A NOTE ON REFLEXIVE MODULES

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For any ring A and left (resp. right) A-module E we let E^* denote the right (resp. left) A-module $\mathrm{Hom}_A(E, A_s)$ (resp. $\mathrm{Hom}_A(E, A_d)$) where A_s (resp. A_d) denotes A considered as a left (resp. right) A-module. Then the mapping $E \to E^{**}$ such that $x \in E$ is mapped onto the mapping $\varphi \to \varphi(x)$ is linear.

Specker [3] has shown that if E is a free Z-module with a denumerable base (where Z denotes the ring of integers) then E is reflexive, i.e. the canonical homomorphism $E \to E^{**}$ is a bijection. In this paper it is shown that a free module E with a denumerable base over a discrete valuation ring A is reflexive if and only if A is not complete and if and only if E is complete when given the topology having finite intersections of the kernels of the linear forms as a fundamental system of neighborhoods of 0. Specker's result can be deduced from these results. We note that this topology has been used and studied by Nunke [2] and Chase [1].

THEOREM 1. *Let A be a discrete valuation ring with prime Π and let E be a free A-module with a denumerable base. Then E is reflexive if and only if A is not complete.*

Proof. Let $(a_i)_{i\in N}$ (N the set of natural numbers) be a base of E and let $E_j = \{\varphi \mid \varphi \in E^*,\ \varphi(a_i) = 0,\ i = 0, 1, 2, \cdots, j-1$. Let $a'_j \in E^*$ be such that $a'_j(a_j) = 1$, $a'_j(a_k) = 0$ if $j \neq k$. Then clearly a'_0, $a'_1, \cdots a'_{j-1}$ generate a supplement of E_j in E^*. For each $x \in E$ the canonical image of x in E^{**} annihilates some E_j and conversely if $\psi \in E^{**}$ annihilates E_j then ψ is the canonical image of $\sum_{i=0,1,\cdots,j-1} \psi(a'_i)ai$. Hence $E \to E^{**}$ is a surjection if and only if each $\psi \in E^{**}$ annihilates some E_j. If $E \to E^{**}$ is not a surjection let $\psi \in E^{**}$ be such that $\psi(E_j) \neq 0$ for each $j \in N$ and let $\varphi_j \in E_j$ be such that $\psi(\varphi_j) \neq 0$. We can suppose that $\varphi_j \in \Pi^j E_j$ and that $\psi(\varphi_j) \in \Pi^m_j A$ but $\psi(\varphi_j) \notin \Pi^{m_j+1} A$ where $m_{i+1} > m_i$ for all $i \in N$. To show A complete it suffices to show that every series $\sum_{j\in N} \beta_j \Pi^{mj}$, $\beta_j \in A$ converges. We can find a scalar multiple of φ_j say φ'_j such that $\psi(\varphi'_j) = \beta_j \Pi^m_j$. Then let $\varphi \in E^*$ be such that $\varphi(x) = \sum_{j\in N} \varphi'_j(x)$ for all $x \in E$. This sum is defined since for a fixed $x \in E$ and M sufficiently large positive integer we have $\varphi_{M+i}(x) = 0$ for all $i \in N$. Furthermore, since $\varphi'_j \in \Pi^j E_j$ it is clear that the series $\sum \varphi'_j$ converges to φ when E^* is given the topology having

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the submodules $\Pi^n E^*$, $n \in N$ as a fundamental system of neighborhoods of 0. Under this topology $\psi: E^* \to A$ is continuous. Hence

$$\sum_{j \in N} \psi(\varphi_j') = \sum_{j \in N} \beta_j \Pi^{m} j$$

converges to $\psi(\varphi)$. Thus A is complete.

Conversely if A is complete let $(a_i')_{i \in N}$ as defined above be a subfamily of the family $(a_i')_{i \in N_1}$, $N_1 \supset N$ where $(a_i' + \Pi E^*)_{i \in N_1}$ is a base of the $A/\Pi\ A$ module $E^*/\Pi\ E^*$. Then if E' is the submodule of E^* generated by the family $(a_i')_{i \in N_1}$ it is easy to see that E' is free with base $(a_i')_{i \in N_1}$ and that E' is a dense pure submodule of E^*, i.e. E^*/E' is divisible and torsion free. Then, since A is complete the map $E^{**} \to E'^*$ which maps an element of E^{**} onto its restriction to E' is a bijection. But this clearly implies the existence of a $\psi \in E^{**}$ such that $\psi(a_i') \neq 0$ for all $i \in N_1$ and hence for all $i \in N$. Thus $E \to E^{**}$ is not a surjection.

