BASIC SEQUENCES AND THE PALEY-WIENER CRITERION

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1. Introduction. Throughout the paper X will denote a complete metric linear space (i.e., a complete topological linear space with topology derived from a metric d with the property that $d(x, y) = d(x - y, 0)$, for all $x, y \in X$) or some specialization thereof over the real or complex field; $\|x\|$ will denote $d(x, 0)$; and if $\{x_n\}$ is a sequence in X, $[x_n]$ will denote the closed linear span of the elements $\{x_n\}_{n \in \omega}$.

A sequence $\{x_n\}$ is said to be a *basic sequence of vectors* if $\{x_n\}$ is a basis of vectors of the space $[x_n]$, i.e., for each $x \in [x_n]$ there corresponds a unique sequence of scalars $\{a_i\}$ such that

$$x = \sum_{i=1}^{\infty} a_i x_i \,, \tag{1.1}$$

the convergence being in the topology of X. We say that the basis is unconditional if the convergence in (1.1) is unconditional. It is well known that if $\{x_n\}$ is a basic sequence of vectors, then every $x \in [x_n]$ can be represented in the form $x = \sum_{i=1}^{\infty} f_i(x) x_i$ where $\{f_i\}$ is the sequence of continuous coefficient functionals biorthogonal to $\{x_i\}$ (Arsove [1, p. 368], Dunford and Schwartz [4, p. 71]).

Similarly, we say that a sequence $\{M_i\}$ of nontrivial subspaces of a complete metric linear space X is a *basis of subspaces* of X, if for each $x \in X$, there corresponds a unique sequence $\{x_i\}$, $x_i \in M_i$ for each i, such that

$$x = \sum_{i=1}^{\infty} x_i \,. \tag{1.2}$$

This concept has been studied by Fage [5], Markus [9], and others in separable Hilbert space and by Grimblyum [6] and McArthur [10] in complete metric linear spaces. We say that the basis of subspaces is *unconditional* if the convergence in (1.2) is unconditional.

If $\{M_i\}$ is a basis of subspaces for X, for each $i \in \omega$ define E_i from X into X by $E_i(x) = x_i$ where $\sum_{i=1}^{\infty} x_i$ is the unique representation of $x \in X$. E_i is a projection (linear and idempotent); $E_i E_j = 0$ if $i \neq j$; the range of E_i is M_i; for each $x \in X$, $x = \sum_{i=1}^{\infty} E_i(x)$ and if $E_i(x) = 0$ for each i, then $x = 0$. $\{M_i\}$ will be called a *Schauder basis of subspaces* if each E_i is continuous.

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A sequence $\{M_i\}$ of non-trivial subspaces of X is a (*unconditional*) *basic sequence of subspaces* if $\{M_i\}$ is a (unconditional) basis of subspaces of $[M_i]$, the closed linear span of $\bigcup_{i\in\omega} M_i$. If $\{M_i\}$ is a basic sequence of subspaces and $x \in [M_i]$ then $x = \sum_{i=1}^{\infty} E_i(x)$, where E_i is now defined on $[M_i]$.

The classical Paley-Wiener theorem can be formulated in X as follows.

1.3. THEOREM. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and let λ be a real number* $(0 < \lambda < 1)$ *such that*

$$\left\| \sum_{n=1}^{m} a_n(x_n - y_n) \right\| \leqq \lambda \left\| \sum_{n=1}^{m} a_n x_n \right\| \tag{1.3a}$$

holds for arbitrary scalars $a_1, \cdots, a_m$. Then (1) *if $\{x_n\}$ is a basis so is $\{y_n\}$;* (2) *if $\{x_n\}$ is fundamental* (i.e., $[x_n] = X$) *so is $\{y_n\}$.*

Recently Arsove [1] showed that Theorem 1.3 is valid in a complete metric linear space. It is the purpose of this paper to show that this result and results similar to those of Pollard [13], Hilding [7], and Nagy [11] (all of which generalize condition 1.3a) are valid for basic sequences of subspaces in X. As a corollary to Theorem 4.3 we obtain a new version of the Paley-Wiener theorem.

