BOUNDS FOR DERIVATIVES IN ELLIPTIC BOUNDARY VALUE PROBLEMS

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I. Introduction. In a recent paper [7], Payne and Weinberger gave pointwise bounds for solutions of second order uniformly elliptic partial differential equations. The bounds for the function and its gradiant involved derivatives of the boundary data. Later [2] the present authors gave a method for obtaining bounds in which no derivatives of the boundary data appeared. Pointwise bounds for derivatives were not dealt with. In [4] the authors gave a method for bounding derivatives for Poisson's equation. The method was, however, restricted to the Laplace operator (or the constant coefficient case) and was not generally applicable.

In this paper we consider the operator

$$Lu \equiv (a^{ij}u_{,i})_{,j} \tag{1.1}$$

where u is a sufficiently smooth function defined in some region R (with boundary C) of Euclidean N dimensional space. Here the notation $u_{,i}$ denotes the partial derivative of u with respect to the cartesian coordinate x^i. In (1.1) the summation convention is used, i.e. $(a^{ij}u_{,i})_{,j} \equiv \sum_{i}^{N}{}_{j=1} (a^{ij}u_{,i})_{,j}$. The coefficient matrix a^{ij} may be a function of position and is assumed to be uniformly positive definite and bounded above. That is there exist positive constant a_0 and a_1 such that

$$a_0 \sum_{i=1}^{N} \xi_i^2 \leqq a^{ij}\xi_i\xi_j \leqq a_1 \sum_{i=1}^{N} \xi_i^2 \tag{1.2}$$

for any real vector $\xi = (\xi_1, \cdots, \xi_N)$. We shall give a method involving the use of a parametrix, for obtaining bounds on any derivative of a function u at an arbitrary interior point P of R. These bounds are in terms of Lu and $\max_{S(P)} |u|$, where $S(P)$ is a sphere containing P. Estimates of this type for very general elliptic operators are described by John [6]. His method does not involve the parametrix and hence the expressions which could be derived would turn out to be quite different. Thus the problem is reduced to that of bounding $\max_{S(P)} |u|$ in terms of quantities which are data of some boundary value problem. We assume throughout that Lu and the coefficients a^{ij} are sufficiently smooth so that all subsequent indicated operations are valid.

In this paper we concern ourselves only with the derivation of appropriate a priori inequalities. The manner of applying such ine-

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qualitites to obtain bounds has been thoroughly discussed in previous papers (see e.g. [2, 4, 7]).

II. Mean value expressions. To obtain the desired bounds we shall first need a certain expression which is in a sense analogous to the solid mean value theorem for harmonic function. One such expression was given in [2]; however, it is quite complicated. We derive now a simpler expression.

Since a fundamental solution corresponding to the operator L is not in general known we make use of a Levi function (or parametrix) (c.f. Miranda [6]).

Let P and Q be two points in R. One possible definition of a parametrix is

$$(2.1)\qquad \begin{aligned} \Gamma(P, Q) &= -(2\pi)^{-1}[a(Q)a(P)]^{1/4}\log\rho\ , && N=2 \\ \Gamma(P, Q) &= 2^{1/2(N-2)}[(N-2)\omega_N]^{-1}[a(Q)a(P)]^{1/4}\rho^{-(N-2)}\ , && N \geqq 3 \end{aligned}$$

where ω_N denotes the surface of the unit sphere in N dimensions,

$$\rho^2 = [a_{ij}(Q) + a_{ij}(P)](x_P^i - x_Q^i)(x_P^j - x_Q^j)\ ,$$

and $a(Q)$ denotes the determinant of the matrix $a_{ij}(Q)$, the inverse of $a^{ij}(Q)$. If the a^{ij} are twice continuously differentiable in the neighborhood of P, this function Γ has the property that

$$(2.2)\qquad L_Q\Gamma = O(r_{PQ}^{-(N-2)}),\ r_{PQ} \to 0$$

where r_{PQ} is the distance from P to Q. An alternate form for a parametrix is

$$(2.3)\qquad \begin{aligned} \bar\Gamma(P, Q) &= (2\pi)^{-1}[a(P)]^{1/2}\log\bar\rho \\ \bar\Gamma(P, Q) &= [(N-2)\omega_N]^{-1}[a(P)]^{1/2}[\bar\rho]^{-(N-2)}\ . \end{aligned}$$

