ON COMPARABLE MEANS

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1. Let $-\infty < a < b < \infty$, and let Φ denote the set of all functions, continuous and strictly monotone in $[a, b]$. For every $\varphi \in \Phi$, every positive integer n, every $x_1, x_2, \cdots, x_n$ of $[a, b]$, and every positive $q_1, q_2, \cdots, q_n$ with $\sum_{\nu=1}^{n} q_\nu = 1$, we consider the mean

$$M_\varphi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n) = \varphi^{-1}(\textstyle\sum_{\nu=1}^{n} q_\nu \varphi(x_\nu)) .$$

Let ψ and χ be elements of Φ. We write

$$(1) \qquad M_\psi \leqq M_\chi$$

if and only if the inequality $M_\psi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n) \leqq M_\chi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n)$ holds for every $n \geqq 1$, every $x_1, x_2, \cdots, x_n$ of $[a, b]$, and every positive $q_1, q_2, \cdots, q_n$ with $\sum_{\nu=1}^{n} q_\nu = 1$.

A well-known necessary and sufficient condition for (1) to hold is that $\chi(\psi^{-1}(x))$ be convex in $[\psi(a), \psi(b)]$ (or $[\psi(b), \psi(a)]$) if χ is increasing, and that $\chi(\psi^{-1}(x))$ be concave there if χ is decreasing.

It is not difficult to see that (1) holds if and only if $M_\psi(x_1, x_2 \mid q_1, q_2) \leqq M_\chi(x_1, x_2 \mid q_1, q_2)$ for every x_1, x_2 of $[a, b]$ and every positive q_1, q_2 with $q_1 + q_2 = 1$, which in turn holds if and only if $M_\psi(x_1, x_2 \mid 1/2, 1/2) \leqq M_\chi(x_1, x_2 \mid 1/2, 1/2)$ for every x_1, x_2 of $[a, b]$.

Similarly, we write

$$(2) \qquad M_\psi < M_\chi$$

if and only if the inequality

$$M_\psi(x_1, x_2, \cdots, x_n \mid q_1, q_2 \cdots, q_n) < M_\chi(x_1, x_2, \cdots, x_n \mid q_1, q_2 \cdots, q_n)$$

holds for every $n \geqq 2$, every $x_1, x_2, \cdots, x_n$ (not all equal) of $[a, b]$, and every positive $q_1, q_2, \cdots, q_n$ with $\sum_{\nu=1}^{n} q_\nu = 1$. A necessary and sufficient condition for (2) to hold is that $\chi(\psi^{-1}(x))$ be strictly convex in $[\psi(a), \psi(b)]$ (or $[\psi(b), \psi(a)]$) if χ is increasing, and that $\chi(\psi^{-1}(x))$ be strictly concave there if χ is decreasing. Also, (2) holds if and only if $M_\psi(x_1, x_2 \mid q_1, q_2) < M_\chi(x_1, x_2 \mid q_1, q_2)$ for every x_1, x_2 $(\neq x_1)$ of $[a, b]$ and every positive q_1, q_2 with $q_1 + q_2 = 1$, which in turn holds if and only if $M_\psi(x_1, x_2 \mid 1/2, 1/2) < M_\chi(x_1, x_2 \mid 1/2, 1/2)$ for every x_1 and x_2 $(\neq x_1)$ of $[a, b]$.

2. In this paper we give simple criteria for the validity of (1)

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and of (2), and then we give a few applications.

THEOREM 1. *Let ψ and χ be elements of Φ differentiable in (a, b), and let $\psi' \neq 0$ there. A necessary and sufficient condition for* (1) *to hold is that χ'/ψ' be nondecreasing in (a, b) if ψ and χ are monotone in the same sense, and that χ'/ψ' be nonincreasing there if ψ and χ are monotone in opposite senses.*

Proof. Consider the function $u(x) \equiv \chi(\psi^{-1}(x))$. Let J denote the open interval joining $\psi(a)$ to $\psi(b)$, and let $\bar{J}$ be the closure of J. For every $\xi \in J$, we have

$$u'(\xi) = \chi'(\psi^{-1}(\xi))/\psi'(\psi^{-1}(\xi)) \; . \tag{3}$$

Suppose that ψ and χ are monotone in the same sense. Then (1) holds if and only if $u(x)$ is convex in $\bar{J}$ in case χ increases, and if and only if $u(x)$ is concave there in case χ decreases. So (1) holds if and only if $u'(x)$ is nondecreasing in J in case ψ increases, and if and only if $u'(x)$ is nonincreasing there in case ψ decreases. From this, with the aid of (3), one easily infers that (1) is equivalent to χ'/ψ' being nondecreasing in (a, b). Similariy one shows that (1) is equivalent to χ'/ψ' being nonincreasing in (a, b), if ψ and χ are monotone in opposite senses.