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2. Basic sequences of subspaces.

Special cases of the following lemma have been used by Hilding [7, p. 93], Nagy [11, p. 76], and others to prove theorems similar to Theorems 2.3 and 2.4.

2.1. LEMMA. *Let $\{M_i\}$ and $\{N_i\}$ be sequences of nontrivial subspaces of the complete metric linear space X. Suppose that for each $i \in \omega$ there exists a one-to-one linear transformation T_i of M_i onto N_i and suppose further that there are positive numbers m, M such that*

$$m \left\| \sum_{i=1}^{p} x_i \right\| \leqq \left\| \sum_{i=1}^{p} T_i(x_i) \right\| \leqq M \left\| \sum_{i=1}^{p} x_i \right\| \tag{2.1a}$$

holds for arbitrary $x_i \in M_i$, $i = 1, \cdots, p$. Then
(i) *there is a linear homeomorphism T of $[M_i]$ onto $[N_i]$ such that the restriction of T to M_i equals T_i for each $i \in \omega$ and such that*

$$m \| x \| \leqq \| T(x) \| \leqq M \| x \| , \text{ for all } x \in [M_i] . \tag{2.1b}$$

(ii) *$\{M_i\}$ is a (unconditional) basic sequence of subspaces if and only if $\{N_i\}$ is a (unconditional) basic sequence of subspaces.*

Proof. Let X_0 denote the space of finite linear combinations of $\bigcup_{i\in\omega} M_i$. These, of course, are reducible to the form $\sum_{i=1}^{n} x_i$, $x_i \in M_i$. If $x_i, x_i' \in M_i$, $i = 1, \cdots, p$ and $\sum_{i=1}^{p} x_i = \sum_{i=1}^{p} x_i'$ then from 2.1a it follows that $\sum_{i=1}^{p} T_i(x_i) = \sum_{i=1}^{p} T_i(x_i')$. Thus we may define a linear transformation S from X_0 into $[N_i]$ by $S(\sum_{i=1}^{p} x_i) = \sum_{i=1}^{p} T_i(x_i)$ and have $m\|x\| \leqq \|S(x)\| \leqq M\|x\|$, for all $x \in X_0$. It is clear that S restricted to M_i is equal to T_i and that S is continuous. Thus defined on a dense subset of $[M_i]$, S has a unique linear extension T to $[M_i]$ satisfying 2.1b. From 2.1b it follows that T is one-to-one and T^{-1} is continuous. We show T is onto $[N_i]$.

Let $y \in [N_i]$. Then $y = \lim_k g_k$ where g_k is of the form $g_k = \sum_{i=1}^{n(k)} y_i^{(k)}$, $y_i^{(k)} \in N_i$, $i = 1, \cdots, n(k)$. For each such $y_i^{(k)}$ there is a unique $x_i^{(k)} \in M_i$ such that $T_i(x_i^{(k)}) = y_i^{(k)}$. Let $h_k = \sum_{i=1}^{n(k)} x_i^{(k)}$. Then from 2.1b, $\|h_p - h_q\| \leqq (1/m)\|g_p - g_q\|$, so $\{h_k\}$ is Cauchy and there is an $x_0 \in [M_i]$ such that $\{h_k\} \to x_0$. Clearly, $T(x_0) = y$.

