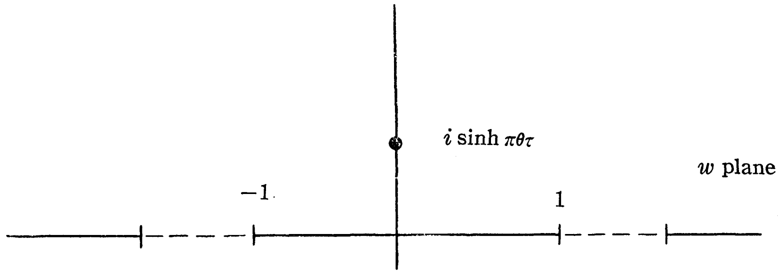


This is trivially less than the harmonic measure of the whole line $Imz = \tau$. But this last is θ, and thus $\gamma_2(\theta) \leqq \theta$.

A second application of the principle of domain extension shows that $\gamma_3(\theta) + \gamma_4(\theta)$ is less than the harmonic measure of the segments connecting $-1/2$ to $-(1/2) + i\tau$ and $1/2$ to $(1/2) + i\tau$ in the half strip bounded by the lines $Rlz = -1/2$, $Rlz = 1/2$, and $Imz = 0$ and lying in the upper half plane.



This is trivially less than the harmonic measure of the two vertical bounding lines. If $w = \sin \pi z$ then the half strip is mapped conformally



onto the half plane $Imw \geqq 0$. The point $i\theta\tau$ goes over into $i \sinh \pi\theta\tau$. Since harmonic measures are invarient under conformal mapping we see that $\gamma_3(\theta) + \gamma_4(\theta)$ is less than the harmonic measure γ of the infinte intervals $(\infty, -1]$ and $[1, \infty)$ with respect to $i \sinh \pi\theta\tau$. But this can be exactly computed using the Poisson formula for the half-plane. We find that

$$\gamma \leqq \frac{2}{\pi} \sinh \pi\tau\theta \int_1^\infty [x^2 + \sinh^2 \pi\tau\theta]^{-1} dx ,$$
$$\leqq \frac{2}{\pi} \sinh \pi\tau\theta .$$

Since $\gamma_3(\theta) = \gamma_4(\theta)$ by symmetry and since $\gamma_3(\theta) + \gamma_4(\theta) < \gamma$ we find that $\gamma_3(\theta)$ and $\gamma_4(\theta)$ are both less than $(1/\pi) \sinh \pi\tau\theta$. Using the mean value theorem we see that $(1/\pi) \sinh \pi\tau\theta \leqq \tau\theta \cosh \pi\tau$, etc.

Let $\boldsymbol{D}_1$ be the subset of $\boldsymbol{D}$ consisting of those functions in $\boldsymbol{D}$ which vanish for $c_1 \leqq z < \infty$ for some $c_1 < 1$, and let $\boldsymbol{D}_1^\wedge = \psi \boldsymbol{D}_1$. Let $\boldsymbol{D}_2$ be the subset of $\boldsymbol{D}$ consisting of those functions in $\boldsymbol{D}$ which vanish for $0 \leqq z \leqq c_2$ for some $c_2 > 1$ and let $\boldsymbol{D}_2^\wedge = \psi \boldsymbol{D}_2$. The principal result of the present section is the following.

THEOREM 11c. *If* $f \in \boldsymbol{D}_1^\wedge$ *or* $\boldsymbol{D}_2^\wedge$ *and if* (as in § 10)

$$a(k, n) = \int_0^{n\pi} f(\zeta) R(k, n, \zeta) \zeta d\zeta$$

then for ν *fixed* $\nu = 0, \pm 1, \pm 2, \cdots$, *we have*

$$a(n, n + \nu) = 0(n^{-r}) \qquad \text{as } n \to \infty$$

for every r.

Proof. We first consider the case $f \in \boldsymbol{D}_1^\wedge$ or $\boldsymbol{D}_2^\wedge$. We have

$$a(k, n) = a_1(k, n) + a_2(k, n)$$

where

$$a_1(k, n) = \int_0^{n\pi/2} f(\zeta) R(k, n, \zeta) \zeta d\zeta ,$$
$$a_2(k, n) = \int_{n\pi/2}^{n\pi} f(\zeta) R(k, n, \zeta) \zeta d\zeta .$$

Using Lemma 11a and the relation

$$\int_0^{n\pi} R(k, n, \zeta)^2 \zeta d\zeta = n$$

it is apparent that $a_2(n, k) = 0(n^{-r})$ as $n \to \infty$ for every $r > 0$. Suppose now $f \in D_1^{\wedge}$. If we set

$$a_1^+(k, n) = \int_0^{n\pi/2} Q_k^+\left(\cos\frac{\zeta}{n}\right)\Theta_n(\zeta) f_1(\zeta) d\zeta \,,$$

$$a_1^-(k, n) = \int_0^{n\pi/2} Q_k^-\left(\cos\frac{\zeta}{n}\right)\Theta_n(\zeta) f_1(\zeta) d\zeta \,,$$

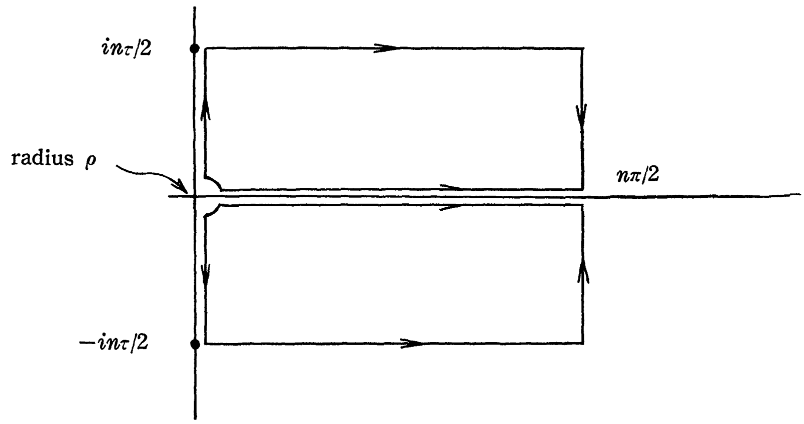
where

$$\Theta_n(\xi) = h_k^{-1/2}\left\{w\left(\cos\frac{\zeta}{n}\right)\right\}^{-1/2}\left\{\sin\frac{\zeta}{n}\right\}^{1/2}\zeta^{\alpha+(1/2)} \,,$$

and where $f_1(\zeta) = \zeta^{-\alpha} f(\zeta)$ then, see (10) of the Appendix,

$$\pi i a_1(k, n) = a_1^-(k, n) - a_1^+(k, n) \,.$$

Note that if β is large $Q_k^{\pm}(\cos \zeta/n) f_1(\zeta)\Theta_n(\zeta)$ may not be integrable near $\zeta = n\pi$. This is the reason for splitting off $a_2(n, k)$. Apply Cauchy's theorem to teach of the curves below and then let $\rho \to 0+$.



We obtain

$$a_1^{\pm}(k, n) = I_1^{\pm} + I_2^{\pm} + I_3^{\pm} \,,$$

where

$$I_1^- = \int_0^{in\tau/2} \,, \qquad I_2^- = \int_{in\tau/2}^{(in\tau/2)+(n\pi/2)} \,, \qquad I_3^- = \int_{(in\tau/2)+(n\pi/2)}^{n\pi/2} \,,$$

$$I_1^+ = \int_0^{-in\tau/2} \,, \qquad I_2^+ = \int_{-in\tau/2}^{(-in\tau/2)+(n\pi/2)} \,, \qquad I_3^+ = \int_{(-in\tau/2)+(n\pi/2)}^{n\pi/2} \,.$$

In all cases the integrand is

$$f_1(\zeta)Q_k\left(\cos\frac{\zeta}{n}\right)\Theta_n(\zeta)d\zeta\ .$$

Let us put

$$\Theta_n^\sharp(t) = h_k^{-1/2}\left\{\cosh\frac{t}{n} - 1\right\}^{-\alpha/2}\left\{\cosh\frac{t}{n} + 1\right\}^{-\beta/2}\left\{t\sinh\frac{t}{n}\right\}^{1/2}t^\alpha$$

for $0 < t < \infty$. Keeping careful track of arguments we find that

$$I_1^- = -\int_0^{n\tau/2} f_1(it)Q_k\left(\cosh\frac{t}{n}\right)\Theta_n^\sharp(t)dt\ ,$$

$$I_1^+ = -\int_0^{n\tau/2} f_1(-it)Q_k\left(\cosh\frac{t}{n}\right)\Theta_n^\sharp(t)dt\ .$$

Since $f_1(\zeta)$ is even $I_1^- - I_1^+ = 0$ and thus

$$\pi i a_1(k, n) = I_2^- + I_2^- - I_2^+ - I_3^+\ .$$

If $h = \psi^{-1}f$ then we have

$$f_1(\zeta) = \int_0^{c_1} h(z)z^{-\alpha}\zeta^{-\alpha}J_\alpha(z\zeta)z^{\alpha+1}dz$$

where $c_1 < 1$. It follows that $f_1(\zeta)$ is an entire function of ζ and that for any c, $c_1 < c < 1$,

$$\left|f_1\left(\xi + \frac{1}{2}in\tau\right)\right| \leqq Ae^{c\tau n/2} \qquad 0 \leqq \xi \leqq n\pi/2\ ;$$

see [1; p. 85]. By (11) of the Appendix we have

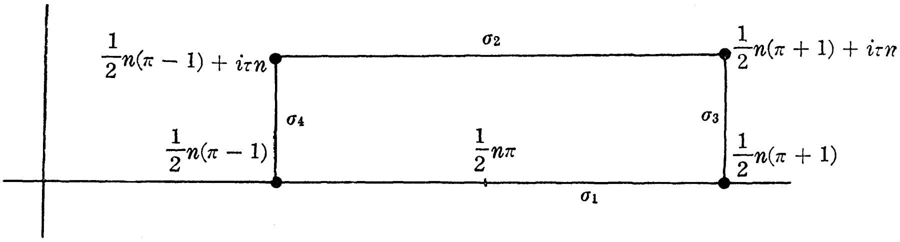
$$\left|Q_{n+\nu}\left[\cos\left(\frac{1}{2}i\tau + \xi n^{-1}\right)\right]\right| \leqq An^q e^{-\tau n/2}\ , \quad 0 \leqq \xi \leqq n\pi/2\ .$$

if ν is fixed. Since

$$\left|\Theta_n^\sharp\left(\frac{1}{2}i\tau + \xi\right)\right| \leqq An^{\alpha+1} \qquad 0 \leqq \xi \leqq n\pi/2$$

we see that I_2^- vanishes exponentially as $n \to \infty$. I_2^+ can be similarly treated.

In order to estimate I_3^- we consider the rectangle below.



