SILOV TYPE C ALGEBRAS OVER A CONNECTED LOCALLY COMPACT ABELIAN GROUP II

ALFRED B. WILLCOX

In 1951 Silov [6] published a structure theory for a class of translation invariant function algebras on compact abelian groups. In 1959 the author extended portions of this structure theory to similar algebras defined on connected locally compact abelian groups [8]. One of the conditions which both Silov and the author employed was that all of the maximal regular ideals of the algebra be determined by the elements of the underlying groups in the usual way. In 1958 de Leeuw [2] published results characterizing the maximal ideals of an algebra of functions on a compact abelian group which satisfies all of Silov's conditions except this one. The results to be reported here constitute, in effect, an additional chapter to [8] motivated by an attempt to generalize de Leeuw's results. We will adopt de Leeuw's terminology, calling an algebra of the type studied in [6] and [8] a *Silov-homogeneous algebra* and an algebra which satisfies the weakened conditions of de Leeuw and the present paper a *homogeneous algebra.*

1. It is appropriate to begin with a brief discussion of an example of a Silov-homogeneous algebra which is a generalization of the group algebra of a locally compact abelian group. Domar, Beurling, Wermer and others have studied algebras of this type and we shall refer to results of Domar [3] in this connection. It is also an example which can be generalized in a natural way to include algebras of the type which we wish to discuss here and for which our results take a particularly simple form.

Let $G = \{s, t, \cdots\}$ be a locally compact abelian group and let $\hat{G} = \{\chi, \cdots\}$ be the group of characters of G. Suppose that p is a real measurable function on $\hat{G}$ which is bounded on compact sets and satisfies the conditions

$$p(\chi) \geqq 1 \tag{1.1}$$

$$p(\chi_1\chi_2) \leqq p(\chi_1)p(\chi_2) \tag{1.2}$$

$$\sum \frac{1}{n^2} \log p(\chi^n) < \infty \tag{1.3}$$

for all $\chi, \chi_1, \chi_2 \varepsilon \hat{G}$.

Received August 5, 1963. This work was supported by National Science Foundation Grant. G 11742.

Let $R(p, \hat{G})$ be the algebra of all Fourier transforms f of those elements $\hat{f}$ of $L_1(\hat{G})$ for which $\|f\| = \int |\hat{f}(\chi)| p(\chi)d\chi < \infty$. That is, $R(p, \hat{G})$ is an algebra of functions $f : f(t) = \int \hat{f}(\chi)\overline{\chi(t)}d\chi$ defined on $\hat{G}$ with norm $\|f\|$ as defined above and with pointwise operations. $R(p, \hat{G})$ is well known to be isomorphic and isometric to the group algebra $L_1(p, \hat{G})$ with a "weight function" p. It is also known [3] that

(a) every maximal regular ideal of $R(p, \hat{G})$ is determined by an element of G ($M_t = \{f : f(t) = 0\}$) and the space of maximal regular ideals of $R(p, \hat{G})$ with the Gelfand topology is homeomorphic to G,

(b) $R(p, \hat{G})$ is a semi-simple completely regular Banach algebra in the sense of Silov (Rickart's terminology [5]),

(c) $R(p, \hat{G})$ is closed under translation by elements of G; if $f \varepsilon R(p, \hat{G})$ and $h \varepsilon G$ then $f_h \varepsilon R(p, \hat{G})$, where $f_h(t) = f(t - h)$ for all $t \varepsilon G$,

(d) the norm in $R(p, \hat{G})$ is translation invariant; $\|f\| = \|f_h\|$ for all f and h,

(e) the elements of $R(p, \hat{G})$ are continuous under translation; $\|f - f_h\| \to 0$ as $h \to 0$,

(f) $R(p, \hat{G})$ is Tauberian in the sense that the set of elements with compact support is dense.

These are the defining conditions for a Silov-homogeneous algebra. In addition, one can show that

(g) $R(p, \hat{G})$ is closed under multiplication by elements of $\hat{G}$; if $f \varepsilon R(p, \hat{G})$ and $\chi \varepsilon \hat{G}$ then the function $\chi f : (\chi f)(t) = \chi(t)f(t)$ is also in $R(p, \hat{G})$. In fact, χf is the Fourier transform of the translate $\hat{f}_\chi$ of the function $\hat{f}$ whose transform is f,

(h) the mapping $f \to \chi f$ is continuous on $R(p, \hat{G})$ for each $\chi \varepsilon \hat{G}$ and the mapping $\chi \to \chi f$ is continuous on $\hat{G}$ for each f with compact support.