To verify (ii) suppose $\{M_i\}$ is basic, i.e., a basic sequence of subspaces. Let $y \in [N_i]$. Then $y = T(x)$ for some $x \in [M_i]$. x has a unique expansion $x = \sum_{i=1}^{\infty} x_i$, $x_i \in M_i$ and $y = \sum_{i=1}^{\infty} T(x_i)$, $T(x_i) \in N_i$. Now if $y = \sum_{i=1}^{\infty} y_i$, $y_i \in N_i$, then $y_i = T(x_i')$ for some unique $x_i' \in M_i$. Hence $0 = T(\sum_{i=1}^{\infty} x_i - x_i')$ which implies $x_i = x_i'$. Since the expansion for y is unique, it follows that $\{N_i\}$ is basic. The converse follows from (i) in the same way. If in the preceding argument $\{M_i\}$ had been assumed an unconditional basis of subspaces for $[M_i]$ then the series $\sum_{i=1}^{\infty} x_i$ would have been unconditionally convergent to x and since T is a linear homeomorphism it follows that $\sum_{i=1}^{\infty} T(x_i)$ would be unconditionally convergent.

2.2. DEFINITION. *Two sequences $\{x_i\}$ and $\{y_i\}$ (in the given order) in X are said to have the property:*

(P–W) (*for Paley-Wiener*) *if there is a real number λ $(0 < \lambda < 1)$ such that $\|\sum_{i=1}^{n} a_i(x_i - y_i)\| \leqq \lambda \|\sum_{i=1}^{n} a_i x_i\|$ holds for arbitrary scalars $a_1, a_2, \cdots, a_n$;*

(P–H) (*for Pollard-Hilding*) *if for each positive real number k, there are real numbers $\lambda_1, \lambda_2 (0 \leqq \lambda_i < \min[1; 2^{1-1/k}], i = 1, 2)$ such that*

$$\left\|\sum_{i=1}^{n} a_i(x_i - y_i)\right\| \leqq \left[\lambda_1 \left\|\sum_{i=1}^{n} a_i x_i\right\|^k + \lambda_2 \left\|\sum_{i=1}^{n} a_i y_i\right\|^k\right]^{1/k}$$

holds for arbitrary scalars $a_1, \cdots, a_n$;
(N) (*for Nagy*) *if there are real numbers λ', μ, ν $(0 \leqq \lambda' < 1, 0 \leqq \nu < 1, 0 \leqq \mu, \mu^2 \leqq [1 - \lambda'][1 - \nu])$ such that*

$$\left\|\sum_{i=1}^{n} a_i(x_i - y_i)\right\|^2 \leqq \lambda' \left\|\sum_{i=1}^{n} a_i x_i\right\|^2 + \mu \left\|\sum_{i=1}^{n} a_i x_i\right\| \cdot \left\|\sum_{i=1}^{n} a_i y_i\right\| + \nu \left\|\sum_{i=1}^{n} a_i y_i\right\|^2$$

holds for arbitrary scalars $a_1, \cdots, a_n$.

If $k = 1$ and $\lambda_1 = \lambda_2$ property P–H reduces to

$$\left\|\sum_{i=1}^{n} a_i(x_i - y_i)\right\| \leqq \lambda\left[\left\|\sum_{i=1}^{n} a_i y_i\right\| + \left\|\sum_{i=1}^{n} a_i x_i\right\|\right] \tag{2.2a}$$

where $\lambda = \lambda_1 = \lambda_2$.

2.3. LEMMA. *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *with property* P–W, P–H *or* N *then* 2.2a *holds, with* λ $(0 < \lambda < 1)$ *an appropriately chosen constant.*

Proof. That property P–W implies 2.2a is obvious. If $\{x_n\}$, $\{y_n\}$ have property P–H, let $\lambda = [\max(\lambda_1, \lambda_2)]^{1/k}$; if $\{x_n\}$, $\{y_n\}$ have property N let $\lambda = [\max(\lambda', \mu, \nu)]^{1/2}$.

