ON THE DIFFERENCE AND SUM OF A BASIC SET OF POLYNOMIALS

W. F. NEWNS

1. Introduction. For any basic set (p_n) of polynomials, the differenced set (u_n) and the sum (v_n) have been defined and studied by Mikhail & Nassif [1, 2], who obtained the best possible bound for the orders of (u_n) and (v_n) when (p_n) has a given order ω. Their method was to estimate directly the expressions for the orders of (u_n) and (v_n).

The object of the present note is to indicate how these results can be obtained by an alternative line of reasoning which the author believes may throw more light on them. He observes also that either approach can be used to go a little further and determine not only the order but the type of the sets. In fact:

THEOREM 1. *If* (p_n) *is of increase* (ω, γ), *then* (u_n) *has increase at most* $\max\{(\omega, \gamma), (1, 1/2\pi)\}$.

THEOREM 2. *Let* (p_n) *have increase* (ω, γ). *Then*

(i) *If* $\limsup D_n/n = \alpha < \infty$, (v_n) *has increase at most* $(\omega + \alpha, \infty)$,

(ii) *If* $D_n^{D_n/n} = 0(n^\alpha)$ *and* $\gamma < \infty$ (*so that* $\omega > 0$), *the increase of* (v_n) *is at most* $(\omega + \alpha, 0)$.

Case (ii) *applies in particular* (*with* $\alpha = 1$) *to simple sets.*

2. Spaces of integral functions. Let f be an integral function, ρ its order. If $0 < \rho < \infty$, the rate of increase of f is (ρ, σ) where σ is the type of f. If $\rho = 0$ we put $\sigma = \infty$, and if $\rho = \infty$ we put $\sigma = 0$ and again define the rate of increase of f as (ρ, σ). We use lexicographic order, so that $(\rho_1, \sigma_1) \leqq (\rho_2, \sigma_2)$ means that either $\rho_1 < \rho_2$ or $\rho_1 = \rho_2$ and $\sigma_1 \leqq \sigma_2$.

The set $I(\rho, \sigma)$ of all integral functions of increase not exceeding (ρ, σ) is a vector space under the usual operations. The space $I(\infty, 0)$ of all integral functions is an $\mathscr{F}$-space under the topology of uniform convergence on compact sets (the compact-open topology). If $\rho < \infty$, $I(\rho, \sigma)$ is an $\mathscr{F}$-space under a (unique) topology $\mathscr{T}(\rho, \sigma)$ finer than that induced on it by the topology $\mathscr{T}(\infty, 0)$ of $I(\infty, 0)$, (c.f. [3] § 5, p. 438). These may be defined as follows. Put

$$|f|_{\infty,r} = \sum |a_k| r^k$$
$$|f|_{\rho,r} = \sum (k/e\rho)^{k/\rho} |a_k| r^k$$

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for $f(z) = \sum a_n z^n$. Then $\mathscr{T}(\rho, 0)$ is defined by the semi-norms $|f|_{\rho,r}$ for all finite r, $\mathscr{T}(\rho, \infty)$ by $|f|_{\rho_1,r}$ for $\rho_1 > \rho$ and all finite r, and $\mathscr{T}(\rho, \sigma)$ for $0 < \sigma < \infty$ by $|f|_{\rho,r}$ for $r < \sigma^{-1/\rho}$.

We denote by $I_0(\rho, \sigma)$ the set of those functions of $I(\rho, \sigma)$ which vanish at the origin: $I_0(\rho, \sigma) = \{g \in I(\rho, \sigma): g(0) = 0\}$. $I_0(\rho, \sigma)$, being a closed subspace of $I(\rho, \sigma)$, is an $\mathscr{F}$-space under the induced structure.