Here $\bar\rho^2 = a_{ij}(P)(x_P^i - x_Q^i)(x_P^j - x_Q^j)$. The function $\bar\Gamma(P, Q)$ is such that if the a^{ij} are continuously differentiable in the neighborhood of P, then

$$(2.4)\qquad L_Q\bar\Gamma = O(r_{PQ}^{-(N-1)}),\ r_{PQ} \to 0\ .$$

Comparing (2.2) and (2.4) we see that Γ is a better approximation to the fundamental solution than is $\bar\Gamma$ near $Q = P$.

Now let $S_a(P)$ be the interior of a sphere of radius a with center at P, and such that $S_a(P) \subset R$. We define the function $f_n(P, Q)$ as follows (for P fixed)

$$(2.5)\qquad \begin{aligned} &\text{(a)}\quad f_n(P, Q) = \begin{cases} 1, & Q = P \\ 0, & r_{PQ} \geqq a \end{cases} \\ &\text{(b)}\quad f_n^{(i)}(P, P) = 0,\ i = 1, 2, \cdots, N-1 \\ &\text{(c)}\quad f_n(P, Q) \in C^{n-1}(E^N) \end{aligned}$$

(continuous derivatives up to and including those of order $n-1$ at each point of Euclidean N-space.) One such function, for example, is the polynomial with values

$$\left[\int_{r_{PQ}}^{a}\rho^{n-1}(a^2-\rho^2)^{n-1}d\rho\right]\left[\int_0^a\rho^{n-1}(a^2-\rho^2)^{n-1}d\rho\right]^{-1},\ r_{PQ}\leqq a\ .$$

Another possible choice is the function

$$\left\{\int_{r_{PQ}}^{a}\exp\left[-\rho^{-2}(a^2-\rho^2)^{-1}\right]d\rho\right\}\left\{\int_0^a\exp\left[-\rho^{-2}(a^2-\rho^2)^{-1}\right]d\rho\right\}^{-1},\ r_{PQ}\leqq a$$

which satisfies (2.5) for all n. Clearly

$$\Gamma_n(P, Q)\equiv f_n(P, Q)\Gamma(P, Q) \tag{2.6}$$

also satisfies (2.2). But $\Gamma_n(P, Q)$ has all derivatives up to and including those of order $n-1$ vanishing on $r_{PQ}=a$. Using (2.1) and (2.2) we find from Green's identity that

$$u(P)=\int_{S_a(P)}u(Q)L_Q\Gamma_n(P, Q)dV_Q-\int_{S_a(P)}\Gamma_n(P, Q)Lu(Q)dV_Q\ , \tag{2.7}$$

provided $n\geqq 2$. This expression is analogous to (5.8) of [2]. In addition to being simpler it possesses the advantage that the integration is taken over spheres, rather than ellipsoids which vary from point to point. We could as well have defined

$$\bar{\Gamma}_n(P, Q)=f_n(P, Q)\bar{\Gamma}(P, Q) \tag{2.8}$$

and obtained

$$u(P)=\int_{S_a(P)}u(Q)L_Q\bar{\Gamma}_n(P, Q)dV_Q-\int_{S_a(P)}\bar{\Gamma}_n(P, Q)Lu(Q)dV_Q\ , \tag{2.9}$$

with $n\geqq 2$.

III. Pointwise bounds. Either (2.7) or (2.9) can be used to obtain bounds in the Dirichlet problem. Using the Schwarz inequality we have

$$\left[\int_{S_a(P)}u(Q)L_Q\bar{\Gamma}_n(P, Q)dV_Q\right]^2\leqq\left[\int_R u^2r_{PQ}^{-q}dV\right]^2\left[\int_{S_a(P)}r_{PQ}^{q}(L\bar{\Gamma}_n)^2dV_Q\right]^2 . \tag{3.1}$$

Equation (2.9) together with (3.1) and the bounds given by Theorem I and II of [2], yield pointwise bounds for u in terms of Lu in R and the values of u on C.