This rectangle is similar to the rectangle of Lemma 11b. Let

$$M_i = \text{l.u.b.}\, |f_1(\zeta)| \quad \zeta \text{ on } \sigma_i \qquad i = 1, 2, 3, 4 .$$

By the principle of harmonic majoration

$$\log\left|f_1\left(\frac{1}{2}n\pi + in\tau\theta\right)\right| \leqq \sum_{i=1}^{4} \gamma_i(\theta) \log M_i .$$

We have $M_1 \leqq An^{-r}$, $M_i \leqq Ae^{c\tau n}$ $i = 2, 3, 4$. By Lemma 11b if τ is sufficiently small

$$\gamma_1(\theta) \geqq \frac{1}{3}, \qquad c\{\gamma_2(\theta) + \gamma_3(\theta) + \gamma_4(\theta)\} \leqq \theta , \qquad 0 \leqq \theta \leqq \frac{1}{2} .$$

For τ so chosen

$$\left|f_1\left(\frac{1}{2}n\pi + i\eta\right)\right| \leqq Ae^{\eta}n^{-r/3} \qquad 0 \leqq \eta \leqq \tau n/2 ,$$

uniformly in n. On the other hand by (11) of the Appendix

$$\left|Q_{n+\nu}\left(\cos\left(\frac{1}{2}\pi + i\eta n^{-1}\right)\right)\right| \leqq An^{q}e^{-\eta} \qquad 0 \leqq \eta \leqq \tau n/2 ,$$

and an elementary argument shows that

$$\left|\Theta_n\left(\frac{1}{2}n\pi + i\eta\right)\right| \leqq An^{\alpha} \qquad 0 \leqq \eta \leqq \tau n/2 .$$

Since r is *arbitrary* these estimates imply that $I_3^- = 0(n^{-r})$ as $n \to \infty$ for every r. I_3^+ can be dealt with similarly.

We now turn to the case $f \in \hat{D_2}$. We have, if $h = \psi^{-1}f$,

$$f(\zeta) = \int_{c_1}^{c_2} h(z)J_\alpha(z\zeta)zdz ,$$

where $1 < c_1 < c_2 < \infty$. Since, see [1; p. 4],

$$J_\alpha(z) = \frac{1}{2}H_\alpha^{(1)}(z) + \frac{1}{2}H_\alpha^{(2)}(z)$$

we have

$$2f(\zeta) = f^{(1)}(\zeta) + f^{(2)}(\zeta) ,$$

where

$$f^{(1)}(\zeta) = \int_{c_1}^{c_2} h(z) H_\alpha^{(1)}(z\zeta) z dz \ ,$$
$$f^{(2)}(\zeta) = \int_{c_1}^{c_2} h(z) H_\alpha^{(2)}(z\zeta) z dz \ ,$$

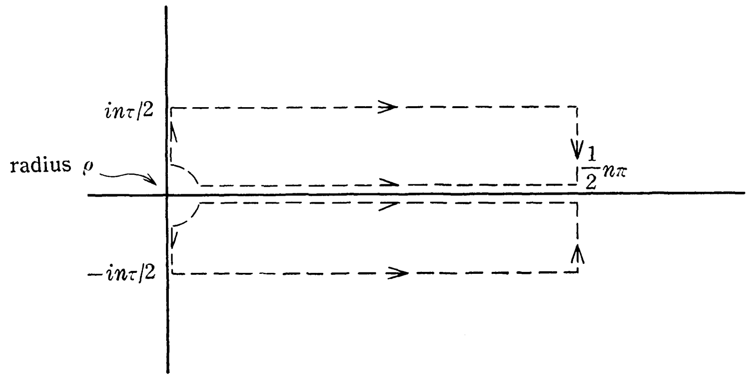
and hence

$$a_1(k, n) = a_1^{(1)}(k, n) + a_1^{(2)}(k, n)$$

where

$$a_1^{(i)}(k, n) = \int_0^{n\pi/2} f^{(i)}(\zeta) R(k, n, \zeta) \zeta d\zeta \ , \qquad i = 1, 2 \ .$$

The functions $f^{(1)}(\zeta)$ and $f^{(2)}(\zeta)$ are analytic in the plane slit from $-\infty$ to 0. Let us apply Cauchy's theorem to one or the other of the two curves below and then let $\rho \to 0+$.



We see that

$$a_1^{(i)}(k, n) = I_1^{(i)} + I_2^{(i)} + I_3^{(i)} \qquad i = 1, 2 \ .$$

$$I_1^{(1)} = \int_0^{in\tau/2} \ , \qquad I_2^{(1)} = \int_{in\tau/2}^{(in\tau/2)+(n\pi/2)} \ , \qquad I_3^{(1)} = \int_{(in\tau/2)+(n\pi/2)}^{n\pi/2} \ ,$$
$$I_1^{(2)} = \int_0^{-in\tau/2} \ , \qquad I_2^{(2)} = \int_{-in\tau/2}^{(-in\tau/2)+(n\pi/2)} \ , \qquad I_3^{(2)} = \int_{(-in\tau/2)+(n\pi/2)}^{n\pi/2} \ .$$

In $I_j^{(i)}$, $j = 1, 2, 3$ the integrand is

$$f^{(i)}(\zeta) P_k^{(\alpha\ \beta)}\left(\cos \frac{\zeta}{n}\right) \Omega_n(\zeta) d\zeta$$

where

$$\Omega_n(\zeta) = h_k^{-1/2} w\left(\cos \frac{\zeta}{n}\right)^{1/2} \left(\sin \frac{\zeta}{n}\right)^{1/2} \zeta^{+1/2} \ .$$

From this we see that if

$$\Omega_n^\sharp(t) = h_k^{-1/2}\left(\cosh\frac{t}{n} - 1\right)^{\alpha/2}\left(\cosh\frac{t}{n} + 1\right)^{\beta/2}\left\{t\sinh\frac{t}{n}\right\}^{1/2}$$

then

$$I_1^{(1)} = -e^{\pi i\alpha/2}\int_0^{\tau n/2} f^{(1)}(te^{\pi i/2})P_k^{(\alpha,\beta)}\left(\cosh\frac{t}{n}\right)\Omega_n^\sharp(t)dt\ ,$$

$$I_1^{(2)} = -e^{-\pi i\alpha/2}\int_0^{\tau n/2} f^{(2)}(te^{-\pi i/2})P_k^{(\alpha\ \beta)}\left(\cosh\frac{t}{n}\right)\Omega_n^\sharp(t)dt\ .$$

Adding we find that

$$I_1^{(1)} + I_1^{(2)} = -\int_0^{\tau n/2}[e^{\pi i\alpha/2}f^{(1)}(te^{\pi i/2}) + e^{-\pi i\alpha/2}f^{(2)}(te^{-\pi i/2})] \cdot P_k^{(\alpha\ \beta)}\left(\cosh\frac{t}{n}\right)\Omega_n^\sharp(t)dt\ .$$

Since arg $t = 0$

$$e^{\pi i\alpha/2}H_\alpha^{(1)}(te^{\pi i/2}) + e^{-\pi i\alpha/2}H_\alpha^{(2)}(te^{-\pi i/2}) = 0\ ,$$

see [1; p. 5]. We have $I_1^{(1)} + I_1^{(2)} = 0$ and thus

$$a_1(n, k) = I_2^{(1)} + I_3^{(1)} + I_2^{(2)} + I_3^{(2)}\ .$$

It follows from [1; p. 85] that if $1 < c < c_2$ then

$$\left|f^{(1)}\left(\frac{1}{2}i\tau n + \xi\right)\right| \leqq Ae^{-c\tau n/2} \qquad 0 \leqq \xi \leqq n\pi/2$$

while by (8) of the Appendix

$$\left|P_{n+\nu}\left(\cos\left[\frac{1}{2}i\tau + \xi n^{-1}\right]\right)\right| \leqq An^q e^{\tau n/2} \qquad 0 \leqq \xi \leqq n\pi/2\ .$$

Since trivially

$$\left|\Omega_n\left(\frac{1}{2}in\tau + \xi\right)\right| \leqq An \qquad 0 \leqq \xi \leqq n\pi/2$$

we see that $I_2^{(1)}$ vanishes exponentially as $n \to \infty$ and thus that $I_2^{(1)} = 0(n^{-r})$ as $n \to \infty$ for every r. Similar considerations apply to $I_2^{(2)}$.

Using Lemmas 11a and 11b we can deal with $I_3^{(1)}$ and $I_3^{(2)}$ very much in the way we dealt with I_3^+ and I_3^-.

12. Convergence of $(S_n^\wedge)^{1/2}F_n^\wedge$ to $(S^\wedge)^{1/2}F^\wedge$. Part II.

LEMMA 12a. *Let $g \in \boldsymbol{D}_1^\wedge$. Then for every nonnegative integer N*

we have

$$\overline{\lim_{n\to\infty}}\int_0^\infty u^{4N}\,|\,F_n^\frown g\cdot(u)\,|^2 u du \leqq \left(\frac{\pi}{2}\right)^{4N}\int_0^\infty u^{4N}|\,g(u)\,|^2\, u du\ .$$

Proof. By (1) of § 10 we have,

$$F_n^\frown g\cdot(u) = n^{-1}\sum_{k=0}^{n} R(k, n, u)a(k, n)$$

where

$$a(k, n) = \int_0^{n\pi} R(k, n, u)g(u)u du\ .$$

Let $g_1(u) = g(u)(1 - \cos u/n)$. Using the recursion formula, (1) of the Appendix, a short computation shows that

$$\left(1 - \cos\frac{u}{n}\right)F_n^\frown g\cdot(u) - F_n^\frown g_1\cdot(u)$$
$$= h_n^{1/2}h_{n+1}^{-1/2}C_{n+1}a(n+1, n)R(n, n, u) - h_{n+1}^{1/2}h_n^{-1/2}A_n a(n, n)R(n+1, n, u)\ .$$

Using Theorem 11c this implies that

$$\left\|\left(1 - \cos\frac{u}{n}\right)F_n^\frown g\cdot(u) - F_n^\frown g_1\cdot(u)\right\| = 0(n^{-r})$$

and thus since $\|\,F_n^\frown\,\| = 1$

$$(1)\qquad \begin{aligned}&\int_0^{n\pi}\left|1 - \cos\frac{u}{n}\right|^2|\,F_n^\frown g\cdot(u)\,|^2 u du \leqq \int_0^{n\pi}|\,F_n^\frown g_1\cdot(u)\,|^2 u du + 0(n^{-r})\ ,\\ &\int_0^{n\pi}\left|1 - \cos\frac{u}{n}\right|^2|\,F_n^\frown g\cdot(u)\,|^2 u du \leqq \int_0^{\infty}\left|1 - \cos\frac{u}{n}\right|^2|\,g(u)\,|^2 u du + 0(n^{-r})\ .\end{aligned}$$

Now

$$(2)\qquad 1 - \cos\frac{u}{n} = 2\sin^2\left(\frac{u}{2n}\right)\begin{cases}\geqq 2\left(\dfrac{2}{\pi}\dfrac{u}{2n}\right)^2 & 0 \leqq u \leqq n\pi\\ \leqq 2\left(\dfrac{u}{2n}\right)^2 & 0 \leqq u < \infty\ .\end{cases}$$

Multiplying (1) through by $(1/4)\pi^4 n^4$ and using (2) we find that

$$\int_0^{n\pi} u^4|\,F_n^\frown g\cdot(u)\,|^2 u du \leqq \left(\frac{\pi}{2}\right)^4\int_0^\infty u^4|\,g(u)\,|^2 u du + 0(n^{-r+2})\ ,$$

which implies our result for $N = 1$. The argument however is valid in general if we use $(1 - \cos u/n)^N$ in place of $(1 - \cos u/n)$.