2.4. THEOREM. *Suppose* $\{M_i\}$ *and* $\{N_i\}$ *are sequences of nontrivial subspaces of* X *and suppose that for each* $i \in \omega$, T_i *is a one-to-one linear transformation of* M_i *onto* N_i. *Suppose further that there is a* $\lambda (0 < \lambda < 1)$ *such that*

$$\left\|\sum_{i=1}^{n} (x_i - T_i(x_i))\right\| \leqq \lambda\left(\left\|\sum_{i=1}^{n} x_i\right\| + \left\|\sum_{i=1}^{n} T_i(x_i)\right\|\right) \tag{2.4a}$$

holds for arbitrary $x_i \in M_i$, $i = 1, \cdots, n$. *Then*
(i) *there is a linear homeomorphism* T *of* $[M_i]$ *onto* $[N_i]$ *such that* T *restricted to* M_i *equals* T_i *for each* i *and such that*

$$[(1 - \lambda)/(1 + \lambda)]\,\|x\| \leqq \|T(x)\| \leqq [(1 + \lambda)/(1 - \lambda)]\,\|x\| \tag{2.4b}$$

for each $x \in [M_i]$;
(ii) $\{M_i\}$ *is a (unconditional) basic sequence of subspaces if and only if* $\{N_i\}$ *is a (unconditional) basic sequence of subspaces.*

Proof.

$$\left\|\sum_{i=1}^{n} T_i(x_i)\right\| \leqq \left\|\sum_{i=1}^{n} (T_i(x_i) - x_i)\right\| + \left\|\sum_{i=1}^{n} x_i\right\|$$
$$\leqq (1 + \lambda)\left\|\sum_{i=1}^{n} x_i\right\| + \lambda\left\|\sum_{i=1}^{n} T_i(x_i)\right\|,$$

i.e.,

$$\left\|\sum_{i=1}^{n} T_i(x_i)\right\| \leqq [(1 + \lambda)/(1 - \lambda)]\left\|\sum_{i=1}^{n} x_i\right\|.$$

Similarly,

$$\left\|\sum_{i=1}^{n} x_i\right\| \leqq [(1+\lambda)/(1-\lambda)]\left\|\sum_{i=1}^{n} T_i(x_i)\right\| .$$

Thus

$$[(1-\lambda)/(1+\lambda)]\left\|\sum_{i=1}^{n} x_i\right\| \leqq \left\|\sum_{i=1}^{n} T_i(x_i)\right\| \leqq [(1+\lambda)/(1-\lambda)]\left\|\sum_{i=1}^{n} x_i\right\| .$$

The conclusions follow from Lemma 2.1.

2.5. COROLLARY. *Suppose* $\{M_i\}$ *and* $\{N_i\}$ *are sequences of non-trivial subspaces of* X *and suppose that for each* $i \in \omega$, T_i *is a one-to-one linear transformation of* M_i *onto* N_i. *Suppose further that* $\{x_i\}$ *and* $\{T_i(x_i)\}$ *have property* P–W, P–H *or* N, *for arbitrary* $x_i \in M_i$ *(observe that since* $x_i \in M_i$ *is arbitrary,* x_i *and* $T_i(x_i)$ *include the scalar* a_i *for each* i*) then the conclusions of Theorem* 2.4 *hold. In particular, if Property* P–W *holds and* $\{M_i\}$ *is a basis of subspaces for* X, *so is* $\{N_i\}$.

Proof. The first part of the corollary follows from Lemma 2.3. Arsove [1, p. 367] has shown how to prove the other assertion of the corollary. We repeat the proof for completeness.

Since Property P–W holds there exists a linear operator T from X into X satisfying $\|x - T(x)\| \leqq \lambda \|x\|$, $x \in X$ and such that T restricted to M_i equals T_i. Let $A = T - I$, where I is the identity operator. A is continuous at each $x \in X$ and furthermore $\|A^n(x)\| \leqq \lambda^n \|x\|$ for each $x \in X$ and positive integer n. Thus a linear operator U of X onto X may be defined by $U(x) = \sum_{n=0}^{\infty}(-A^n(x))$, $x \in X$. It follows that $\|U(x)\| \leqq (1-\lambda)^{-1}\|x\|$, so U is continuous. Given $y \in X$, let $x = U(y)$. Then $y = (I + A)x = T(x)$ so T is onto X. Thus $\{N_i\}$ is a basis of subspaces for X.