(i) $\hat{G}f$ spans $R(p, \hat{G})f$ topologically if f has compact support.

$R(p, \hat{G})$ is not of type C but its C-completion is locally isomorphic to an algebra $TK_\omega(G)$ [8]. An argument similar to that on page 1293 of [8] shows that if $p(\chi^n) = o(n)$ then the C-completion of $R(p, \hat{G})$ is the algebra $C_0(G)$ of all continuous complex functions vanishing at ∞ on G. Thus every closed primary ideal in $R(p, \hat{G})$ is maximal [8, p. 1293].

Now suppose that S is a measurable subsemigroup of $\hat{G}$ which contains the identity and generates $\hat{G}$ (in the sense that $\hat{G}$ is the smallest subgroup containing S). Suppose, further, that p is a real measurable function defined on S satisfying conditions (1.1) and (1.2) for characters in S. Let $R(p, S)$ be the subset of $R(p, \hat{G})$ determined by those $\hat{f}$ which vanish a. e. outside of S^{-1}. The algebras $R(p, S)$ are vanishing algebras in the sense of Simon [7] and others and include the algebras of generalized analytic functions of Arens and Singer [1]. $R(p, S)$

is actually a closed subalgebra of $R(p, \hat{G})$ and, as a Banach algebra, has many, but not all, of the properties which we have listed above.

THEOREM 1. *$R(p, S)$ is a closed translation invariant subalgebra of $R(p, \hat{G})$ which separates points of G and has the following additional properties:*

(a) *$R(p, S)$ is closed under multiplication by elements of S,*

(b) *The algebra $[Sf]$ generated by Sf is dense in $R(p, S)f$,*

(c) *The mapping $f \to \chi f$ is continuous in f for each $\chi \varepsilon S$,*

(d) *If p satisfies condition (1.3) for all $\chi \varepsilon S$ then the mapping $\chi \to \chi f$ is continuous on S for each $f \varepsilon R(p, S)$ with compact support.*

Proof. Let $R = R(p, S)$ and suppose that $f, g \in R$. If f and g are the Fourier transforms of $\hat{f}$ and $\hat{g}$ then fg is the transform of $\hat{f} * \hat{g}$. Since

$$\hat{f} * \hat{g}(\chi) = \int \hat{f}(\chi_1)\hat{g}(\chi\chi_1^{-1})d\chi_1$$

and since S^{-1} is a semigroup, it is easy to see that $\hat{f} * \hat{g}(\chi) = 0$ a. e. for $\chi \notin S^{-1}$. Thus $fg \in R$. Straightforward computations show that $f + g$ and αf belong to R, where α is a complex number, and that R is closed in $R(p, \hat{G})$. From the fact that the translate f_s of f is the transform of the function $\hat{g}(\chi) = \chi(s)\hat{f}(x)$ it follows that R is closed under translation. Since S is a generating subsemigroup of G it separates points of G and it follows easily that R does also.

(a) Let $\chi_1 \in S$. Then

$$\begin{aligned}[\chi_1 f](t) &= \chi_1(t)\int \hat{f}(\chi)\overline{\chi(t)}d\chi \\ &= \int \hat{f}(\chi)\overline{[\chi\chi_1^{-1}](t)}d\chi \\ &= \int \hat{f}(\chi\chi_1)\overline{\chi(t)}d\chi\ .\end{aligned}$$

Since $\chi_1^{-1} \in S^{-1}$ then, for all $\chi \notin S^{-1}$, $\chi\chi_1 \notin S^{-1}$. Thus $\hat{f}(\chi\chi_1) = 0$ for almost every $\chi \notin S^{-1}$. Thus $\chi_1 f \in R$.

(b) Let $f \in R$ and let I be the closure of the space $[Sf]$. I is then the transfrom of the closure $\hat{I}$ in $L_1(p, \hat{G})$ of the space spanned by the translates of $\hat{f}$ by members of S. Thus $\hat{I}$ is S-invariant in $L_1(p, \hat{G})$. An argument analogous to the by-now-classical one for group algebras [4, 31F] with the use of Domar's representation of the linear functionals on $R(p, \hat{G})$ as Borel measures on G [3, p. 10] shows that I is an ideal in R. Thus, if $g \in R$, $gf \in I$ so gf is a limit of linear combinations of elements of Sf.