THEOREM 12b. *If* $f \in D_1^\frown$ *then*

$$\lim_{n\to\infty} \| (S^\wedge)^{1/2} F^\wedge f - (S_n^\wedge)^{1/2} F_n^\wedge f \| = 0 .$$

Proof. An elementary argument gives

$$\| (S^\wedge)^{1/2} F^\wedge f - (S_n^\wedge)^{1/2} F_n^\wedge f \|^2 \leqq I_1 + 2I_2 + 2I_3$$

where

$$I_1 = \int_0^T | u^\omega F^\wedge f \cdot (u) - s_n(u)^{1/2} F_n^\wedge f \cdot (u) |^2 u du ,$$

$$I_2 = \int_T^\infty u^{2\omega} | F^\wedge f \cdot (u) |^2 u du ,$$

$$I_3 = \int_T^\infty s_n(u) | F_n^\wedge f \cdot (u) |^2 u du .$$

By (2) of § 9 (if $T \geqq 1$) then

$$I_3 \leqq A \int_T^\infty u^{2\omega+1} | F_n^\wedge \cdot (u) |^2 u du .$$

Choose N so that if $a = 4N - (2\omega + 1)$ then $a > 0$. We then have

$$I_3 \leqq A T^{-a} \int_0^\infty u^{4N} | F_n^\wedge f \cdot (u) |^2 u du .$$

It is now evident from Lemma 12a that if T is sufficiently large then

$$\overline{\lim_{n\to\infty}}\, I_3 < \varepsilon/4 . \tag{3}$$

Since $F^\wedge f \cdot (u) = f(u)$ for $f \in \boldsymbol{D}_1^\wedge$ we see using Lemma 11a that for all sufficiently large T

$$I_2 < \varepsilon/4 . \tag{4}$$

Suppose now that T has been chosen so that (3) and (4) hold. Since $\lim_{n\to\infty} s_n(u)^{1/2} = u^\omega$ boundedly for $0 \leqq u \leqq T$ and since by Theorem 10b $F_n^\wedge f \cdot (u) \to F^\wedge f \cdot (u)$ in $\boldsymbol{H}^\wedge$ we have

$$\lim_{n\to\infty} I_1 = 0 . \tag{5}$$

Combining (3), (4) and (5) gives

$$\overline{\lim_{n\to\infty}} \| (S^\wedge)^{1/2} F^\wedge f - (S_n^\wedge)^{1/2} F_n^\wedge f \| \leqq \varepsilon ,$$

but ε is arbitrary, etc.

LEMMA 12c. *Let $g \in \boldsymbol{D}_2^\wedge$; then for every nonnegative integer N we have*

$$\overline{\lim_{n\to\infty}} \int_0^\infty u^{4N} | F_n^\wedge g \cdot (u) |^2 u du \leqq \left(\frac{\pi}{2}\right)^{4N} \int_0^\infty u^{4N} | g(u) |^2 u du .$$

THEOREM 12d. *If* $f \in D_2^{\wedge}$ *then*

$$\lim_{n\to\infty} \| (S^{\wedge})^{1/2} F^{\wedge} f - (S_n^{\wedge})^{1/2} F_n^{\wedge} f \| = 0 .$$

Note that for $f \in D_2^{\wedge}$, $F^{\wedge} f = 0$.

The demonstrations of Lemma 12c and Theorem 12d are so much like those of Lemma 12a and Theorem 12b that they are omitted.

13. Convergence of $(S_n^{\wedge})^{1/2}F_n^{\wedge}$ to $(S^{\wedge})^{1/2}F^{\wedge}$, Part III.

If

$$f(u) = \int_0^\infty f^*(z) J_\alpha(zu) z dz$$

$$g(u) = \int_0^\infty g^*(z) J_\alpha(zu) z dz$$

and if

$$h^*(z) = \int_0^\infty \{f(u) g(u) u^{-\alpha}\} J_\alpha(uz) u du$$

then $h^*(z)$ is a "convolution" of $f^*(z)$ and $g^*(z)$. Indeed if $\alpha \geqq -1/2$ then there exists a very interesting formula for h^* in terms of f^* and g^*, and it is possible using this formula to read off simple properties concerning supports such as those proved below. See, for example, [3] or [4]. However these arguments are not available if $-1 < \alpha < -1/2$.

Let $\delta(z)$ be a nonnegative function in D_1 such that

$$(1) \qquad \int_0^\infty \delta(z) z^{\alpha+1} dz = 2^\alpha \Gamma(\alpha + 1) \qquad \alpha > -1 .$$

We define

$$\Delta(u) = \int_0^\infty \delta(z) J_\alpha(zu) z dz .$$

Let also $\Delta_1(u) = u^{-\alpha}\Delta(u)$. We know from Lemma 11a that $\Delta_1(u) = 0(u^{-r})$ as $u \to +\infty$ for every r. It is easily seen using (1) that $\Delta_1(0) = 1$. Also $\Delta_1(u)$ is the restriction to the real axis of an even entire function $\Delta_1(w)$ which satisfies $|\Delta_1(w)| \leqq Ae^{|v|}(1 + |w|^q)$, $w = u + iv$, where $q = \max(0, -1/2 - \alpha)$.

LEMMA 13a. *Let* $f^*(z) \in H$. *If*

1. $f^*(z)$ *vanishes for* $c \leqq z < \infty$;
2. $f(u) = \int_0^\infty f^*(z) J_\alpha(uz) z dz$;
3. $f^*(\lambda, z) = \int_0^\infty f(u) \Delta_1(\lambda u) J_\alpha(uz) u du$,

then $z^{-\alpha}f^*(\lambda,z)$ *is the restriction to* $0 \leqq z < \infty$ *of an even infinitely differentiable function and* $f^*(\lambda, z) = 0$ *for* $c + \lambda \leqq z < \infty$.

Proof. We will merely sketch the demonstration. Since

$$J_\alpha(z) = \frac{1}{2}[H_\alpha^{(1)}(z) + H_\alpha^{(2)}(z)]$$

we have

$$2f^*(\lambda, z) = \int_0^\infty [H_\alpha^{(1)}(zu) + H_\alpha^{(2)}(zu)]f_1(u)\Delta_1(\lambda u)u^{\alpha+1}du .$$

It is easily seen that this can be rewritten as

$$2f^*(\lambda, z) = \int_{-\infty}^\infty H_\alpha^{(1)}(zu)f_1(u)\Delta_1(\lambda u)u^{\alpha+1}du$$

where zu has argument 0 for $0 < u < \infty$ and argument π for $-\infty < u < 0$. By Cauchy's theorem if

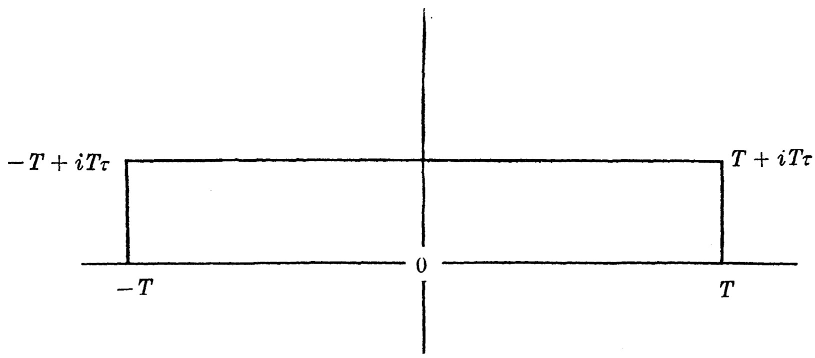
$$I = \int_{-T}^T H_\alpha^{(1)}(zu)f_1(u)\Delta_1(\lambda u)u^{\alpha+1}du$$

then

$$I = I_1 + I_2 + I_3 ,$$

where

$$I_1 = \int_{-T}^{-T+iT\tau} , \qquad I_2 = \int_{-T+iT\tau}^{T+iT\tau} , \qquad I_3 = \int_{T+iT\tau}^{T} .$$



Fixing τ conveniently we can show by arguments like those in § 11 that if $z > c + \lambda$, $I_1, I_2, I_3 \to 0$ as $T \to \infty$. Using the fact that $f(u)\Delta_1(\lambda u) = 0(u^{-r})$ as $u \to \infty$ for every r we see from the formula defining $f^*(\lambda, z)$ that $z^{-\alpha}f^*(\lambda, z)$ is the restriction to $0 \leqq z < \infty$ of an even infinitely differentiable function. By continuity $f^*(\lambda, z) = 0$

for $z = c + \lambda$.

LEMMA 13b. *Let $f^*(z) \in \boldsymbol{H}$. If*
1. *$f^*(z)$ vanishes for $0 \leqq z \leqq c$, $c > 0$;*
2. $f(u) = \int_0^\infty f^*(z) J_\alpha(uz) z dz \qquad (M_2)$;
3. $f^*(\lambda, u) = \int_0^\infty f(u) \Delta_1(\lambda u) J_\alpha(uz) u du$;

then $z^{-\alpha} f^*(\lambda, z)$ is the restriction to $0 < z < \infty$ of an even infinitely differentiable function and $f^*(\lambda, z) = 0$ for $0 \leqq z \leqq c - \lambda$ if $c - \lambda > 0$.

Proof. Again we merely sketch the proof. We have

$$2f(u) = f^{(1)}(u) + f^{(2)}(u)$$

where

$$f^{(1)}(u) = \int_0^\infty H_\alpha^{(1)}(zu) f(z) z dz ,$$
$$f^{(2)}(u) = \int_0^\infty H_\alpha^{(2)}(zu) f(z) z dz ,$$

and thus

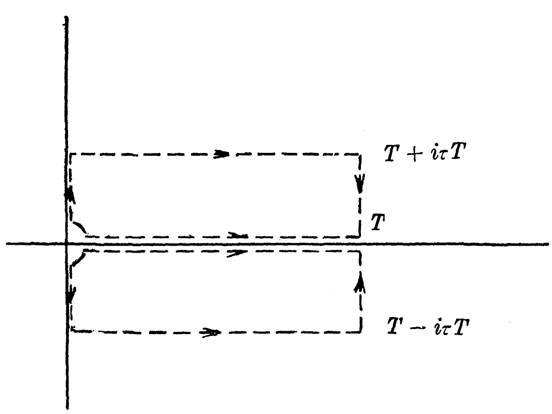
$$2f^*(\lambda, z) = \lim_{T \to \infty} [I^{(1)} + I^{(2)}]$$

where

$$I^{(1)} = \int_0^T f^{(1)}(u) \Delta_1(\lambda u) J_\alpha(uz) u du$$
$$I^{(2)} = \int_0^T f^{(2)}(u) \Delta_1(\lambda u) J_\alpha(uz) u du .$$

By Cauchy's theorem

$$I^{(1)} = I_1^{(1)} + I_2^{(1)} + I_3^{(1)} ,$$
$$I^{(2)} = I_1^{(2)} + I_2^{(2)} + I_3^{(2)} ,$$



where

$$I_1^{(1)} = \int_0^{i\tau T}, \qquad I_2^{(1)} = \int_{i\tau T}^{i\tau T+T}, \qquad I_3^{(1)} = \int_{i\tau T+T}^{T},$$
$$I_1^{(2)} = \int_0^{-i\tau T}, \qquad I_2^{(2)} = \int_{-i\tau T}^{-i\tau T+T}, \qquad I_3^{(2)} = \int_{-i\tau T+T}^{T}.$$

It is easily verified that $I_1^{(1)} + I_1^{(2)} = 0$, and arguments like those used in § 11 suffice to prove that $I_2^{(1)}$, $I_3^{(1)}$, $I_2^{(2)}$, and $I_3^{(2)} \to 0$ as $T \to \infty$ if τ is suitably chosen, for $0 \leqq z < c - \lambda$, etc.