One can modify Theorem 1 by replacing in it (1) by (2), "nondecreasing" by "strictly increasing," and "nonincreasing" by "strictly decreasing."

3. Given a function ψ, one may construct by means of Riemann-Stieltjes integrals functions χ such that $M_\psi \leqq M_\chi$. In fact, we have the following

THEOREM 2. *Let ψ be a real function, continuous in $[a, b]$ and differentiable in (a, b). Let $m(x)$ be nondecreasing or nonincreasing in $[a, b]$, continuous in (a, b), and suppose $m(x)\psi'(x) \neq 0$ throughout (a, b). Let C be a real constant, and for every $x \in [a, b]$ let*

$$\chi(x) = C + \int_a^x m(t) d\psi(t) \; .$$

Then ψ and χ belong to Φ. If $m(x)$ is positive in (a, b) and nondecreasing in $[a, b]$, or negative in (a, b) and nonincreasing in $[a, b]$, then $M_\psi \leqq M_\chi$. Otherwise, $M_\chi \leqq M_\psi$.

Proof. Since $\psi' \neq 0$ in (a, b), by a well known property of the derivative, ψ' is either positive throughout (a, b), or negative throughout (a, b). Thus ψ is strictly monotone in $[a, b]$. Also, by well-known properties of the Riemann-Stieltjes integral, χ is continuous in $[a, b]$, and $\chi'(x) = m(x)\psi'(x)$ throughout (a, b) (and so χ is strictly monotone in $[a, b]$). If $m(x)$ is positive in (a, b) and nondecreasing in $[a, b]$, then ψ and χ are monotone in the same sense in $[a, b]$, χ'/ψ' is nondecreasing in (a, b), and hence by Theorem 1, $M_\psi \leqq M_\chi$. Similarly the rest of Theorem 2 follows.

Theorem 2 can be modified by replacing in it "nondecreasing" by "strictly increasing," "nonincreasing" by "strictly decreasing," "$M_\psi \leqq M_\chi$" by "$M_\psi < M_\chi$," and "$M_\chi \leqq M_\psi$" by "$M_\chi < M_\psi$."

4. A converse of Theorem 2 is the following

THEOREM 3. *Let ψ and χ be elements of Φ differentiable in (a, b), and suppose $\psi' \neq 0$ there. Suppose, furthermore, that $M_\psi \leqq M_\chi$. Then there exists a function $m(x)$, nondecreasing in (a, b) if ψ and χ are monotone in the same sense, and nonincreasing there if ψ and χ are monotone in opposite senses, such that throughout* $[a, b]$

$$\chi(x) = \chi(a) + \int_a^x m(t)\psi'(t)dt \qquad \text{(a Lebesgue integral)}\,. \tag{4}$$

Proof. For every $x \in (a, b)$, let $m(x) = \chi'(x)/\psi'(x)$. By Theorem 1, $m(x)$ has the monotonicity property steated in Theorem 3. Now for every $x \in [a, b]$

$$\chi(x) - \chi(a) = \int_a^x \chi'(t)dt = \int_a^x m(t)\psi'(t)dt$$

(cf. [5], Theorems 269 (p. 188) and 264 (p. 183)).

REMARK. Observe that the integral in (4) can be written, under appropriate conditions, as a Riemman-Stieltjes integral: $\int_a^x m(t)\,d\psi(t)$. [Cf. loc. cit, Theorem 322.1 (p. 254), and 322 (p. 253)].

Theorem 3 remains valid if we replace in it "$M_\psi \leqq M_\chi$" by "$M_\psi < M_\chi$," "nondecreasing" by "strictly increasing," and "nonincreasing" by "strictly decreasing."

5. It is known that if the end-point a is positive and $r < s$, $rs \neq 0$, then $M_{x^r} < M_{x^s}$, and $M_{x^{-|r|}} < M_{\log x} < M_{x^{|r|}}$. Consequently, if $a > 0$ then for every real r $(\neq 0, 1)$, $M_{(x^r)'} < M_{x^r}$, and $M_{(\log x)'} < M_{\log x}$. The question thus arises: Under what conditions on a function φ does one have $M_{\varphi'} < M_\varphi$ (or $M_{\varphi'} \leqq M_\varphi$)?