(c) and (d) are obvious since R is a subalgebra of $R(p, G)$.

We will refer to the parts of Theorem 1 as conditions (1, a), (1, b) and (1, c).

In general, not every maximal regular ideal in $R(p, S)$ is determined by an element of G. It is natural, therefore, to look for a characterization of these ideals in such an algebra. If $t \in G$ then the set of all elements of $R(p, S)$ which vanish at t is a maximal regular ideal, and distinct elements of G determine distinct maximal regular ideals because $R(p, S)$ separates points of G. But if there are other maximal regular ideals in $R(p, S)$ what do they look like? An answer to this question is not hard to find by more-or-less standard techniques of harmonic analysis. The maximal regular ideals are determined uniquely by the homomorphisms φ of S into the multiplicative group of the complexes which take the identity into 1 and satisfy the condition

$$|\varphi(\chi)| \leqq p(\chi)$$

for all $\chi \in S$. However, we will delay the proof of this fact in order to proceed to the more general class of algebras to which we referred in the opening paragraph. We will see that these algebras share with $R(p, S)$ weakened forms of conditions (1, a), (1, b) and (1, c) and will show that these properties result in a preliminary form of the above characterization of the maximal regular ideals.

2. Let G be a connected locally compact abelian group. An algebra R of functions on G will be called a *homogeneous algebra* on G if:

(2.1) R is a Banach algebra of continuous complex valued functions vanishing at ∞ on G, having the usual pointwise operations and in which convergence in the norm implies pointwise convergence,

(2.2) R is completely regular on G; i. e., R contains functions which are 1 on arbitrary compact sets and 0 on arbitrary disjoint closed sets,

(2.3) R is closed under translation by elements of G and has a translation invariant norm,

(2.4) the mapping $t \to f_t$ ($f_t(s) = f(s - t)$ for all s) is continuous from G to R for each fixed $f \in R$,

(2.5) the set R' of elements of R with compact support in G is dense in R,

(2.6) if f_n and g_n are sequences of elements of R such that g_n has support in a fixed compact set C and for each $t_0 \in C$ there exists an $h \in G$ such that $|g_n(t)| \leqq |(f_n)_h(t)|$ holds for all t in a neighborhood of t_0, then $f_n \to 0$ implies $g_n \to 0$ in R.

If we assume, in addition to the six properties above, that every maximal regular ideal of R consists of the set of elements which vanish at a specified $t \in G$ then R is a Silov-homogeneous algebra in the sense of [8]. Actually, we have assumed a bit more. Condition (2.6) is, under these conditions, similar to the type C condition which was employed in [8]. For our present purposes, however, we require a condition on the algebra analogous to the type C condition but which involves only the underlying group G. This analog could be stated as follows: For $f \in R$ and $t \in G$ define $\|f\|_t$ to be the infinum of $\|g\|$ for all g which agree with f on a neighborhood of t, and define $|||f||| = \sup \|f\|_t : t \in G$. Then the norm in R is stronger that the uniform norm and weaker than $|||\cdot|||$ (hence is equivalent to $|||\cdot|||$).

It is not hard to show that, given conditions (2.1) – (2.5), this condition implies condition (2.6). However, (2.6) may actually be weaker and is in a form which will be most easily applied in the proof of the next theorem.

Examples of algebras which satisfy the above conditions can be found among the algebras $TK_\omega(G)$ discussed in [8]. These are certain algebras of functions on G with values in a primary B-algebra K. If such an algebra is (isomorphic to) an algebra of complex functions on G and is completely regular on G in the sense of (2.2) then it is a homogeneous algebra. Another example is the algebra of complex continuous functions vanishing at ∞ on the reals which are boundary values of analytic functions on the half-plane.