THEOREM 13c. *Let $f \in \boldsymbol{D}[(S^\frown)^{1/2}F^\frown]$; then given $\varepsilon > 0$ there exists $h \in \boldsymbol{D}[(S^\frown)^{1/2}F^\frown]$ such that:*

(1) $$\| f - h \| < \varepsilon, \qquad \| (S^\frown)^{1/2}F^\frown\{f - h\} \| < \varepsilon,$$

(2) $$(S_n^\frown)^{1/2}F_n^\frown h \to (S^\frown)^{1/2}F^\frown h \qquad as\ n \to \infty.$$

Proof. It is obviously sufficient to consider two cases. $F^\frown f = f$ and $F^\frown f = 0$.

Suppose that $F^\frown f = f$. By assumption $f \in \boldsymbol{D}[(S^\frown)^{1/2}F^\frown]$ so that

$$\| (S^\frown)^{1/2}F^\frown f \|^2 = \int_0^\infty u^{2\omega} | f(u) |^2 u du < \infty .$$

For $0 < \theta < 1$ let $g(u) = f(\theta u)$. Then if $f^* = \psi^{-1}f$, $g^* = \psi^{-1}g$ we have

$$g^*(z) = \int_0^\infty f(\theta u) J_\alpha(uz) u du \qquad (M_2)$$
$$= \int_0^\infty f(u) J_\alpha(uz\theta^{-1})\theta^{-2} u du \qquad (M_2)$$
$$= f^*(z\theta^{-1})\theta^{-2} .$$

$F^\frown f = f$ implies that $f^*(z) = 0$ for $z > 1$. It follows that $g^*(z) = 0$ for $z > \theta$. Consequently $F^\frown g = g$ as well. It is also evident that $g \in \boldsymbol{D}[(S^\frown)^{1/2}F^\frown]$. Since

$$\| f - g \|^2 = \int_0^\infty | f(u) - f(\theta u) |^2 u du ,$$
$$\| (S^\frown)^{1/2}F^\frown\{f - g\} \|^2 = \int_0^\infty u^{2\omega} | f(u) - f(\theta u) |^2 u du ,$$

it is apparent that by taking θ sufficiently near 1 we can insure that

$$\| f - g \| < \varepsilon/2, \quad \| (S^\frown)^{1/2}F^\frown\{f - g\} \| < \varepsilon/2 .$$

We next define $h(u) = g(u)\Delta_1(\lambda u)$. If $\lambda > 0$ is so small that $\lambda + \theta < 1$ then by Lemma 13a if $h^* = \psi^{-1}h$ $h^*(z) = 0$ for $z > 1$, and thus $F^\frown h = h$. Since

$$\| g - h \|^2 = \int_0^\infty | g(u) |^2 | 1 - \Delta(\lambda u) |^2 u du ,$$

$$\| (S^\frown)^{1/2} F^\frown \{g - h\} \|^2 = \int_0^\infty u^{2\omega} | g(u) |^2 | 1 - \Delta(\lambda u) |^2 u du ,$$

it is evident that we can choose $\lambda > 0$ so small that $\lambda + \theta < 1$, and that

$$\| g - h \| < \varepsilon/2 , \qquad \| (S^\frown)^{1/2} F^\frown \{g - h\} \| < \varepsilon/2 .$$

Thus h satisfiies (1). By Lemma 13a $h \in \boldsymbol{D}_1^\frown$, and therefore by Theorem 12b (2) holds as well.

Suppose that $F^\frown f = 0$. Then, if $f^* = \psi^{-1} f$, $f^*(z) = 0$ for $0 < z < 1$. Choose $1 < c_1 < c_2 < \infty$ so that if $g^*(z) = f^*(z)$ for $c_1 < z < c_2$ and $g^*(z) = 0$ otherwise then $\| f^* - g^* \| < \varepsilon/2$. Let $g = \psi g^*$. Clearly $F^\frown g = 0$. We have

$$\| f - g \| = \| f^* - g^* \| < \varepsilon/2 ,$$

while

$$\| (S^\frown)^{1/2} F^\frown \{f - g\} \| = 0 .$$

Next let $h(u) = \Delta_1(\lambda u) g(u)$, where $\lambda > 0$ is so small that $c_1 - \lambda > 1$, which implies using Lemma 13b, that $F^\frown h = 0$, and so small that $\| g - h \| < \varepsilon/2$. Then $\| f - h \| < \varepsilon$ and $\| (S^\frown)^{1/2} F^\frown \{f - h\} \| = 0$, so that (1) holds. By Lemmas 13a and 13b $h \in \boldsymbol{D}_2^\frown$ and thus Theorem 12d can be applied to verify (2).

14. The asymptotic formula. Let $S_F^\frown$ be constructed from $F^\frown$ and $S^\frown$ as in § 3. Note that if $S^\frown = \{f \mid f \in \boldsymbol{H}, F^\frown f \in \boldsymbol{D}(S^\frown)^{1/2}\}$ then $S^\frown$ is dense in $\boldsymbol{H}^\frown$ so that $S_F^\frown$ is a self-adjoint transformation on $\boldsymbol{H}$ itself. Let

$$S_F^\frown = \int_{0-}^\infty \lambda d\Psi^\frown(\lambda)$$

be the spectral resolution of $S_F^\frown$ on $\boldsymbol{H}$, and let

$$S_{n\,F}^\frown = \int_{0-}^\infty \lambda d\Psi_n^\frown(\lambda)$$

be the spectral resolution of $S_{n,F}^\frown = F_n^\frown S_n^\frown F_n^\frown$. It follows from Theorems 9b, 10a and 13c combined with Theorem 6a that

$$(1) \qquad \Psi_n^\frown(\lambda) \to \Psi^\frown(\lambda) \qquad 0 \leqq \lambda < \infty$$

for evey $\lambda \notin \sigma_p(S_F^\frown)$.

Let us define

$$R^{\wedge} = S_F^{\wedge}\big|_{N^{\wedge}}, \qquad N^{\wedge} = F^{\wedge}H^{\wedge},$$
$$R_n^{\wedge} = S_{n,F}^{\wedge}\big|_{N_n^{\wedge}}, \qquad N_n^{\wedge} = F_n^{\wedge}H^{\wedge}.$$

Since, as is easily seen, $R^{\wedge} > 0$, $R_n^{\wedge} > 0$, we have the spectral resolutions

$$R^{\wedge} = \int_0^{\infty} \lambda dE^{\wedge}(\lambda) \qquad \text{on } N^{\wedge},$$

where

$$E^{\wedge}(\lambda) = \Psi^{\wedge}(\lambda) - \Psi^{\wedge}(0) \qquad 0 \leqq \lambda < \infty,$$

and

$$R_n^{\wedge} = \int_0^{\infty} \lambda dE_n^{\wedge}(\lambda) \qquad \text{on } N_n^{\wedge}$$

where

$$E_n^{\wedge}(\lambda) = \Psi_n^{\wedge}(\lambda) - \Psi_n^{\wedge}(0) \qquad 0 \leqq \lambda < \infty.$$

Since $\Psi^{\wedge}(0) = I - F^{\wedge}$, $\Psi_n^{\wedge}(0) = I - F_n^{\wedge}$, it follows from (1) that

$$(2) \qquad E_n^{\wedge}(\lambda) \to E^{\wedge}(\lambda) \qquad 0 \leqq \lambda < \infty$$

for all $\lambda \notin \sigma_p(R)$.

LEMMA 14a. *With the above definitions let $f_n \in N_n^{\wedge}$, and let $\|f_n\| = 1$, $(R_n^{\wedge} f_n | f_n) \leqq m < \infty$ for $n \in \mathfrak{p}$. We assert that if $f_n \rightharpoonup f$ as $n \to \infty$ in $\mathfrak{p}_1$ then $f \neq 0$.*

Proof. If $f_n \in N_n^{\wedge}$ then

$$f_n(u) = n^{-1} \sum_{k=1}^{n} R(k, n, u)a(k, n) \qquad 0 \leqq u \leqq n\pi$$

and $f_n(u) = 0$ for $u > n\pi$. We have

$$1 = \|f_n\|^2 = n^{-1} \sum_{k=0}^{n} |a(k, n)|^2 .$$

By Schwartz's inequality

$$|f_n(u)|^2 \leqq n^{-1} \sum_{k=0}^{n} R(k, n, u)^2 .$$

Since, see § 10 for a similar estimate, if $0 \leqq k \leqq n$

$$|R(k, n, u)| \leqq M\left(\frac{k+1}{n}\right)^{\alpha+(1/2)} u^{\alpha} \qquad 0 \leqq u \leqq a_2 < \infty,$$

it follows that

$$(3) \qquad |f_n(u)| \leqq Mu^{\alpha} \qquad 0 \leqq u \leqq a_2 .$$

Next

$$|f_n'(u)|^2 \leqq n^{-1} \sum_{k=0}^{n} R'(k, n, u)^2 .$$

Since, as is easily verified,

$$|R'(k, n, u)| \leqq M\left(\frac{k+1}{n}\right)^{\alpha+(1/2)} u^{\alpha-1} \qquad 0 < u \leqq a_2$$

we have

$$(4) \qquad |f_n'(u)| \leqq Mu^{\alpha-1} \qquad 0 < u \leqq a_2 .$$

It follows from (3) and (4) that the $\{f_n(u)\}_1^\infty$ are uniformly bounded and equicontinuous on any interval $0 < a_1 \leqq u \leqq a_2 < \infty$. Therefore since $f_n(u) \rightharpoonup f(u)$ as n in $\mathfrak{p}_1$ we have (if $f(u)$ is suitably redefined on a set of measure zero)

$$(5) \qquad \lim_{\mathfrak{p}_1} f_n(u) = f(u) \qquad \text{uniformly for } a_1 \leqq u \leqq a_2 .$$

Given any number $m_1 > 0$ we assert that there exists a number $a_2 > 0$ and an integer N such that if $n \geqq N$

$$(6) \qquad s_n(u) \geqq m_1 \qquad a_2 \leqq u \leqq n\pi .$$

The inequality (6) is an immediate consequence of the relations

$$s_n(u) = \left(2n \sin \frac{u}{2n}\right)^{2\omega} \left[L\left(2 \sin^2 \frac{u}{2n}\right) \Big/ L(n^{-2})\right] ,$$

$$s_n(u) \geqq M(\varepsilon)\left(2n \sin \frac{u}{2n}\right)^{2\omega} \left[\left(2n^2 \sin^2 \frac{u}{2n}\right)^{\varepsilon} + \left(2n^2 \sin^2 \frac{u}{2n}\right)^{-\varepsilon}\right]^{-1} .$$

See § 9. We have

$$\int_0^{a_2} s_n(u)\,|f_n(u)|^2\, udu + \int_{a_2}^{\infty} s_n(u)\,|f_n(u)|^2\, udu = (R_n^{\wedge} f_n \,|\, f_n) ,$$

$$\int_{a_2}^{\infty} s_n(u)\,|f_n(u)|^2\, udu \leqq m .$$

By (6) if $n \geqq N$

$$\int_{a_2}^{\infty} s_n(u)\,|f_n(u)|^2\, udu \geqq m_1 \int_{a_2}^{\infty} |f_n(u)|^2\, udu .$$

Therefore if $n \geqq N$ we have

$$m_1 \int_{a_2}^{\infty} |f_n(u)|^2\, udu \leqq m$$

and thus, since $||f_n|| = 1$,

$$\int_0^{a_2} |f_n(u)|^2 u du \geqq 1 - \frac{m}{m_1} . \tag{7}$$

The relations (3), (5), and (7) imply that

$$\int_0^{a_2} |f(u)|^2 u du \geqq 1 - \frac{m}{m_1} > 0$$

and thus that $f \neq 0$ in $\boldsymbol{H}^{\wedge}$, as desired.