3. Rate of increase of a basic set. As with functions, we define the rate of increase of a basic set to be the pair (ω, γ) where ω is the order, γ the type if $0 < \omega < \infty$ and similar conventions where $\omega = 0$ or ∞. We again use lexicographic order and recall the following result [6]:

THEOREM 3. *A basic set (p_n) is of increase not exceeding (ω, γ) if and only if it is effective for $I(\rho, \sigma)$ in $\mathscr{T}(\infty, 0)$ for all $(\rho, \sigma) < (1/\omega, 1/\gamma)$.*

4. The difference operator. For any integral function g we put $\Delta g = f$, where

$$f(z) = g(z+1) - g(z) .$$

THEOREM 4. *The difference operator Δ is a continuous linear mapping of $I(\rho, \sigma)$ onto itself.*

A proof that Δ is a linear mapping of $I(\rho, \sigma)$ onto itself will be found in [5] (pp. 21–24) and [4][1] (Theorem I). Continuity of Δ for the compact-open topology (induced by $\mathscr{T}(\infty, 0)$) is easily checked. Continuity for $\mathscr{T}(\rho, \sigma)$ now follows from the closed graph theorem.

Clearly Δ is not a bijection: its kernel contains not only constants but any function of period 1. Since the only functions of period 1 and increase less than $(1, 2\pi)$ are constants, we have:

THEOREM 5. *If $(\rho, \sigma) < (1, 2\pi)$, then Δ is an isomorphism between the $\mathscr{F}$-spaces $I_0(\rho, \sigma)$ and $I(\rho, \sigma)$.*

Under the hypotheses of Theorem 5, $\Delta: I_0(\rho, \sigma) \to I(\rho, \sigma)$ has a continuous inverse $\mathscr{S}: I(\rho, \sigma) \to I_0(\rho, \sigma)$. If $f = \Delta g$ we have $g = \mathscr{S} f$ and call g the *sum* of f.

5. The differenced set. In defining the differenced set (u_n) of a given basic set (p_n), there is no loss of generality in taking $p_0(z) = 1$.

[1] For this reference, which he had failed to trace, the author is indebted to Dr. J. M. Whittaker.

Then

$$u_n = \Delta p_{n+1}$$

and the set (u_n) is basic with respect to the representation

$$z^n = \sum_0^\infty \Pi_{k+1}(\phi_{n+1}) u_k(z) ,$$

where

$$\phi_{n+1}(z) = \mathscr{S} z^n .$$

To prove Theorem 1, let the increase of (p_n) be (ω, γ). If ω is infinite there is nothing to prove, so we suppose $\omega < \infty$. Let $(\rho, \sigma) < \min \{(1/\omega, 1/\gamma), (1, 2\pi)\}$, $\sigma < \infty$ and let $f \in I(\rho, \sigma)$. Then $g = \mathscr{S} f \in I(\rho, \sigma)$ and (Theorem 3)

$$g = \sum_0^\infty \Pi_k(g) p_k \qquad (\mathscr{T}(\infty, 0)) .$$

Since Δ is continuous in $\mathscr{T}(\infty, 0)$,

$$f = \Delta g = \sum_0^\infty \Pi_k(g) \Delta p_k = \sum_1^\infty \Pi_k(g) u_{k-1} = \sum_0^\infty \Pi_{k+1}(g) u_k ,$$

showing that f is represented in $\mathscr{T}(\infty, 0)$ by a series of the required form. To prove that this is the basic series of f, it is obvious that $f \to \Pi_{k+1}(g)$ is continuous (being composed of the continuous functions $\mathscr{S}$ and Π_{k+1}) and hence the series is basic under the inverse matrix

$$\Pi_{k+1}(\mathscr{S} z^n) = \Pi_{k+1}(\phi_{n+1}) .$$

Theorem 1 now follows from Theorem 3.

REMARK. Nothing in this argument depends on the $p_n(z)$ being polynomials. They may be integral functions of any order.

6. **The sum of a basic set.** Given a basic set (p_n) of polynomials,[2] the sum (v_n) is defined by

$$v_n = \mathscr{S} p_{n-1} \quad (n > 0) , \qquad v_0 = 1.$$

This set is basic with respect to the representation

$$z^n = \sum_1^\infty \Pi_{k-1}(\vartheta_{n-1}) v_k \qquad (n > 0) ,$$

where $\vartheta_{n-1}(z) = \Delta z^n$.

[2] In the definition of (v_n) we could allow the (p_n) to be integral functions of increase $<(1, 2\pi)$. However, Theorem 2 applies only to sets of polynomials.