In case G is compact abelian a Silov-homogeneous algebra contains all of the characters of G and these elements generate the algebra. In case G is connected, locally compact and abelian a type C Silov-homogeneous algebra is closed under multiplication by elements of $\hat{G}$ (hence, by complete regularity, contains these characters modulo compact sets), and, for each $f \in R$, Rf is generated by $\hat{G}f$. In neither of these cases does a homogeneous algebra necessarily contain all characacters (mod compact sets). However, as deLeeuw points out in the compact case, R is generated by a semigroup of characters which also generates $\hat{G}$ in the group sense. We have something like this here.

THEOREM 2. *Let R be a homogeneous algebra on a connected locally compact abelian group G. There exists a generating subsemigroup S of $\hat{G}$ containing the identity such that*

(a) *R' is closed under multiplication by elements of S,*

(b) *For each $f \in R'$, Sf generates Rf topologically,*

(c) *For each compact subset C of G and each $\chi \in S$, the mapping $f \to \chi f$ is bounded on the set R_C of all elements of R with support C.*

Proof. Since the argument repeats certain of the constructions of [8] we will omit many of the details. It is well known that $G = E_m \times G_c$, where E_m is the m-dimensional vector group and G_c is compact. Denote an element of G by (s, t) where $s = (x_1, x_2, \cdots, x_m) \in E_m$

and $t \in G_c$. Consider an m-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ of positive real numbers, and define

$$C_\alpha = \{(s, t) : |x_i| \leqq \alpha_i , \quad t \in G_c\}$$

and

$$D_\alpha = \{(s, 1) : s = (k_1\alpha_1, k_2\alpha_1, \cdots, k_m\alpha_m),\ k_i \text{ an integer}\} ,$$

where in the definition of D_α 1 denotes the identity element of G_c. If we make the usual identifications, C_α is isomorphic and homeomorphic to the quotient group G/D_α. Moreover, G is covered by the compact sets $C_{k\alpha}$, $k = 1, 2, \cdots$.

If f is an element of R which vanishes outside of C_α then by adding together the appropriate translates of f we may obtain an element g of R which agrees with f on C_α and is D_α-periodic on $C_{k\alpha}$ for any prechosen k. That is, g is periodic on $C_{k\alpha}$ and any element of D_α is a period. We call g a D_α-periodic extension of f to $C_{k\alpha}$.

Let $C = C_\alpha$, $D = D_\alpha$, and $C_k = C_{k\alpha}$, $k = 2, 3, \cdots$. Denote by $k(C)$ the set of elements of R which vanish on C and by $\bar{R}$ the difference algebra $R/k(C)$. By complete regularity on G, $\bar{R}$ has an identity $\bar{e}$. Let $\bar{R}_\alpha$ be the subalgebra of $\bar{R}$ generated by $\bar{e}$ and the images in $\bar{R}$ of all f in R with support in C and their translates by elements of C, D-periodically extended to C_2. $\bar{R}_\alpha$ is a homogenous space of functions on G/D in the sense of Silov, so is generated by a set $\bar{S}$ of characters of G/D [6, 2.7]. Each element of $\bar{S}$ is uniquely associated with a character χ of G such that $\chi(t) = 1$ on D. Call the set of characters of G determined in this way S_α.

By complete regularity of R, $\bar{R}_\alpha$, and hence $\bar{S}$, separates points of G/D. It follows that $\bar{S}$ generates the character group of G/D and hence that S_α generates the subgroup of $\hat{G}$ consisting of those elements which are 1 on D.

If $\chi \in S_\alpha$ let $\bar{\chi}$ be the corresponding element of $\bar{R}_\alpha$. Then $\bar{\chi} = \bar{f} + \mu\bar{e}$ where $\bar{f} = \lim \bar{f}_n$ and each f_n is a suitably extended translate of an element of R supported by C. Let f'_n be a Cauchy sequence in R with $\bar{f}_n = \bar{f}'_n$ and let f''_n be a D-periodic extension of f_n to C_4. If g is an element of R which is 1 on C_2 and 0 outside of C_4 then the sequence $g_n = f''_n g$ satisfies condition (2.6) relative to the Cauchy sequence $f'_n/\|g\|$. It follows that g_n is Cauchy and converges to an element of R which is $\chi(t) - \mu$ on C_2. Thus R contains an element which is $\chi(t)$ on C_2. The above construction could be repeated for each C_k so we conclude that R contains elements which are $\chi(t)$ on arbitrary compact sets. Thus R' is closed under multiplication by elements of S_α. It is also clear from the above that $S_\alpha f$ generates Rf for any f which vanishes outside of $C = C_\alpha$.