Applying Theorem 7a we now see that if $0 < \mu_1 \leqq \mu_2 \leqq \mu_3 \cdots$, $\lim_{k\to\infty} \mu_k = \infty$, are the eigen values of $S_F^{\wedge}$ then

$$\lim_{n\to\infty} 2^{\omega} n^{2\omega} L(n^{-2})^{-1}[t(1) - \lambda_{n,k}] = \mu_k \qquad k = 1, 2, \cdots .$$

We have thus proved the following.

THEOREM 14b. *Under the assumptions of* § 8 *we have*

$$\lambda_{n,k} = t(1) - 2^{-\omega} L(n^{-2}) n^{-2\omega}[\mu_k + o(1)]$$

as $n \to \infty$ *for each fixed* $k = 1, 2, \cdots$.

If we take $\omega = 1$, $L(y) = \sigma$ then we obtain as a very special case of Theorem 14b formula (7) of § 1.

15. Maximum at an interior point. We will next take up the case where $t(x)$ has a unique absolute maximum at x_0, $-1 < x_0 < 1$. We assume that $t(x)$ is continuous for $-1 \leqq x \leqq 1$ and that

$$t(x) < t(x_0) \qquad -1 \leqq x \leqq 1, \; x \neq x_0 .$$

We further assume that

$$t(x_0) - t(x) = \begin{cases} \sigma_1 |x - x_0|^{\omega} L(x - x_0) & x \to x_0^+ \\ \sigma_2 |x - x_0|^{\omega} L(x - x_0) & x \to x_0^- \end{cases}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $\omega > 0$ and $L(y)$ is a positive even function defined for $-2 \leqq y \leqq 2$ and continuous there except at $y = 0$. At $y = 0$ $L(y)$ is slowly oscillating.

In what follows we will again find it necessary to work with four Hilbert spaces.

$\boldsymbol{L}$ is, as before, the Hilbert space of complex valued functions $f(k)$ defined for $k = 0, 1, 2, \cdots$, with inner product

$$(f \mid g)_L = \sum_{k=0}^{\infty} f(k) g(k)^*$$

Similarly $L^\wedge$ is, as before, the space of Lebesgue measurable functions on $-1 \leqq x \leqq 1$ with inner product

$$(f \mid g)_{L^\wedge} = \int_{-1}^{1} f(x)g(x)^* w_{\alpha,\beta}(x)dx$$

where $w_{\alpha\,\beta}(x)$ is defined in § 1.

$E^\wedge$ and E are Hilbert spaces of Lebesgue measurable functions on $(-\infty, \infty)$ with inner products

$$(f \mid g)_{E^\wedge} = \int_{-\infty}^{\infty} f(u)g(u)^* du \ ,$$
$$(f \mid g)_{E} = \int_{-\infty}^{\infty} f(z)g(z)^* dz \ .$$

We have the following maps between these spaces. There is, as before, a mapping ϕ from L to $L^\wedge$ defined by

$$\phi f\cdot(x) = \sum_{k=0}^{\infty} f(k)h_k^{-1/2}P_k^{(\alpha,\beta)}(x) \ .$$

The series on the right is the limit of the partial sums in the metric of $L^\wedge$. The inverse mapping is

$$\phi^{-1}f\cdot(k) = \int_{-1}^{1} f(x)h_k^{-1/2}P_k^{(\alpha,\beta)}(x)w_{\alpha,\beta}(x)dx \ .$$

These mappings are unitary.

There is a mapping ψ from E to $E^\wedge$ defined by

$$\psi f\cdot(u) = \int_{-\infty}^{\infty} e^{2\pi iuz}f(z)dz \ ,$$

where the integral on the right is the limit of the partial integrals in the metric of $E^\wedge$. The inverse mapping is

$$\psi^{-1}f\cdot(z) = \int_{-\infty}^{\infty} e^{-2\pi iuz}f(u)du \ ,$$

where etc. These mappings are also unitary.

Let $0 < \xi_0 < \pi$ be such that $\cos\xi_0 = x_0$. We set

$$\theta_n(u) = [1-\cos(2\pi un^{-1}+\xi_0)]^{\alpha/2}[1+\cos(2\pi un^{-1}+\xi_0)]^{\beta/2}\sin^{1/2}(2\pi un^{-1}+\xi_0) \ .$$

For each $n = 1, 2, \cdots$ we define a mapping χ_n from $L^\wedge$ to $E^\wedge$ by setting

$$\chi_n f\cdot(u) = \begin{cases} f(\cos[2\pi un^{-1}+\xi_0])\theta_n(u)(2\pi/n)^{1/2} & 0 \leqq 2\pi un^{-1}+\xi_0 \leqq \pi \\ 0 & \text{otherwise} \ . \end{cases}$$

Clearly χ_n is an isometric map of $L^\wedge$ into $E^\wedge$. We define a mapping χ_n^* from $E^\wedge$ to $L^\wedge$ by

$$\chi_n^* f\cdot(x) = f\Big(\frac{n}{2\pi}[-\xi_0 + \arccos x]\Big)(1 - x^2)^{-1/4} w_{\alpha,\beta}(x)^{-1/2}(n/2\pi)^{1/2} .$$

χ_n^* is a partially isometric mapping of $E^\wedge$ onto $L^\wedge$. χ_n^* is isometric on $\chi_n L^\wedge$ and zero on $(\chi_n L^\wedge)^\perp$, the orthogonal complement of $\chi_n L^\wedge$ in $E^\wedge$. Moreover $\chi_n^*\chi_n = I$ and $\chi_n\chi_n^* = I$ on $\chi_n L^\wedge$ and $\chi_n\chi_n^* = 0$ on the orthogonal complement of $\chi_n L^\wedge$.

We now introduce various operators on these Hilbert spaces.

a. E_n is defined on L by

$$E_n f\cdot(k) = \begin{cases} f(k) & 0 \leqq k \leqq n \\ 0 & \text{otherwise} . \end{cases}$$

E_n induces the following additional operators:

$E_n^\wedge$ on $L^\wedge$ defined by $E_n^\wedge = \phi E_n \phi^{-1}$;
$F_n^\wedge$ on $E^\wedge$ defined by $F_n^\wedge = \chi_n E_n^\wedge \chi_n^*$.

b. $T^\wedge$ is defined on $L^\wedge$ by

$$T^\wedge f\cdot(x) = [t(x_0) - t(x)]f(x) .$$

Starting from $T^\wedge$ we obtain the following related operators:

T on L defined by $T = \phi^{-1}T^\wedge\phi$;
$T_n^\wedge$ on $E^\wedge$ defined by $T_n = \chi_n T^\wedge \chi_n^*$:
$S_n^\wedge$ on $E^\wedge$ defined by $S_n^\wedge = [n^\omega/L(n^{-1})]T_n^\wedge$.

c. $S^\wedge$ is defined on $E^\wedge$ by

$$S^\wedge f(u) = f(u)s(u)$$

where

$$s(u) = \begin{cases} \sigma_1(-2\pi[\sin\xi_0]u)^\omega & u \leqq 0 \\ \sigma_2(2\pi[\sin\xi_0]u)^\omega & u > 0 . \end{cases}$$

d. F is defined on E by

$$Ff\cdot(z) = \begin{cases} f(z) & |z| \leqq 1 \\ 0 & |z| > 1 . \end{cases}$$

We introduce

$$F^\wedge \text{ on } E^\wedge \text{ defined by } F^\wedge = \psi F \psi^{-1} .$$

If $\lambda_{n,1} \geqq \lambda_{n,2} \geqq \cdots \geqq \lambda_{n\ n+1}$ are the eigen values of C_n, see § 1, then

$$t(x_0) - \lambda_{n,1} \leqq t(x_0) - \lambda_{n,2} \leqq \cdots \leqq t(x_0) - \lambda_{n,n+1}$$

are the eigen values of the following operators:

$$E_n T E_n \Big|_{E_n \boldsymbol{L}} ;$$
$$E_n^{\wedge} T^{\wedge} E_n^{\wedge} \Big|_{E_n^{\wedge} \boldsymbol{L}^{\wedge}} ;$$
$$F_n^{\wedge} T_n^{\wedge} F_n^{\wedge} \Big|_{F_n^{\wedge} \boldsymbol{E}^{\wedge}} .$$

The eigen values of

$$F_n^{\wedge} S_n^{\wedge} F_n^{\wedge} \Big|_{F_n^{\wedge} \boldsymbol{E}^{\wedge}}$$

are, in increasing order, $\{[t(x_0) - \lambda_{n,k}]n^{\omega}L(n^{-1})^{-1}\}_{k=1}^{n+1}$. Our program in what follows is like that carried out in sections 8–14, in that we will show that $F_n^{\wedge} S_n^{\wedge} F_n^{\wedge}$ "converges" to $S_F^{\wedge}$ as $n \to \infty$ and thus that if

$$0 < \mu_1 \leqq \mu_2 \leqq \cdots, \lim_{k \to \infty} \mu_k = \infty ,$$

are the eigen values of

$$S_F^{\wedge} \Big|_{F^{\wedge} \boldsymbol{E}^{\wedge}} ,$$

then

$$\lim_{n \to \infty} (t(x_0) - \lambda_{n,k})n^{\omega}L(n^{-1})^{-1} = \mu_k \qquad k = 1, 2, \cdots ,$$

or equivalently

$$\lambda_{n,k} = t(x_0) - n^{-\omega}L(n^{-1})[\mu_k + o(1)]$$

as $n \to \infty$.

