LEAST SQUARES AND INTERPOLATION IN ROOTS OF UNITY

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We mention the Erdös-Turán theorem [2] that if $F(\theta)$ is a real continuous function with period 2π, and if $t_n(\theta)$ is the unique trigonometric polynomial of order n that coincides with $F(\theta)$ in $2n+1$ points equally spaced over an interval of length 2π, then $t_n(\theta)$ converges to $F(\theta)$ on that interval in the mean of second order. It is the purpose of the present note to prove the analogue in the complex domain, and to discuss some related remarks.

THEOREM 1. *Let the function $f(z)$ be analytic in $D\colon |z|<1$, continuous in $D+C\,(C\colon |z|=1)$, and let $p_n(z)$ be the polynomial of degree n coinciding with $f(z)$ in the $(n+1)$ st roots of unity. Then the sequence $p_n(z)$ converges to $f(z)$ on C in the mean of second order. Consequently we have*

$$\lim_{n\to\infty} p_n(z) = f(z) \qquad \textit{uniformly in } |z| \leqq r\,(<1)\,. \tag{1}$$

If we set

$$I_n = \int_C |f(z) - p_n(z)|^2\, |dz|\,, \tag{2}$$

we have

$$p_n(z) \equiv \sum_{k=1}^{n+1} f(\omega^k) A_k(z)\,,$$

$$A_k(z) \equiv \frac{\omega^k(z^{n+1}-1)}{(n+1)(z-\omega^k)}\,, \qquad \omega = e^{2\pi i/(n+1)}\,,$$

and shall show

$$\lim_{n\to=\infty} I_n = 0\,. \tag{3}$$

We introduce the notation

$$f(z) - t_n(z) \equiv \Delta(z)\,, \quad E_n = \max\,[|\Delta z|, z \text{ on } C]\,,$$

where $t_n(z)$ is the polynomial of degree n of best Tchebycheff approximation to $f(z)$ on C, and denote by $P_n(z)$ the polynomial of degree n that coincides with $\Delta(z)$ in the $(n+1)$ st roots of unity. Then we have $P_n(z) \equiv p_n(z) - t_n(z)$, whence

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$$I_n = \int_C |\Delta(z) - P_n(z)|^2 |dz|$$
$$\leqq 2\int_C |\Delta z|^2 |dz| + 2\int_C |P_n(z)|^2 |dz| = I_n + I_n'' .$$

There follow the relations $I_n' \leqq 4\pi E_n^2$,

$$I_n'' = 2\int_C \left| \sum_{k=1}^{n+1} \Delta(\omega^k) A_k(z) \right|^2 |dz|$$
$$\leqq 2\sum_{k=1}^{n+1}\sum_{j=1}^{n+1} |\Delta(\omega^k)\bar{\Delta}(\omega^j)| \left| \int_C A_k(z)\bar{A}_j(z)\, |dz| \right|$$
$$\leqq 2E_n^2 \sum_{k=1}^{n+1}\sum_{j=1}^{n+1} \left| \int_C A_k(z)\bar{A}_j(z)\, |dz| \right| .$$

However, we have

$$A_k(z) \equiv \frac{\omega^k}{n+1}(z^n + \omega^k z^{n-1} + \cdots + \omega^{nk}) ,$$
$$\int_C A_k(z)\bar{A}_j(z)\, |dz| = \frac{2\pi\omega^{k-j}}{(n+1)^2}(1 + \omega^{k-j} + \omega^{2(k-j)} + \cdots + \omega^{n(k-j)})$$
$$= 2\pi\delta_{jk}/(n+1) ,$$

where δ_{jk} is the Kronecker δ. It is well known [4, Theorem 5, p. 36] that $E_n \to 0$ as $n \to \infty$, so (3) holds.

Equation (1) follows from (3) by the Cauchy integral formula

$$(4) \qquad [f(z) - p_n(z)]^2 = \frac{1}{2\pi i}\int_C \frac{[f(t) - p_n(t)]^2\, dt}{t - z} , \qquad |z| < 1 .$$

With the hypothesis of Theorem 1, the conclusion (1) is due to Fejér (1918). Theorem 1 is related to various other results concerning convergence of polynomials interpolating in roots of unity; for instance (Runge) if $f(z)$ in Theorem 1 is analytic in $|z| \leqq 1$, equation (1) holds uniformly in $|z| \leqq 1$. Further references to the subject are given by Curtiss [1].