Proceeding heuristically, let f be given (with $f(0) = 0$) and put $g = \Delta f$. Then

$$g = \sum_0^\infty \Pi_k(g) p_k \tag{1}$$

and we obtain

$$f = \mathscr{S} g = \sum_0^\infty \Pi_k(g) \mathscr{S} p_k = \sum_0^\infty \Pi_k(g) v_{k+1} \tag{2}$$

a series with continuous coefficients (composed of Δ and Π_{k-1}) which is therefore basic under

$$\Pi_{k-1}(\Delta z^n) = \Pi_{k-1}(\vartheta_{n-1}) .$$

This argument is valid for all $f \in I_0(\rho_0, \sigma_0)$ only if (ρ_0, σ_0) satisfies certain requirements. For equation (1) to hold in (say) $\mathscr{T}(\infty, 0)$ we need $(\rho_0, \sigma_0) < (1/\omega, 1/\gamma)$. For $\mathscr{S}$ to be well-defined, we need $(\rho_0, \sigma_0) < (1, 2\pi)$. But to apply $\mathscr{S}$ to (1) to obtain (2), we need (1) to hold in a topology $\mathscr{T}(\rho_1, \sigma_1)$ in which $\mathscr{S}$ is continuous, i.e. one for which $(\rho_1, \sigma_1) < (1, 2\pi)$. The problem arises as to which (ρ_0, σ_0) will satisfy these requirements and the answer is given by:

THEOREM 6. *Suppose that* (p_n) *is effective for* $I(\rho, \sigma)$ *in* $\mathscr{T}(\infty, 0)$, $(0 < \rho < \infty,\ 0 < \sigma < \infty)$, *and that* $D_n^{D_n/n} = 0(n^\beta)$. *Given* $\rho_1 (0 < \rho_1 < \infty)$ *put* $(1/\rho_0) = (1/\rho) + (\beta/\rho_1)$. *Then* (p_n) *is effective for* $I(\rho_0, \sigma_0)$ *in* $\mathscr{T}(\rho_1, 0)$ *for all finite* σ_0.

We first complete the proof of Theorem 2. For case (i), let $\rho_0 < (1/\omega + \alpha)$ and choose $\beta > \alpha$ such that $\rho_0 < (1/\omega + \beta)$. Put $(1/\rho_0) = (1/\rho) + \beta$ so that $\rho < (1/\omega)$. The hypotheses of Theorem 6 hold with $\rho_1 = 1$ and so the heuristic argument above holds for (ρ_0, σ_0) for any finite σ_0. This being true for any $\rho_0 < (1/\omega + \alpha)$, case (i) follows from Theorem 3.

For case (ii), we put $\rho = (1/\omega)$, $\beta = \alpha$ and choose $\sigma < (1/\gamma)$. We conclude similarly that (v_n) is effective for $I(\rho_0, \sigma_0)$ in $\mathscr{T}(\infty, 0)$ when $(1/\rho_0) = \omega + \alpha$ and σ_0 is finite. By Theorem 3, this is equivalent to the stated result.

We now prove Theorem 6. Put

$$\gamma = \begin{cases} \sup\{\beta - D_n/n\} & (e\rho_1 \geqq 1) \\ \inf\{\beta - D_n/n\} & (e\rho_1 < 1) . \end{cases}$$

Since $D_n \geqq n$ and $\limsup D_n/n \leqq \beta$, γ is finite. Also we are dealing with a Cannon set so that effectiveness is equivalent to absolute effectiveness. Let $0 < \sigma_0 < \infty$. We have to prove ([3], §§ 7, 8): given

$r_1 < \infty$, there exist M and $r_0 < \sigma_0^{-1/\rho_0}$ such that

$$\sum_l |\pi_{nl}| \sum_k \left(\frac{k}{e\rho_1}\right)^{k/\rho_1} |p_{lk}|\, r_1^k \leqq M\left(\frac{n}{e\rho_0}\right)^{n/\rho_0} r_0^n \; .$$