3. **Basic sequences of vectors.** If X has a basis of vectors $\{x_n\}$, then $\{x_n\}$ induces in a natural way a basis of subspaces $\{M_i\}$ for X. We have only to define M_i to be the span of the single element x_i (denoted by $sp(x_i)$). From the remarks in the introduction we have $x = \sum_{i=1}^{\infty} f_i(x)x_i$ for each $x \in X$, so $E_i(x) = f_i(x)x_i$. Since $h(a) = ax_i$ is a linear homeomorphism of the scalar field into X and $f_i(x)$ is a continuous linear functional it follows that E_i is continuous for each $i \in \omega$ and so $\{M_i\}$ is a Schauder basis of subspaces for X. Thus, for Schauder bases of vectors, we obtain the following theorems as corollaries to the theorems of § 2.

3.1. THEOREM. *Suppose* $\{x_i\}$ *and* $\{y_i\}$ *are nontrivial (i.e.,* $x_i \neq 0$, $y_i \neq 0$, *for each* $i \in \omega$) *sequences in* X *and suppose there is a* $\lambda (0 < \lambda < 1)$ *such that*

$$\left\| \sum_{i=1}^{n} a_i(x_i - y_i) \right\| \leqq \lambda \left(\left\| \sum_{i=1}^{n} a_i x_i \right\| + \left\| \sum_{i=1}^{n} a_i y_i \right\| \right) \tag{3.1a}$$

holds for arbitrary scalars $a_1, \cdots, a_n$. *Then,*
(i) *there exists a linear homeomorphism* T *of* $[x_i]$ *onto* $[y_i]$ *such that* $T(x_i) = y_i$ *for each* $i \in \omega$, *and*
(ii) $\{x_i\}$ *is a (unconditional) basic sequence of vectors if and only if* $\{y_i\}$ *is a (unconditional) basic sequence of vectors.*

Proof. Let $M_i = sp(x_i)$ and $N_i = sp(y_i)$. Define a linear operator T_i from M_i onto N_i by $T_i(ax_i) = ay_i$ where a is an arbitrary scalar. Clearly, T_i is one-to-one and continuous. 3.1a can be rewritten

$$\left\| \sum_{i=1}^{n} (x_i' - T_i(x_i')) \right\| \leqq \lambda \left(\left\| \sum_{i=1}^{n} x_i' \right\| + \left\| \sum_{i=1}^{n} T_i(x_i') \right\| \right) \tag{3.1b}$$

for arbitrary $x_i' \in M_i$, $i = 1, \cdots, n$. The conclusions follow from Theorem 2.4.

Thus in particular, if $\{x_n\}$ and $\{y_n\}$ are nontrivial sequences in X with property P–W, P–H or N, the conclusions of 3.1 are valid.

We have remarked that if $\{x_n\}$ and $\{y_n\}$ have property P–W and $\{x_n\}$ is a basis of vectors for X, then $\{y_n\}$ is a basis of vectors for X. From 3.1 it follows that if $\{x_n\}$ is an unconditional basis of vectors for X, then $\{y_n\}$ is an unconditional basis of vectors for X.

4. Basic sequences in Banach spaces.

From Grinblyum [6] the following can be derived (a proof is given in [10]).

4.1. LEMMA. *Let* $\{M_i\}$ *be sequence a of nontrivial closed subspaces in a Banach space* X. $\{M_i\}$ *is a Schauder basis of subspace for* $[M_i]$ *if and only if there is a* $K \geqq 1$ *such that for arbitrary* $p, q \in \omega$, $p \leqq q$ *we have* $\| \sum_{i=1}^{p} x_i \| \leqq K \| \sum_{i=1}^{q} x_i \|$, *for arbitrary* $x_i \in M_i$, $i = 1, \cdots, q$.