In order to bound the first derivatives of u we can use (2.7), with $n\geqq 3$, to obtain

$$\frac{\partial u(P)}{\partial x_P^i} = \int_{S_a(P)} u(Q) L_Q \frac{\partial \Gamma_n(P, Q)}{\partial x_P^i} d V_Q - \frac{\partial}{\partial x_P^i}\left[\int_{S_a(P)} \Gamma_n(P, Q) L u(Q) d V_Q\right]. \tag{3.2}$$

Hence we have

$$\left|\frac{\partial u(P)}{\partial x_P^i}\right| \leqq \max_{Q \in S_a(P)} |u(Q)| \int_{S_a(P)} \left| L_Q \frac{\partial \Gamma_n(P, Q)}{\partial x_P^i}\right| d V_Q + \left|\frac{\partial}{\partial x_P^i}\left[\int_{S_a(P)} \Gamma_n(P, Q) L u(Q) d V_Q\right]\right|. \tag{3.3}$$

Now if a is so chosen that we can obtain a bound for $\max_{Q \in S_a(P)} |u(Q)|$ then (3.3) provides a bound for $|\partial u(P)/\partial x_P^i|$. If, for example, the least distance from P to the boundary C is r_0, then we could choose $a = (1/2)r_0$. Thus the closure $\overline{S_a(P)}$ of $S_a(P)$ is a compact subset of R and hence only interior bounds for u are required. Note that we could not replace (3.2) by a similar expression involving $\bar{\Gamma}_n$ since the integrals on the right would not exist.

We note from (3.2) that

$$\int_{S_a(P)} L_Q \frac{\partial \Gamma_n(P, Q)}{\partial x_P^i} d V_Q = 0 . \tag{3.4}$$

Thus if $n \geqq 4$ we have the representation

$$\frac{\partial^2 u(P)}{\partial x_P^i \partial x_P^j} = \int_{S_a(P)} [u(Q) - u(P)] L_Q \frac{\partial^2 \Gamma_n(P, Q)}{\partial x_P^i \partial x_P^j} d V_Q - \frac{\partial^2}{\partial x_P^i \partial x_P^j}\left[\int_{S_a(P)} \Gamma_n(P, Q) L u(Q) d V_Q\right] \tag{3.5}$$

since

$$[u(Q) - u(P)] L_Q \frac{\partial^2 \Gamma_n(P, Q)}{\partial x_P^i \partial x_P^j} = O(r_{PQ}^{-(N-1)}) \tag{3.6}$$

for $r_{PQ} \to 0$. From (3.5) we see that

$$\left|\frac{\partial^2 u(P)}{\partial x_P^i \partial x_P^j}\right| \leqq \max_{Q \in S_a(P)} \left|\frac{u(Q) - u(P)}{r_{PQ}}\right| \int_{S_a(P)} r_{PQ} \left| L_Q \frac{\partial^2 \Gamma_n(P, Q)}{\partial x_P^i \partial x_Q^j}\right| d V_Q + \left|\frac{\partial^2}{\partial x_P^i \partial x_P^j}\left[\int_{S_a(P)} \Gamma_n(P, Q) L u(Q) d V_Q\right]\right|. \tag{3.7}$$

Now

$$\max_{Q \in S_a(P)} \left|\frac{u(Q) - u(P)}{r_{PQ}}\right| \leqq \max_{Q \in S_a(P)} |\operatorname{grad} u(Q)| . \tag{3.8}$$