Put $s = \rho_0^{1/\rho_0}\rho^{-1/\rho}\rho_1^{-\beta/\rho_1}c^{1/\rho_1}(e\rho_1)^{\;/\rho_1}\sigma^{-1/\rho}\sigma_0^{1/\rho_0}$ where c is chosen large enough for $s \geqq 1$ and $D_n^{D_n/n} \leqq cn^\beta$. The left-hand member of the inequality to be proved may be written

$$\sum_l |\pi_{nl}| \sum_k \left(\frac{k}{e\rho_1}\right)^{k/\rho_1} s^{-k}\, |p_{lk}|\, (r_1 s)^k \; .$$

The largest value of k appearing in this is D_n. Since the sequence $(k/e\rho_1)^{k/\rho_1}s^{-k}$ increases to ∞ from some point on, we have $(k/e\rho_1)^{k/\rho_1}s^{-k} \leqq A(D_n/e\rho_1)^{D_n/\rho_1}s^{-D_n}$ for some A and all n. Also

$$\sum_l |\pi_{nl}| \sum_k |p_{lk}|\, (r_1 s)^k \leqq B \sum_l |\pi_{nl}|\, M_l(R) = B\omega_n(R)$$

for $R > r_1 s$, and since (p_n) is effective for $I(\rho, \sigma)$ in $\mathscr{T}(\infty, 0)$, there exist C and $r < \sigma^{-1/\rho}$ such that

$$\omega_n(R) \leqq C\left(\frac{n}{e\rho}\right)^{n/\rho} r^n \; .$$

Finally, since $D_n \geqq n$ and $s \geqq 1$ we have $s^{-D_n} \leqq s^{-n}$. Thus the left-hand member of the inequality to be proved does not exceed

$$\begin{aligned} &A\left(\frac{D_n}{e\rho_1}\right)^{D_n/\rho_1} s^{-n} BC\left(\frac{n}{e\rho}\right)^{n/\rho} r^n \\ &\quad \leqq ABCc^{n/\rho_1}\left(\frac{n}{e\rho_1}\right)^{\beta n/\rho_1}(e\rho_1)^{(\beta n - D_n)/\rho_1}s^{-n}\left(\frac{n}{e\rho}\right)^{n/\rho} r^n \leqq M\left(\frac{n}{e\rho_0}\right)^{n/\rho_0} r_0^n \; , \end{aligned}$$

where $r_0 = \rho_0^{1/\rho_0}c^{1/\rho_1}\rho_1^{-\beta/\rho_1}(e\rho_1)^{\;/\rho_1}s^{-1}\rho^{-1/\rho}r < \sigma_0^{-1/\rho_0}$, as required.

7. **Examples.** Let (ν_n) be a sequence of even nonnegative integers, (γ_n) a sequence of real numbers and ω a nonnegative real number. Consider the set

$$\begin{aligned} p_{2n}(z) &= z^{2n} \; , \\ p_{2n+1}(z) &= z^{2n+1} + ((2n+1)!)^{\omega}\gamma_n^{2n+1}z^{\nu_n} \; . \end{aligned}$$

EXAMPLE (i). $\nu_n = 2n$, $\gamma_n = \log(2n+1)$. It will be found that (p_n) has increase (ω, ∞) and (v_n) has increase $(\omega + 1, \infty)$.

EXAMPLE (ii). $\nu_n = 2n$, $\gamma_n = (1/\log(2n+1))$ $(n > 0)$, $\omega > 0$. Here (p_n) is of increase $(\omega, 0)$ and (v_n) of increase $(\omega + 1, 0)$.

EXAMPLE (iii). Choose ν_n so that $\nu_n/2n \to \alpha \geqq 1$, but $((\nu_n!)^{1/2n}/(2n)^{\alpha}) \to$

∞. Put $\gamma_n = \sqrt{((2n)^\alpha/(\nu_n!)^{1/2n})}$, $\omega > 0$. Here $\limsup D_n/n = \alpha$ and (p_n) is of increase $(\omega, 0)$, but (v_n) is of increase $(\omega + \alpha, \infty)$.

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THE UNIVERSITY, LIVERPOOL