THEOREM 4. *A necessary and sufficient condition for a real* *function* φ *to fulfill the conditions* (α)–(γ) *below is that* $\varphi(x)$ *should be (throughout* $[a, b]$) *of one of the forms* $A + \int_a^x \exp C(t)dt$, $A - \int_a^x \exp C(t)dt$, $A + \int_a^x \exp\{-C(t)\}dt$, $A - \int_a^x \exp\{-C(t)\}dt$, *where* A *is a real number, and* $C(t)$ *is a function, continuous and convex in* $[a, b]$, *differentiable in* (a, b), *and satisfying there* $C'(x) < 0$.

(α) φ *is twice differentiable in* (a, b), $\varphi'(a)$ *and* $\varphi'(b)$ *exist as right and left hand derivatives, respectively,* $\varphi'(a)\varphi'(b) \neq 0$, *and* φ' *is continuous in* $[a, b]$.

(β) $\varphi'\varphi'' \neq 0$ *throughout* (a, b) (*and hence* φ *and* φ' *are strictly monotone in* $[a, b]$).

(γ) $M_{\varphi'} \leqq M_\varphi$.

Proof.

Necessity. By Theorem 1, φ'/φ'' is either positive and nondecreasing in (a, b), or negative and nonincreasing there. Thus, φ''/φ' is either positive and nonnincreasing in (a, b), or negative and nondecreasing there. In the first case we set $C(x) = -\log|\varphi'(x)|$ (in $[a, b]$). Then $C(x)$ is continuous in $[a, b]$ and $C'(x) < 0$ in (a, b). Also $C'(x)$ is nondecreasing in (a, b), and, therefore, $C(x)$ is convex in $[a, b]$. Either for every $x \in [a, b]$, $\varphi(x) = \varphi(a) + \int_a^x \exp\{-C(t)\}dt$, or for every $x \in [a, b]$, $\varphi(x) = \varphi(a) - \int_a^x \exp\{-C(t)\}dt$. In the second case, we set $C(x) = \log|\varphi'(x)|$ (in $[a, b]$). Then $C(x)$ is continuous in $[a, b]$, $C'(x) < 0$ in (a, b), and, again, $C(x)$ is convex in $[a, b]$. Either for every $x \in [a, b]$, $\varphi(x) = \varphi(a) + \int_a^x \exp C(t)dt$, of for every $x \in [a, b]$, $\varphi(x) = \varphi(a) - \int_a^x \exp C(t)dt$.

Sufficiency. (α) and (β) clearly hold. Also, by the convexity of $C(t)$, $C'(t)$ is nondecreasing in (a, b). Now, either throughout (a, b), $\varphi'/\varphi'' = \{C'(t)\}^{-1}$, or throughout (a, b), $\varphi'/\varphi'' = -\{C'(t)\}^{-1}$. In the first case, φ' and φ are monotone in opposite senses, and φ'/φ'' is nonincreasing in (a, b). In the second case, φ' and φ are monotone in the same sense, and φ'/φ'' is nondecreasing in (a, b). In either case, by Theorem 1, $M_{\varphi'} \leqq M_\varphi$.

Theorem 4 can be modified by replacing in it "convex" by "strictly convex," and "$M_{\varphi'} \leqq M_\varphi$" by "$M_{\varphi'} < M_\varphi$."

THEOREM 5. *Let* φ *be strictly monotone in* $[a, b]$ *and three-times differentiable in* (a, b). *Let* φ' *be continuous in* $[a, b]$ (*where* $\varphi'(a)$

and $\varphi'(b)$ *are right and left hand derivatives, respectively). Let* $\varphi'' \neq 0$ *throughout* (a, b). *A necessary and sufficient condition for* $M_{\varphi'} \leqq M_\varphi$ *to hold is that* $\varphi''^2 \geqq \varphi'\varphi'''$ *throughout* (a, b) *if* φ' *and* φ *are monotone in the same sense, and that* $\varphi''^2 \leqq \varphi'\varphi'''$ *throughout* (a, b) *if* φ' *and* φ *are monotone in opposite senses.*

Theorem 5 follows easily from Theorem 1 by considering the derivative of φ'/φ''.

Similarly, under the hypotheses of Theorem 5, $M_{\varphi'} < M_\varphi$ holds, if $\varphi''^2 > \varphi'\varphi'''$ throughout (a, b) and φ and φ' are monotone in the same sense, and also if $\varphi''^2 < \varphi'\varphi'''$ throughout (a, b) and φ and φ' are monotone in opposite senses.