Because the material of §§ 15–19 is in large part like the material of §§ 8–14 we will only give in detail those arguments which differ from those given there. These occur primarily in § 16 and § 17. In the later sections we will simply list the various results since the details can be easily supplied.

16. Convergence of $(S_n^{\wedge})^{1/2}$ to $(S^{\wedge})^{1/2}$ (interior maximum). We suppose throughout that $t(x)$ satisfies the assumptions of § 15. Let $0 < \xi_0 < \pi$ be such that $\cos \xi_0 = x_0$.

It follows from § 15 that $T_n^{\wedge} f(u) = t_n(u)f(u)$ where $t_n(u) = t(\cos \xi_0) - t[\cos (2\pi n^{-1}u + \xi_0)]$ for $0 \leqq 2\pi n^{-1}u + \xi_0 \leqq \pi$ and is zero otherwise. Consequently $S_n^{\wedge} f(u) = s_n(u)f(u)$ where $s_n(u) = t_n(u)n^{\omega}L(n^{-1})^{-1}$. Let $s(u)$ be defined as in § 15.

LEMMA 16a. *With the above definitions*

$$(1)\qquad \lim_{n\to\infty} s_n(u) = s(u) \qquad -\infty < u < \infty ,$$

and for any $\varepsilon > 0$ *there is a constant* $M(\varepsilon)$ *independent of* u, $-\infty < u < \infty$, *and* $n = 1, 2, \cdots$ *such that*

$$(2)\qquad s_n(u) \leqq M(\varepsilon)[|u|^{\varepsilon} + |u|^{-\varepsilon}]\,|u|^{\omega} .$$

Proof. It follows from the assumptions of § 15 that if $\lambda(u) = \cos(2\pi u + \xi_0) - \cos \xi_0$ then

$$t(\cos \xi_0) - t\cos(2\pi u + \xi_0) \backsim \begin{cases} \sigma_2\,|\lambda(u)|^{\omega}\, L(\lambda(u)) & u \to 0+ \\ \sigma_1\,|\lambda(u)|^{\omega}\, L(\lambda(u)) & u \to 0- . \end{cases}$$

Since $\lambda(u) \backsim 2\pi u \sin \xi_0$ as $u \to 0$ we find using (4) of § 9 that

$$(3)\qquad [t(\cos \xi_0) - t\cos(2\pi u + \xi_0)] \backsim \begin{cases} \sigma_2(2\pi \sin \xi_0)^{\omega} u^{\omega} L(u) & \text{as } u \to 0+ \\ \sigma_1(2\pi \sin \xi_0)^{\omega}(-u)^{\omega} L(n) & \text{as } u \to 0- . \end{cases}$$

Thus for u fixed, $u \neq 0$, we see that as $n \to \infty$

$$(4)\qquad s_n(u) \backsim \begin{cases} \sigma_2(2\pi \sin \xi_0)^{\omega} u^{\omega} L(un^{-1})/L(n^{-1}) & u > 0 \\ \sigma_1(2\pi \sin \xi_0)^{\omega}(-u)^{\omega} L(un^{-1})/L(n^{-1}) & u < 0 . \end{cases}$$

A second application of (4) of § 9 yields (1). It follows from (3) that if b is a sufficiently large positive constant then

$$t(\cos \xi_0) - t[\cos(2\pi u + \xi_0)] \leqq b\,|u|^{\omega}\, L(u) ,$$

and thus

$$s_n(u) \leqq b\,|u|^{\omega}\, L(un^{-1})/L(n^{-1}) .$$

Using (5) of § 9 we obtain our desired result.

THEOREM 16b. *$(S^{\wedge})^{1/2}$ is the closure of the strong limit of $(S_n^{\wedge})^{1/2}$ as $n \to \infty$.*

Proof. Let $f \in \boldsymbol{D}[(S^{\wedge})^{1/2}]$ and $\varepsilon > 0$ be given. If $\delta > 0$ is sufficiently small then it is evident that if $f_\delta(u) = e^{-\delta u^2} f(u)$ then

$$\| f - f_\delta \|_{E^{\wedge}} \leqq \varepsilon , \qquad \| (S^{\wedge})^{1/2}(f - f_\delta) \|_{E^{\wedge}} \leqq \varepsilon .$$

Moreover it is evident from (1) and (2) that

$$(S_n^{\wedge})^{1/2} f_\delta \to (S^{\wedge})^{1/2} f_\delta \qquad \text{in } \boldsymbol{E}^{\wedge} \text{ as } n \to \infty .$$

17. Convergence of $F_n^{\wedge}$ (interior maximum).

THEOREM 17a. *If $F_n^{\wedge}$ and $F^{\wedge}$ are defined as in § 15 then $F_n^{\wedge}$*

converges strongly to $F^\wedge$ *as* $n \to \infty$.

Proof. Let us write

$$R(k, n, u) = h_k^{-1/2} P_k^{(\alpha,\beta)}(\cos 2\pi u n^{-1} + \xi_0)\theta_n(u)\sqrt{2\pi} \ .$$
$$Q(k, n, u) = \cos(2\pi(k + \eta)n^{-1}u + k\xi_0 + \zeta) \ ,$$

where $\eta = (\alpha + \beta + 1)/2$, $\zeta = \xi_0(\alpha + \beta + 1)/2 - (\alpha + (1/2))\pi/2$. It follows from (6) of the Appendix that

(1) $$R(k, n, u) - 2Q(k, n, u) \to 0 \qquad \text{as } k \to \infty$$

uniformly in n and u, if for some fixed $\varepsilon > 0$

$$\varepsilon \leqq 2\pi u n^{-1} + \xi_0 \leqq \pi - \varepsilon \ .$$

Starting from the definition of $F_n^\wedge$ we find that

(2) $$F_n^\wedge f \cdot (u) = \frac{1}{n} \sum_{k=0}^{n} R(k, n, u) a(k, n)$$

where

(3) $$a(k, n) = \int_{I_n} f(v) R(k, n, v)\, dv \ .$$

Here $I_n = \{v \mid 0 \leqq 2\pi n^{-1} v + \xi_0 \leqq \pi\}$.

Let us now assume that $f(u)$ is continuous for $-\infty < u < \infty$ and vanishes except for $|u| \leqq a$. We first show that under this assumption $F_n^\wedge f \cdot (u) \to F^\wedge f \cdot (u)$ uniformly in any set $|u| \leqq b < \infty$. It follows from (1) that there exists a constant M such that if n is sufficiently large.

(4) $$|R(k, n, u)| \leqq M$$

for $|u| \leqq a$, and $k = 0, 1, \cdots$. Let us set

$$\textstyle\sum_1 (\delta, n, u) = \frac{1}{n} \sum_{0 \leqq k < n\delta} R(k, n, u) a(k, n) \ ,$$

$$\textstyle\sum_2 (\delta, n, u) = \frac{1}{n} \sum_{n\delta \leqq k \leqq n} R(k, n, u) a(k, n) \ .$$

Using (4) and the corresponding inequality for $|u| \leqq b$ we find that for all large n

(5) $$\textstyle\sum_1 (\delta, n, u) \leqq M\delta \qquad \text{if } |u| \leqq b \ .$$

Let $g = \psi^{-1} f$ so that

$$g(z) = \int_{-\infty}^{\infty} f(u) e^{-2\pi i u z} du \ .$$

Using (1), but writing the cosine in complex form, we find that

$$\lim_{k\to\infty}\left\{a(n, k) - e^{-i(k\xi_0+\zeta)}g\left(\frac{k+\eta}{n}\right) - e^{i(k\xi_0+\zeta)}g\left(-\frac{k+\eta}{n}\right)\right\} = 0 .$$

Using (1) again we see that as $n \to \infty$

$$(6) \qquad |\textstyle\sum_2(\delta, n, u) - \sum_I - \sum_{II} - \sum_{III} - \sum_{IV}| \to 0$$

uniformly for $|u| \leqq b$ where

$$\sum_I = n^{-1}\sum_{\delta n\leqq k\leqq n} e^{2\pi i(k+\eta)un^{-1}}g\left(\frac{k+\eta}{n}\right) ,$$

$$\sum_{II} = n^{-1}\sum_{\delta n\leqq k\leqq n} e^{2i(k\xi_0+\zeta)}e^{2\pi i(k+\eta)un^{-1}}g\left(-\frac{k+\eta}{n}\right) ,$$

$$\sum_{III} = n^{-1}\sum_{\delta n\leqq k\leqq n} e^{-2\pi i(k+\eta)un^{-1}}g\left(-\frac{k+\eta}{n}\right) ,$$

$$\sum_{IV} = n^{-1}\sum_{\delta n\leqq k\leqq n} e^{-2i(k\xi_0+\zeta)}e^{-2\pi i(k+\eta)un^{-1}}g\left(\frac{k+\eta}{n}\right) .$$

$\sum_I$ and $\sum_{III}$ are Riemann sums so that

$$(7) \qquad \lim_{n\to\infty}\sum_I = \int_\delta^1 e^{2\pi iuz}g(z)dz ,$$

$$(7') \qquad \lim_{n\to\infty}\sum_{III} = \int_{-1}^{-\delta} e^{2\pi iuz}g(z)dz ,$$

for $|u| \leqq b$. Since we can easily prove that the functions $\sum_I(\delta, n, u)$ are for n sufficiently large equicontinuous for $|u| \leqq b$ it follows that (7) holds uniformly for $|u| \leqq b$. Similar remarks apply to $\sum_{III}$. If we sum $\sum_{II}$ by parts, the summation being applied to $e^{-2ik\xi_0}$, it is easily seen that $\lim_{n\to\infty}\sum_{II} = 0$ uniformly for $|u| \leqq b$ and similarly for $\sum_{IV}$. Given $\varepsilon > 0$ let us choose δ so small that

$$|\textstyle\sum_1(\delta, n, u)| < \varepsilon/2 , \qquad \left|\displaystyle\int_{-\delta}^{\delta} g(z)e^{2\pi iuz}dz\right| < \varepsilon/2 ,$$

for $|u| \leqq b$. It then follows on collecting our estimates that $|F_n^\wedge f\cdot(u) - F^\wedge f\cdot(u)| < \varepsilon$ for $|u| \leqq b$, for all sufficiently large n. The demonstration can be completed as in § 10.

18. Convergence of $(S_n^\wedge)^{1/2}F_n^\wedge$ to $(S^\wedge)^{1/2}F^\wedge$ (interior maximum). The considerations here are parallel to those of § 11, § 12, and § 13 but somewhat simpler.

Let $\boldsymbol{D}$ be the set of functions $h(z)$ in $\boldsymbol{E}$ which are infinitely differentiable and have compact support, and let $\boldsymbol{D}^\wedge = \psi\boldsymbol{D}$. Let $\boldsymbol{D}_1$ be the subset of $\boldsymbol{D}$ consisting of those functions which have support in

$|z| \leqq c_1$ for some $c_1 < 1$, and let $D_1^{\wedge} = \psi D_1$. Let D_2 be the subset of D consisting of those functions which have support in $|z| \geqq c_2$ for some $c_2 > 1$, and let $D_2^{\wedge} = \psi D_2$.