Clearly we can use (3.3) with a smaller value of a to bound the right hand side of (3.8). Thus we can bound an arbitrary second derivative of u in terms of Lu in R and the maximum of $|u|$ over a compact subset of R. In order to treat an arbitrary third derivative we note from (3.5) that

$$(3.9)\quad \int_{S_a(P)}(x_Q^\alpha - x_P^\alpha)L_Q\frac{\partial^2\Gamma_n(P,Q)}{\partial x_P^i\partial x_P^j}dV_Q = \frac{\partial^2}{\partial x_P^i\partial x_P^j}\left[\int_{S_a(P)}\Gamma_n(P,Q)Lx_Q^\alpha dV_Q\right]$$

for $\alpha, i, j = 1, \cdots, N$. Combining (3.9) and (3.5) we have

$$\begin{aligned}\frac{\partial^2 u(P)}{\partial x_P^i\partial x_P^j} &= \int_{S_a(P)}[u(Q) - u(P) - (x_Q^\alpha - x_P)u_{,\alpha}(P)]L_Q\frac{\partial^2\Gamma_n(P,Q)}{\partial x_P^i\partial x_P^j}dV_Q\\ (3.10)\qquad &- \frac{\partial^2}{\partial x_P^i\partial x_P^j}\left[\int_{S_a(P)}\Gamma_n(P,Q)Lu(Q)dV_Q\right]\\ &- u_{,\alpha}(P)\frac{\partial^2}{\partial x_P^i\partial x_P^j}\left[\int_{S_a(P)}\Gamma_n(P,Q)Lx_Q^\alpha dV_Q\right]\end{aligned}$$

where we have summed over α from 1 to N. It follows from (3.10) that if $n \geqq 5$

$$\begin{aligned}\frac{\partial^3 u(P)}{\partial x_P^i\partial x_P^j\partial x_P^k} &= \int_{S_a(P)}[u(Q) - u(P) - (x_Q^\alpha - x_P^\alpha)u_{,\alpha}(P)]L_Q\frac{\partial^3\Gamma_n(P,Q)}{\partial x_P^i\partial x_P^j\partial x_P^k}dV_Q\\ (3.11)\qquad &- \frac{\partial^3}{\partial x_P^i\partial x_P^j\partial x_P^k}\left[\int_{S_a(P)}\Gamma_n(P,Q)Lu(Q)dV_Q\right]\\ &- u_{,\alpha}(P)\frac{\partial^3}{\partial x_P^i\partial x_P^j\partial x_P^k}\left[\int_{S_a(P)}\Gamma_n(P,Q)Lx_Q^\alpha dV_Q\right].\end{aligned}$$

The first integral on the right may be bounded as

$$\begin{aligned}(3.12)\qquad &\left|\int_{S_a(P)}[u(Q) - u(P) - (x_Q^\alpha - x_P^\alpha)u_{,\alpha}(P)]L_Q\frac{\partial^3\Gamma_n(P,Q)}{\partial x_P^i\partial x_P^j\partial x_P^k}dV_Q\right|\\ &\leqq \max_{\substack{Q\in S_a(P)\\ \alpha\,\beta=1\cdots N}}|u_{,\alpha\beta}(Q)|\int_{S_a(P)}r_{PQ}^2\left|L_Q\frac{\partial^3\Gamma_n(P,Q)}{\partial x_P^i\partial x_P^j\partial x_P^k}\right|dV_Q\,.\end{aligned}$$

Now (3.11) and (3.12) can be used to reduce the problem of bounding third derivatives to that of bounding second derivatives. It is clear how to proceed to higher derivatives. In each of the preceding bounds certain differentiability assumptions must be made. These conditions become more and more stringent the more derivatives of u that we wish to bound. Some conditions of this nature are of course required since in general u cannot be expected to be smooth.

Thus for an arbitrary derivative at P the method described above yields a bound in terms of Lu in R and the maximum of $|u|$ on a compact subset (for example $S_a(P)$ for some a) of R. These bounds, together with bounds for $|u|$ in $S_a(P)$ in terms of data in various boundary value problems, yield pointwise bounds for derivatives at interior points in terms of the respective data. For such bounds see [1, 2, 3, 4, 5, 7, 8].

The techniques which we have used here to bound derivatives of solutions to boundary value problems at interior points in terms of the operator and bounds for the solution itself, will carry over quite naturally to higher order equations and to equations of other than elliptic type.

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