4.2. LEMMA. *Let* $\{M_i\}$ *be a sequence of nontrivial closed subspaces of a Banach space* X. $\{M_i\}$ *is an unconditional Schauder basis of subspaces of* $[M_i]$ *if and only if there is a* $K \geqq 1$ *such that for arbitrary finite sets of positive integers* F, F' *with* $F \subset F'$ *we have* $\| \sum_{i \in F} x_i \| \leqq K \| \sum_{i \in F'} x_i \|$, *for arbitrary* $x_i \in M_i$.

4.3. THEOREM. *Suppose* $\{M_i\}$ *and* $\{N_i\}$ *are sequences of closed nontrivial subspaces of a Banach space* X.

(1) *If there is a* $\lambda (0 < \lambda < 1)$ *such that for an arbitrary finite set of integers* F' *and arbitrary* $y_i \in N_i$, $i \in F'$, *there exists* $x_i \in M_i$, $i \in F'$ *such that*

$$\left\| \sum_{i \in F} (y_i - x_i) \right\| \leqq \lambda \left[\left\| \sum_{i \in F} x_i \right\| + \left\| \sum_{i \in F} y_i \right\| \right] \tag{4.3a}$$

holds for arbitrary $F \subset F'$ *then* $\{N_i\}$ *is an unconditional (Schauder) basic sequence of subspaces if* $\{M_i\}$ *is an unconditional (Schauder) basic sequence of subspaces;*
(2) *if there is a* $\lambda (0 < \lambda < 1)$ *such that for arbitrary* $q \in \omega$ *and arbitrary* $y_1, \cdots, y_q$, $y_i \in N_i$, $i = 1, \cdots, q$ *there exist* $x_1, \cdots, x_q$, $x_i \in M_i$, $i = 1, \cdots, q$ *such that*

$$\left\| \sum_{i=1}^{p} (y_i - x_i) \right\| \leqq \lambda \left[\left\| \sum_{i=1}^{p} x_i \right\| + \left\| \sum_{i=1}^{p} y_i \right\| \right] \tag{4.3b}$$

holds for all $p \leqq q$ *then* $\{N_i\}$ *is a (Schauder) basic sequence of subspaces if* $\{M_i\}$ *is a (Schauder) basic sequence of subspaces.*

Proof. We prove (2). The proof of (1) is analogous using Lemma 4.2 instead of 4.1.

Suppose $\{M_i\}$ be a basis of subspaces for $[M_i]$. By Lemma 4.1 there is a $K \geqq 1$ such that

$$\left\| \sum_{i=1}^{p} x_i \right\| \leqq K \left\| \sum_{i=1}^{q} x_i \right\|, \ x_i \in M_i, \ p \leqq q \ .$$

We have

$$\left\| \sum_{i=1}^{p} y_i \right\| \leqq \left\| \sum_{i=1}^{p} (y_i - x_i) \right\| + \left\| \sum_{i=1}^{p} x_i \right\|$$

and from (4.4b) it follows that

$$\left\| \sum_{i=1}^{p} y_i \right\| \leqq \frac{1+\lambda}{1-\lambda} \left\| \sum_{i=1}^{p} x_i \right\| .$$

Also

$$\left\| \sum_{i=1}^{q} x_i \right\| \leqq \frac{1+\lambda}{1-\lambda} \left\| \sum_{j=1}^{q} y_i \right\| .$$

Thus we have

$$\left\| \sum_{i=1}^{p} y_i \right\| \leqq \left[\frac{1+\lambda}{1-\lambda} \right]^2 K \left\| \sum_{i=1}^{q} y_i \right\| .$$

Thus by Lemma 4.1, $\{N_i\}$ is a basis of subspaces for $[N_i]$.