THEOREM 18a. *If $f \in D_1^{\wedge}$ or D_2 and if, as in § 17,*

$$a(k, n) = \int_{I_n} f(\zeta) R(k, n, \zeta) d\zeta ,$$

then for ν fixed, $\nu = 0, \pm 1, \pm 2, \cdots$, we have

$$a(n, n + \nu) = 0(n^{-r}) \qquad \textit{as } n \to \infty$$

for every r.

Proof. We will carry out only the first steps of the demonstration since it will be evident in a moment that the arguments used in § 11 apply almost without change.

We recall that $I_n = \{\zeta \mid -\gamma_1 n \leqq \zeta \leqq \gamma_2 n\}$ where $\gamma_2 = (\pi - \xi_0)/2\pi$, $\gamma_1 = \xi_0/2\pi$. Choose δ_1, $0 < \delta_1 < \gamma_1$ and δ_2, $0 < \delta_2 < \gamma_2$. Then

$$a(k, n) = a_1(k, n) + a_2(k, n) + a_3(k, n)$$

where

$$a_1(k, n) = \int_{-\delta_1 n}^{\delta_2 n} f(\zeta) R(k, n, \zeta) d\zeta ,$$

$$a_2(k, n) = \int_{\delta_2 n}^{\gamma_2 n} f(\zeta) R(k, n, \zeta) d\zeta ,$$

$$a_3(k, n) = \int_{-\gamma_1 n}^{-\delta_1 n} f(\zeta) R(k, n, \zeta) d\zeta .$$

Since $f(\zeta) \in D^{\wedge}$ we have

$$f(\zeta) = \int_{-\infty}^{\infty} g(z) e^{2\pi i z \zeta} dz$$

where $g = \psi^{-1} f$ is infinitely differentiable with compact support. Repeated integration by parts shows that

$$(1) \qquad f(\zeta) = 0(|\zeta|^{-r}) \qquad \zeta \to \pm \infty$$

for every r. Using

$$\int_{I_n} R(k, n, \zeta)^2 d\zeta = n$$

and Schwartz's inequality we see that

$$|a_2(k, n)|^2 \leqq n \int_{\delta_2 n}^{\gamma_2 n} |f(\zeta)|^2 d\zeta ,$$

and using (1) that $a_2(k, n) = 0(n^{-r})$ as $n \to \infty$ uniformly in k. Similar considerations apply to $a_3(k, n)$. If we set

$$a_1^+(k, n) = \int_{-\delta_1 n}^{\delta_2 n} h_k^{-1/2} Q_k^+[\cos(2\pi\zeta n^{-1} + \xi_0)]\Theta_n(\zeta) f(\zeta) d\zeta ,$$
$$a_1^-(k, n) = \int_{-\delta_1 n}^{\delta_2 n} h_k^{-1/2} Q_k^-[\cos(2\pi\zeta n^{-1} + \xi_0)]\Theta_n(\zeta) f(\zeta) d\zeta ,$$

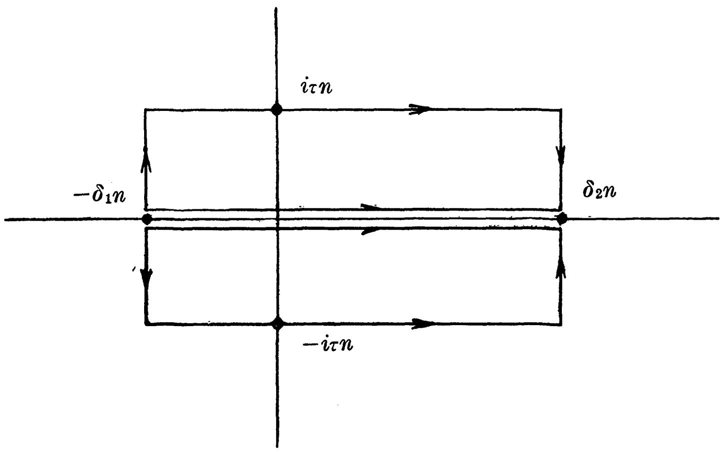
where

$$\Theta_n(\zeta) = [1 - \cos(2\pi\zeta n^{-1} + \xi_0)]^{-\alpha/2}[1 + \cos(2\pi\zeta n^{-1} + \xi_0)]^{-\beta/2} \cdot \sin^{1/2}(2\pi\zeta n^{-1} + \xi_0)\sqrt{2\pi} .$$

then

$$-\pi i a_1(k, n) = a_1^-(k, n) - a_1^+(k, n) .$$

Let us apply Cauchy's theorem to each of the curves below.



We find that

$$a_1^{\pm}(k, n) = I_1^{\pm} + I_2^{\pm} + I_3^{\pm}$$

where

$$I_1^- = \int_{-\delta_1 n}^{-\delta_1 n + i\tau n} , \qquad I_2^- = \int_{-\delta_1 n + i\tau n}^{\delta_2 n + i\tau n} , \qquad I_3^- = \int_{\delta_2 n + i\tau n}^{\delta_2 n} ,$$
$$I_1^+ = \int_{-\delta_1 n}^{-\delta_1 n - i\tau n} , \qquad I_2^+ = \int_{-\delta_1 n - i\tau n}^{\delta_2 n - i\tau n} , \qquad I_3^+ = \int_{\delta_2 n - i\tau n}^{\delta_2 n} .$$

In all cases the integrand is

$$f(\zeta) h_k^{-1/2} Q_k [\cos(2\pi\zeta n^{-1} + \xi_0)]\Theta_n(\zeta) d\zeta .$$

It is sufficient to verify that each of these six integrals is $0(n^{-r})$ as

$n \to \infty$. Since the methods of § 11 now apply almost without change the remainder of the proof for the case $f \in \boldsymbol{D}_1^\frown$ is omitted, as well as the proof for the case $f \in \boldsymbol{D}_2^\frown$.

LEMMA 18b. *Let $g \in \boldsymbol{D}_1^\frown$ or $\boldsymbol{D}_2^\frown$. Then for any nonnegative integer N and some finite constant A_N we have*

$$\lim_{n\to\infty} \int_{I_n} u^{2N} \,|\, F_n^\frown g\cdot(u) \,|^2\, du \leqq A_N \int_{-\infty}^{\infty} u^{2N} \,|\, g(u) \,|^2\, du \;.$$

We will only sketch the proof of this result. Let

$$\rho_n(u) = [\cos(2\pi u n^{-1} + \xi_0) - \cos \xi_0]^N \;.$$

Using the recursion formula and Theorem 18a it is easy to see that if $g_n(u) = \rho_n(u) g(u)$

$$\rho_n(u) F_n^\frown g\cdot(u) = F_n^\frown g_n\cdot(u) + 0(n^{-r})$$

for $u \in I_n$. It follows that

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int_{I_n} \rho_n(u)^2 \,|\, F_n^\frown g\cdot(u) \,|^2\, du &\leqq \overline{\lim_{n\to\infty}} \int_{I_n} |\, F_n^\frown g_n\cdot(u) \,|^2\, du \;, \\ &\leqq \overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} |\, g_n(u) \,|^2\, du \;, \end{aligned}$$

since $F_n^\frown$ is a projection; that is

$$\overline{\lim_{n\to\infty}} \int_{I_n} \rho_n(u)^2 \,|\, F_n^\frown g\cdot(u) \,|^2\, du \leqq \overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} \rho_n(u)^2 \,|\, g(u) \,|^2\, du \;.$$

We have

$$\cos(2\pi u n^{-1} + \xi_0) - \cos \xi_0 = -2 \sin(\pi u n^{-1}) \sin(\pi u n^{-1} + \xi_0) \;.$$

Since $0 \leqq 2\pi u n^{-1} + \xi_0 \leqq \pi$ if $u \in I_n$ we have

$$0 < \xi_0/2 \leqq \pi u n^{-1} + \xi_0 \leqq (\pi + \xi_0)/2 \qquad \text{for } u \in I_n \;.$$

It follows that there exist finite positive constants A_1 and A_2 such that

$$\begin{aligned} |\,(\cos 2\pi u n^{-1} + \xi_0) - \cos \xi_0 \,| &\leqq A_1 \,|\, u \,| \qquad & -\infty < u < \infty \;, \\ &\geqq A_2 \,|\, u \,| & u \in I_n \;, \end{aligned}$$

etc.

THEOREM 18c. *Let $f \in \boldsymbol{D}_1^\frown$ or $\boldsymbol{D}_2^\frown$. Then*

$$\lim_{n\to\infty} \|\, (S^\frown)^{1/2} F^\frown f - (S_n^\frown)^{1/2} F_n^\frown f \,\| = 0 \;.$$

Note that if $f \in D_2^\frown$, $F^\frown f = 0$.

THEOREM 18d. *$(S^\frown)^{1/2}F^\frown$ is the closure of the strong limit of $(S_n^\frown)^{1/2}F_n^\frown$ as $n \to \infty$.*

Note that the demonstration of Theorem 18d is simpler than that of its analogue, Theorem 13c, in that, because convolution is possible in E, the analogues of Lemmas 13a and 13b are completely trivial.

19. The asymptotic formula (interior maximum). In this section we will complete the theory for the case of an interior maximum giving some details. Let $S_F^\frown$ be constructed from $F^\frown$ and $S^\frown$ as in § 3. Note that if $S^\frown = \{f \mid f \in E^\frown, F^\frown f \in D(S^\frown)^{1/2}\}$ then $S^\frown$ is dense in $E^\frown$ so that $S_F^\frown$ is a self-adjoint transformation on $E^\frown$ itself. Let

$$S_F^\frown = \int_{0-}^{\infty} \lambda d\Psi^\frown(\lambda)$$

be the spectral resolution of $S_F^\frown$ on $E^\frown$, and let

$$S_{n,F}^\frown = \int_{0-}^{\infty} \lambda d\Psi_n^\frown(\lambda)$$

be the spectral resolution of $S_{n\,F}^\frown = F_n^\frown S_n^\frown F_n^\frown$. It follows from Theorems 16b, 17a, and 18b, combined with Theorem 6a that

$$\Psi_n^\frown(\lambda) \to \Psi^\frown(\lambda) \qquad 0 \leqq \lambda < \infty \tag{1}$$

for every $\lambda \notin \sigma_p(S_F^\frown)$.

Let us define

$$R^\frown = S_F^\frown\big|_{N^\frown} \qquad N^\frown = F^\frown E^\frown ,$$
$$R_n^\frown = S_{n,F}^\frown\big|_{N_n^\frown} \qquad N_n^\frown = F_n^\frown E^\frown .$$

Since, as is easily seen, $R^\frown > 0$, $R_n^\frown > 0$, we have the spectral resolutions

$$R^\frown = \int_0^\infty \lambda dE^\frown(\lambda) \text{ on } N^\frown,\ R_n^\frown = \int_0^\infty \lambda dE_n^\frown(\lambda) \text{ on } N_n^\frown$$

where

$$E^\frown(\lambda) = \Psi^\frown(\lambda) - \Psi^\frown(0) \qquad 0 \leqq \lambda < \infty ,$$
$$E_n^\frown(\lambda) = \Psi_n^\frown(\lambda) - \Psi_n^\frown(0) \qquad 0 \leqq \lambda < \infty .$$

Since $\Psi^\frown(0) = I - F^\frown$, $\Psi_n^\frown(0) = I - F_n^\frown$ it follows from (1) that

$$E_n^\frown(\lambda) \to E^\frown(\lambda) \qquad 0 \leqq \lambda < \infty \tag{2}$$

for all $\lambda \notin \sigma_p(R^\frown)$.