There exist numerous other results, somewhat similar to Theorem 1, where now a sequence of polynomials $P_n(z, 1/z)$ of respective degrees n in z and $1/z$ converges on C in the second-order mean to a given function $f(z)$ defined merely on C. The function $f(z)$ can be expressed on C as $f(z) \equiv f_1(z) + f_2(z)$, where $f_1(z)$ is of the Hardy-Littlewood class H_2 and $f_2(z)$ of the analogous class G_2 for the region $|z| > 1$ (we suppose $f_2(\infty) = 0$; compare e.g. [4, § 6. 11]). Any function of class H_2 is orthogonal on C to any function of class G_2, so if we set $P_n(z, 1/z) \equiv p_n(z) + q_n(1/z)$, where $p_n(z)$ and $q_n(1/z)$ are polynomials of respective degrees n in their arguments, $q_n(0) = 0$, we have

$$(5) \qquad \lim_{n\to\infty} p_n(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t - z} \equiv f_1(z) \equiv \frac{1}{2\pi i}\int_C \frac{f_1(t)dt}{t - z}, \quad z \text{ interior to } C ,$$

$$(6) \qquad \lim_{n\to\infty} q_n(1/z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} \equiv f_2(z) \equiv \frac{1}{2\pi i}\int_C \frac{f_2(t)dt}{t-z}\,, \qquad z \text{ exterior to } C\,,$$

with uniformity of approach for z on an arbitrary compact set in the respective regions. This remark concerning (5) and (6) applies for instance in the case of the Erdös-Turán theorem, where we set $F(\theta) \equiv f(e^{i\theta})$ and $t_n(\theta) \equiv p_n(e^{i\theta}, e^{-i\theta})$ on C.

A second remark concerning (5) and (6) is as follows. By the orthogonality relations we have for the second-order norms on C

$$\|f - P_n\|^2 = \|f_1 - p_n\|^2 + \|f_2 - q_n\|^2 .$$

Consequently the rapidity of convergence in the mean on C of p_n to f_1 and of q_n to f_2 is not less than that of P_n to f. If the positive numbers $\varepsilon_1, \varepsilon_2, \cdots$ are given and approach zero, there is a corresponding class K of functions $f(z)$ belonging to L_2 on C such that for each $f(z)$ there exist polynomials $P_n(z, 1/z)$ with

$$(7) \qquad \|f(z) - P_n(z, 1/z)\| = O(\varepsilon_n)\,;$$

here the $P_n(z, 1/z)$ may be taken as the partial sums of the Fourier or Laurent development of $f(z)$ on C. It follows that every function $f(z)$ in K can be written $f(z) \equiv f_1(z) + f_2(z)$, with f_1 and f_2 in H_2 and G_2 respectively, where the partial sums $p_n(z)$ of the Taylor development of $f_1(z)$ satisfy

$$(8) \qquad \|f_1(z) - p_n(z)\| = O(\varepsilon_n)$$

and the partial sums $q_n(1/z)$ of the Laurent development of $f_2(z)$ satisfy

$$(9) \qquad \|f_2(z) - q_n(1/z)\| = O(\varepsilon_n)\,.$$

Thus f_1 and f_2 belong to K on C.

As a particular case of this application, we mention the class of functions $L(2, k, \alpha)$, $o < \alpha < 1$, namely the class of functions whose kth derivatives on C satisfy there a square-integrated Lipschitz condition of order α; this class was first studied by Hardy and Littlewood, theorems proved in detail by Quade [3]. An alternative definition of the class is (7) with $\varepsilon_n = 1/n^{k+\alpha}$. It follows that every function $f(z)$ of class $L(2, k, \alpha)$, $o < \alpha < 1$, can be expressed on C as $f_1(z) + f_2(z)$, where the latter two functions, of respective classes H_2 and G_2 satisfy (8) and (9) with the same values of ε_n; thus $f_1(z)$ and $f_2(z)$ likewise belong to $L(2, k, \alpha)$ on C. The case $\alpha = 1$ can be similarly treated, where the integrated Lipschitz conditions are replaced by the condition

$$(10) \qquad \int_0^{2\pi} |F(\theta + h) + F(\theta - h) - 2F(\theta)|^2\, d\theta = O(h^2)\,,$$

and $F(\theta) \equiv f^{(k)}(z)$ is continuous on C; this class was introduced by Zygmund, and is characterized by (7) with $\varepsilon_n = 1/n^{k+1}$. We have as before $f(z) \equiv f_1(z) + f_2(z)$ if $f(z)$ is given, and the corresponding classes of $f_1(z)$ and $f_2(z)$ are characterized by (8) and (9) with the same values of ε_n, and by (10). These classes of analytic functions are studied in [5].

Added in proof. A second proof of Theorem 1, due to G. H. Curtiss, will appear shortly.

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