COROLLARY. *If A is an integral domain with a prime Π such that the discrete valuation ring A_π is not complete then free A-modules with denumerable bases are reflexive.*

Proof. There exist canonical injections of E, E^* and E^{**} in E_π, E_π^*, and E_π^{**} and furthermore if for $x \in E$, $\varphi \in E^*$, and $\psi \in E^{**}$ we let $\bar{x}$, $\bar{\varphi}$, and $\bar{\psi}$ denote the image of x, φ, and ψ in E_π, E_π^*, and E_π^{**} then $\varphi(x) = \bar{\varphi}(\bar{x})$ and $\psi(\varphi) = \bar{\psi}(\bar{\varphi})$. Then if $(a_i)_{i \in N}$ is a base of E, $(\bar{a}_i)_{i \in N}$ is a base of E_π and if $(a_i')_{i \in N}$ is defined as above we get $\bar{a}_i'(\bar{a}_i) = 1$, $\bar{a}_i'(\bar{a}_j) = 0$ if $i \neq j$. Then if $\psi \in E^{**}$ is such that $\psi(E_j) = 0$ for each j then $\bar{\psi}$ is not in the image E_π under the canonical homomorphism since $\bar{\psi}((E_\pi)_j) \neq 0$ where E_j and $(E_\pi)_j$ are defined as above.

THEOREM 2. *If A is a left Noethrian hereditary ring, then a left A module E is reflexive if and only if E is complete when endowed with the topology having the finite intersections of the kernels of the linear forms as a fundamental system neighborhoods of* 0.

Proof. Clearly E is separated with the topology described in the theorem if and only if the map $E \to E^{**}$ is an injection hence we suppose that E is separated. For each finite subset X of E^* consider the subset X° of E^{**} consisting of all $\psi \in E^{**}$ such that $\psi(X) = 0$. Let E^{**} be endowed with the topology having the submodules X° as a fundamental system of neighborhoods of 0 where X ranges through all finite subsets of E^*. Then it is immediate that E^{**} is complete with this topology. If we can establish that the canonical map $E \to E^{**}$ maps E isomorphically onto a dense subset of E^{**} then it will follow immediately that E is complete if and only if E is reflexive.

Let X be a finite subset of E^*. Then clearly the intersection of the kernels of the elements in X is mapped onto the intersection of X° with the canonical image of E in E^{**} hence E is mapped isomorphically onto a subset of E^{**}. Thus it only remains to prove that the image of E in E^{**} is dense in E^{**}. If $\psi \in E^{**}$ and $X = \{\varphi_1, \varphi_2, \cdots \varphi_n\}$ is a finite set of elements of E^* consider the map $E \to \prod_{i=1,\cdots,n} A_i$ such that $x \to (\varphi_i(x))_{i=1,\cdots,n}$ where $A_i = A_s$. Since A is left hereditary the kernel of this map $E_1 = \bigcap_{i=1,\cdots,n} \varphi_i^{-1}(0)$ is a direct summand of E so let $E = E_1 + E_2$ (direct). Then since A is left Noetherian E_2 is a finitely generated projective module so it is relfexive. Now $E^* = E_i^\circ + E_2^\circ$ (direct) and $E^{**} = E_1^{\circ\circ} + E_2^{\circ\circ}$ (direct). Clearly $E_2^{\circ\circ}$ is isomorphic to E_2^{**} and the restriction of the canonical homomorphism $E \to E^{**}$ maps E_2 isomorphically onto $E_2^{\circ\circ}$. If $\psi = \psi_1 + \psi_2$ where $\psi_1 \in E_1^{\circ\circ}$ let $x \in E_2$ be such that $x \to \psi_2$ under the map $E \to E^{**}$. Then since $\psi - \psi_2 \in E_1^{\circ\circ}$ and since $X = \{\varphi_1, \varphi_2, \cdots, \varphi_n\} \subset E_1^\circ$ we get $\psi - \psi_2 \in X^\circ$. This completes the proof.

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