Lemma 19a. *With the above definitions let $f_n \in N_n^\frown$, and let $\|f_n\| = 1$, $(R_n^\frown f_n \mid f_n) \leqq m$ for $n \in \mathfrak{p}$. We assert that if $f_n \rightharpoonup f$ as $n \to \infty$ in $\mathfrak{p}_1$ then $f \neq 0$.*

Proof. If $f_n \in N_n^\frown$ then

$$f_n(u) = \frac{1}{n} \sum_{k=0}^{n} R(k, n, u) a(k, n) \qquad u \in I_n$$

and $f_n(u) = 0$ if $u \notin I_n$. Here $R(k, n, u)$ is defined as in § 17, and

$$a(k, n) = \int_{I_n} R(k, n, \zeta) f_n(\zeta) d\zeta .$$

we have

$$1 = \|f_n\|^2 = \frac{1}{n} \sum_{k=0}^{n} |a(n, k)|^2 ,$$

and therefore by Schwarz's inequality

$$|f_n(u)|^2 \leqq \frac{1}{n} \sum_{k=0}^{n} R(k, n, u)^2 .$$

y (1) of § 17 if $|u| \leqq a < \infty$ then there exists a constant M such ıat $|R(k, n, u)| \leqq M$ for $k = 0, 1, \cdots$ provided n is sufficiently large. follows that for all large n

$$|f_n(u)| \leqq M \qquad |u| \leqq a . \tag{3}$$

ext

$$|f_n'(u)|^2 \leqq \frac{1}{n} \sum_{k=0}^{n} R'(k, n, u)^2 .$$

/e assert that if $|u| \leqq a$ then for all sufficiently large n and a suitable ɔnstant M, $|R'(k, n, u)| \leqq M$ for $k = 0, 1, \cdots, n$. This inequality ɜn be reduced by means of the formula

$$2 \frac{d}{dx} P_n^{(\alpha,\beta)}(x) = (n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1,\beta+1)}(x) ,$$

o the one given above. See [1; p. 170]. We may therefore apply ›chwarz's inequality again to obtain for all sufficiently large n

$$|f_n'(u)| \leqq M \qquad |u| \leqq a . \tag{4}$$

t follows from (3) and (4) that the $\{f_n(u)\}_1^\infty$ are uniformly bounded ,nd equicontinuous on any interval $|u| \leqq a < \infty$. Therefore if $f_n \rightharpoonup f$

as $n \to \infty$ in $\mathfrak{p}_1$ we have

$$(5) \qquad \lim_{\mathfrak{p}_1} f_n(u) = f(u) \qquad |u| \leqq a$$

uniformly, provided $f(u)$ is suitably redefined on a set of measure zero.

Given $m_1 > 0$ it is easy to see that there exists a number $a > 0$ and an integer N such that if $n \geqq N$

$$(6) \qquad s_n(u) \geqq m_1 \qquad \{u \in I_n \mid |u| \geqq a\} .$$

The remainder of the proof follows the lines § 14 so closely it is omitted.

THEOREM 19b. *If*

$$0 < \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots , \quad \lim_{k\to\infty} \mu_k = \infty ,$$

are the eigen values of $R^\wedge$ *then for each* $k = 1, 2, \cdots$

$$\lambda_{n\,k} = t(x_0) - n^{-\omega} L(n^{-1})[\mu_k + o(1)]$$

as $n \to \infty$.

Let us consider as an example the case where $t(x)$ has a unique absolute maximum at x_0 and is twice continuously differentiable in a neighborhood of x_0. Then $t'(x_0) = 0$. We assume that $t''(x_0) = -\sigma^2 < 0$. Then in terms of the notation of § 15, $\omega = 2$, $\sigma_1 = \sigma_2 = \sigma^2/2$, $L \equiv 1$. Consequently

$$s(u) = \pi\sigma^2(\sin^2 \xi_0)u^2 \qquad -\infty < u < \infty ,$$

and the eigen values $0 < \mu_1 \leqq \cdots$ of $R^\wedge$ are easily seen to be the eigen values of the differential operator $R = \psi^{-1}R^\wedge\psi$ defined by

$$Rf\cdot(z) = -\frac{\sigma^2}{4}\sin^2 \xi_0 f''(z) ,$$

the domain $D(R)$ consisting of those functions $f(z)$ with support in $-1 \leqq z \leqq 1$ which are such that $f(z)$ and $f'(z)$ are absolutely continuous for $-1 < z < 1$, $f''(z) \in L^2(-1, 1)$ and $f(1-) = f(-1+) = 0$. Since $\mu_k = \sigma^2(\sin \xi_0)^2 k^2/8$ we find that

$$\lambda_{n,k} = t(x_0) - \sigma^2(1 - x_0^2)k^2/8n^2 + o(n^{-2})$$

as $n \to \infty$ for each $k = 1, 2, \cdots$. See (6) of § 1.

APPENDIX

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined in § 1 satisfy the recursion formula, [1; p. 168],

$$(1)\qquad xP_n^{(\alpha,\beta)}(x) = A_n P_{n+1}^{(\alpha,\beta)}(x) + B_n P_n^{(\alpha,\beta)}(x) + C_n P_{n-1}^{(\alpha,\beta)}(x)$$

where

$$(2)\qquad A_n = 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta+1)^{-1} \cdot(2n+\alpha+\beta+2)^{-1},$$

$$(3)\qquad B_n = -(\alpha^2-\beta^2)(2n+\alpha+\beta+2)^{-1}(2n+\alpha+\beta)^{-1},$$

$$(4)\qquad C_n = 2(n+\alpha)(n+\beta)(2n+\alpha+\beta)^{-1}(2n+\alpha+\beta+1)^{-1}.$$

We have the following limit relation

$$(5)\qquad \lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}(\cos zn^{-1}) = (z/2)^{-\alpha} J_\alpha(z)$$

uniformly for z in any bounded subset of the complex z plane, [1; p. 173].

We also have

$$(6)\qquad h_n^{1/2} P_n^{(\alpha,\beta)}(\cos\theta)[w_{\alpha,\beta}(\cos\theta)]^{1/2}\sin^{1/2}\theta - \sqrt{\frac{2}{n}}\cos(N\theta+\gamma)\to 0$$

as $n\to\infty$ uniformly for $\varepsilon \leqq \theta \leqq \pi-\varepsilon$, if $\varepsilon > 0$. Here

$$N = n + (\alpha+\beta+1)/2, \qquad \gamma = -\left(\alpha+\frac{1}{2}\right)\pi/2.$$

See [12; p. 190].

Let $q = \max(\alpha, \beta, -1/2)$; then

$$(7)\qquad |P_n^{(\alpha,\beta)}(\cos u)| \leqq A(n+1)^q \qquad -\infty < u < \infty$$

where A depends upon α and β, [12, p. 163]. Furthermore if $w = u + iv$ then, see [12; p. 190],

$$|P_n^{(\alpha,\beta)}(\cos w)| \leqq A(n+1)^{-1/2} e^{|v|n}$$

uniformly for $|v| \geqq v_0 > 0$. Here A depends only upon α, β and v_0. Applying Hadamard's three lines theorem to $P_n^{(\alpha,\beta)}(\cos w)$ we find that for all w

$$(8)\qquad |P_n^{(\alpha,\beta)}(\cos w)| \leqq A(n+1)^q e^{|v|n}$$

where A and q are independent of n and w. The inequality (8), although crude, has the advantage that it holds uniformly in n and w.

We set

$$(9)\qquad Q_n(z) = \frac{1}{2}\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)(z-t)^{-1} w_{\alpha,\beta}(t)\,dt.$$

for all complex $z \notin [-1, 1]$. We then have

$$Q_n(z) = (1-z)^\alpha (z+1)^\beta Q_n^{(\alpha,\beta)}(z) ,$$

where $Q_n^{(\alpha,\beta)}(z)$ is the standard Jacobi function of the second kind, [1; p. 170]. We will use $Q_n(z)$ rather than $Q_n^{(\alpha,\beta)}(z)$ because it is single valued in the z-plane slit from -1 to 1. If we set

$$Q_n^+(x) = \lim_{\varepsilon \to 0+} Q_n(x + i\varepsilon)$$
$$Q_n^-(x) = \lim_{\varepsilon \to 0+} Q_n(x - i\varepsilon)$$

then for $-1 < x < 1$

$$Q_n^-(x) - Q_n^+(x) = \pi i P_n^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) . \tag{10}$$

By an argument analogous to that used to prove (8), see [12; p. 219], we can show that if $v = I_m w \neq 0$ then

$$| Q_n(\cos w) \sin^2 w | \leqq A(n+1)^q e^{-|v|(n-2)} \tag{11}$$

where A and q are independent of n and w. Like (8) this inequality is quite crude, but it is important because it is uniform in n and w.

BIBLIOGRAPHY

1. A. Erdélyi et al., *Higher transcendental functions* vol. 2, New York 1953.
2. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Berkeley and Los Angeles 1958.
3. D. Guy, *Hankel multiplier transformations and weighted p-norms*, Trans. Amer. Math. Soc., **95** (1960), 137-189.
4. I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, Jour. d Anal. Math., **8** (1960-61), 307-336.
5. ———, *Extreme eigen values of Toeplitz forms associated with orthogonal polynomials*, to appear in Jour. d Anal. Math.
6. M. Kac, W. L. Murdock and G. Szegö, *On the eigen values of certain Hermitian forms*, J. Rat. Mech. Anal., **2** (1953), 767-800.
7. T. Kato, *Quadratic Forms in Hilbert spaces and asymptotic perturbation series*, Mimeographed Notes, Berkeley 1955.
8. R. Nevanlinna, *Eindeutige analytische functionen* Berlin 1936.
9. S. V. Parter, *Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations*, Trans. Amer. Math. Soc., **99** (1961), 153-193.
10. ———, *On the extreme eigenvalues of truncated Toeplitz matrices*, Bull. Amer. Math. Soc., **67** (1961), 191-196.
11. ———, *On the extreme eigenvalues of Toeplitz matrices*, Trans. Amer. Math. Soc., **100** (1961), 263-276.
12. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Col. Pub., 23, New York, 1939.
13. B. von Szegy-Nagy, *Spectraldarstellung Linearer Transformationen des Hilbertschen Raumes*, Berlin 1942.
14. H. Widom, *On the eigenvalues of certain Hermitian operators*, Trans. Amer. Math.

Soc., **88** (1958), 491-522.
15. H. Widom, *Stable procesess and integral equations,* Trans. Amer. Math. Soc., **98** (1961), 430-439.
16. ———, *Extreme eigenvalues of translation kernels,* Trans. Amer. Math. Soc., **100** (1961), 252-262.
17. ———, *Extreme eigenvalues of N-dimensional convolution operators,* Trans. Amer. Math. Soc., **106** (1963), 391-414.