As an example, let $a = 0$, $b = \pi/2$, $\varphi(x) \equiv \cos x$. φ and φ' are monotone in the same sense in $[0, \pi/2]$, and $\varphi''^2 = \cos^2 x > -\sin^2 x = \varphi'\varphi'''$ throughout $(0, \pi/2)$. Therefore, $M_{-\sin x} < M_{\cos x}$, i.e., $M_{\sin x} < M_{\cos x}$.

6. In a previous paper [3] the authors studied, for given positive $q_1, q_2, \cdots, q_n$ (with $\sum_{\nu=1}^n q_\nu = 1$), the ratio

$$(5)\quad \begin{cases} F(x_1, x_2, \cdots, x_n) \\ \quad = M_\chi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n)/M_\psi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n) \end{cases}$$

where $0 < a$, $\psi(x) \equiv x^r$, $\chi(x) \equiv x^s$ $(r < s,\ rs \neq 0)$.

Their purpose was to find an upper bound for F in

$$I = \{(x_1, x_2, \cdots, x_n):\ a \leqq x_k \leqq b,\ k = 1, 2, \cdots, n\} .$$

A crucial step was to show that if X^* is a point of I such that $F(X^*) = \max\{F(X) : X \in I\}$, then X^* is necessarily a vertex of I. In particular, X^* cannot be an interior point of I. This last property holds under quite general conditions:

THEOREM 6. *Let* ψ *and* χ *be elements of* Φ, *differentiable in* (a, b), *and satisfying* $\psi'\chi' \neq 0$ *there. Assume* $0 \notin [a, b]$, $M_\psi < M_\chi$. *Let* $q_1, \cdots, q_n$ $(n > 1)$ *be given positive numbers with* $\sum_{\nu=1}^n q_\nu = 1$, *and let* I *be as in the last paragraph. Let* F *of* (5) *attain its maximum in* I *at a point* $X^* = (x_1^*, \cdots, x_n^*)$ *of* I. *Then* X^* *is not an interior point of* I.

Proof. Suppose that some x_j^* satisfies $a < x_j^* < b$. Then $(\partial F/\partial x_j)_{x_\nu = x_\nu^*,\ \nu=1,2,\cdots,n} = 0$, i.e.,

$$\Big[\psi^{-1}\Big(\sum_{\nu=1}^n q_\nu \psi(x_\nu^*)\Big)\Big]^{-2}\Big[q_j\chi'(x_j^*)\psi^{-1}\Big(\sum_{\nu=1}^n q_\nu\psi(x_\nu^*)\Big)\Big/\chi'\Big(\chi^{-1}\Big(\sum_{\nu=1}^n q_\nu\chi(x_\nu^*)\Big)\Big)$$
$$- q_j\psi'(x_j^*)\chi^{-1}\Big(\sum_{\nu=1}^n q_\nu\chi(x_\nu^*)\Big)\Big/\psi'\Big(\psi^{-1}\Big(\sum_{\nu=1}^n q_\nu\psi(x_\nu^*)\Big)\Big)\Big] = 0 .$$

Thus

$$\chi'(x_j^*)/\psi'(x_j^*) = \left[\chi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \chi(x_\nu^*)\Big)\chi'\Big(\chi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \chi(x_\nu^*)\Big)\Big)\right]$$
$$\times \Big/ \left[\psi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \psi(x_\nu^*)\Big)\psi'\Big(\psi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \psi(x_\nu^*)\Big)\Big)\right].$$

Let C denote the right hand side of the last equality. If both x_j^* and x_k^* are interior points of $[a, b]$, then $\chi'(x_j^*)/\psi'(x_j^*) = C = \chi'(x_k^*)/\psi'(x_k^*)$, and hence, by the strict monotonicity of χ'/ψ' [see the end of § 2], $x_j^* = x_k^*$. Thus, if X^* were an interior point of I, we would have $x_1^* = x_2^* = \cdots = x_n^*$, and therefore

$$1 = F(x_1^*, x_2^*, \cdots, x_n^*) = \max\{F(X) : X \in I\} > 1 .$$

References

1. E. F. Beckenbach, *On the inequality of Kantorovich* (*abstract*), Notices, Amer. Math. Society, **10**, No. 5 (1963), 440.
2. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag (1961).
3. G. T. Cargo and O. Shisha, *Bounds on Ratios of Means*, Journal of Research of the National Bureau of Standards, Vol. 66B, No. 4, pp. 169–170 (1962).
4. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (second edition), Cambridge University Press (1952).
5. H. Kestelman, *Modern theories of integration* (second revised edition), Dover Publications, 1960.

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