4.4. COROLLARY. *Let $\{x_i\}$ and $\{y_i\}$ be non-trivial sequences in a Banach space X.*
(1) *If there is a $\lambda(0 < \lambda < 1)$ such that for an arbitrary finite set of indices F' and arbitrary scalars $\{a_i\}$, $i \in F'$, there exist scalars $\{b_i\}$, $i \in F'$, such that*

$$\left\| \sum_{i \in F} (a_i y_i - b_i x_i) \right\| \leqq \lambda \left[\left\| \sum_{i \in F} a_i y_i \right\| + \left\| \sum_{i \in F} b_i x_i \right\| \right] \tag{4.4a}$$

holds for arbitrary $F \subset F'$ then $\{y_i\}$ is an unconditional (Schauder) basic sequence of vectors if $\{x_i\}$ is an unconditional (Schauder) basic sequence of vectors;
(2) *if there is a $\lambda(0 < \lambda < 1)$ such that for arbitrary $q \in \omega$ and arbitrary scalars $a_1, \cdots, a_q$ there are scalars $b_1, \cdots, b_q$ such that*

$$\left\| \sum_{i=1}^{p} (a_i y_i - b_i x_i) \right\| \leqq \lambda \left[\left\| \sum_{i=1}^{p} b_i x_i \right\| + \left\| \sum_{i=1}^{p} a_i y_i \right\| \right] \tag{4.4b}$$

holds for all $p \leqq q$ then $\{y_i\}$ is a (Schauder) basic sequence of vectors if $\{x_i\}$ is a(Schauder) basic sequence of vectors.

Proof. Let $M_i = sp(x_i)$, $N_i = sp(y_i)$ and apply the preceeding theorem.

4.4 is a new form of the Paley-Wiener theorem for we no longer require the coefficients of x_i and y_i to be the same. We could now define properties similar to properties P–W, P–H and N by merely asserting the existence of a scalar b_i to replace the coefficient of x_i in each of the properties defined in 2.2. It is easy to see that these new forms of properties P–W, P–H and N imply the hypotheses of corollary 4.5.

It is unknown to the author whether $[x_n]$ is linearly homeomorphic to $[y_n]$ or not.

REFERENCES

1. M. G. Arsove, *The Paley-Wiener theorem in metric linear spaces*, Pacific J. Math., **10** (1960), 365-379.
2. C. Bessaga and A. Pelczynski, *On bases and unconditional conditional convergence of series in Banach spaces*, Studia Math., **17** (1958), 151-164.
3. M. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1958.
4. N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Interscience Publishers, New York, 1958.
5. M. K. Fage, *The rectification of bases in Hilbert space*, Dokl. Akad. Nauk., SSSR (N.S.) **75** (1950), 1053-1056 (In Russian).
6. M. M. Grinblyum, *On the representation of a space of type B in the form of a direct sum of subspaces*, Dokl. Akad. Nauk. SSSR (N.S.) **70** (749-752.

7. S. H. Hilding, *Note on completeness theorems of Paley-Wiener type*, Ann. of Math., (2) **49** (1948), 953-955.
8. M. Krein, D. Milman and M. Rutman, *On a property of a basis in a Banach space*, Khark. Zap. Matem. Obsh. (4), **16** (1940), 182. (In Russian with English Resume).
9. A. S. Markus, *A basis of root vectors of a dissipative operator*, Soviet Math.-Doklady, Amer. Math. Soc. Trans. **1** (1960), 599-602.
10. C. W. McArthur. *Infinite direct sums in complete metric linear spaces* (to appear).
11. B. Sz. Nagy, *Expansion theorems of Paley-Wiener type*, Duke Math. J., **14** (1947), 975-978.
12. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, New York, 1934.
13 H. Pollard, *Completeness theorems of Paley-Wiener type*, Ann. of Math. (2) **45** (1944), 738-739.

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