Let S be the set of all $\chi \in \hat{G}$ which belong to R on arbitrary compact sets. This is, $\chi \in S$ if and only if given any compact set C in

G there exists an element f of R such that $f(t) = \chi(t)$ on C. We have seen that $S_\alpha \subset S$ so S is not vacuous. Clearly S is a subsemigroup of $\hat{G}$. Moreover, since $G = E_m \times G_c$ it follows that any character χ of G is identically 1 on some subgroup D_α of the type discussed above. Thus, as we have seen, $\chi = \chi_1\chi_2^{-1}$ where $\chi_1, \chi_2 \in S_\alpha \subset S$, so S is a generating subsemigroup of $\hat{G}$. By complete regularity of R, S contains the identity character. It also follows from what we have proved above that R' is closed under multiplication by S and that, for $f \in R'$ Sf generates Rf.

If $g(t) = \chi(t)$ on C and f vanishes outside of C, then

$$\| \chi f \| = \| gf \| \leqq \| g \| \| f \| ,$$

so the mapping $f \to \chi f$ is bounded on R_C. This completes the proof of the theorem.

Conditions (2, a), (2, c) and (2, c) are weakened forms of conditions (1, a), (1, b) and (1, c) which are satisfied by the algebras $R(p, S)$ of § 1. We are now ready to prove a lemma which can be considered as providing the algebraic part of our principle results.

3. If S is a semigroup of characters of G denote by $P(S)$ the set of complex linear combinations of elements of S. Let R be an algebra of continuous complex functions vanishing at ∞ on a connected locally compact abelian group G, and suppose that R contains a dense subset R' such that

(3, a) R' is closed under multiplication by a semigroup S of characters of G containing the identity,

(3, b) for $f \in R'$, $P(S)f$ is dense in Rf,

(3, c) for each $f \in R'$ the mapping $g \to \chi g$ is continuous on $P(S)f$.

Both algebras $R(p, S)$ and homogeneous algebras satisfy these conditions as we have seen. In $R(p, S)$ the subset R' may to taken to be the entire algebra.

Let $\mathrm{Hom}(S, C)$ be the set of all homomorphisms of S into the multiplicative group of complex numbers which carry the identity into 1. If $\varphi \in \mathrm{Hom}(S, C)$ and $\gamma \in P(S)$ define $\varphi(\gamma) = \sum \alpha_i \varphi(\chi_i)$, where $\gamma = \sum \alpha_i \chi_i$. We will call an element φ of $\mathrm{Hom}(S, C)$ an *R-semicharacter* of S if there exists an element $f \in R'$ such that

$$(3.1) \qquad | \varphi(\gamma) | \leqq \| \gamma f \| \text{ for all } \gamma \in P(S) \text{ and}$$

$$(3.2) \qquad \text{if } \lim \gamma_n f = f^2 \text{ then } \lim \varphi(\gamma_n) \text{ exists and is not } 0 .$$

In the following, if M is a maximal regular ideal in R we will use the notation $f(M)$ for the image of f in the difference algebra R/M, considered as the complex field.

LEMMA 3. *Under the conditons stated above, there is a one-to-one correspondence between the set of maximal regular ideals of R and the set of R-semicharacters of S. The maximal regular ideal M corresponds to the R-semicharacter φ if and only if*

$$g(M) = \lim \varphi(\gamma_n) \text{ whenever } \gamma_n f \to gf ,$$

where f is as in the definition of the semicharacter φ, and

$$\varphi(\chi) = \frac{(\chi f)(M)}{f(M)} ,$$

where f is any element of R' such that $f(M) \neq 0$.

Proof. Let φ be an R-semicharacter of S. Let $f \in R'$ satify the conditions, relative to φ, of the definition. If $g \in R$ then, by hypothesis, there exists a sequence γ_n in $P(S)$ such that $\lim \gamma_n f = gf$. Define $\bar{\varphi}(g) = \lim \varphi(\gamma_n)$. $\bar{\varphi}$ is clearly a linear functional on R. Suppose that $\lim \gamma_n f = gf$, $\lim \sigma_n f = hf$ and $\lim \rho_n f = ghf$. Then $\lim (\gamma_n f)(\sigma_n f) = \lim (\gamma_n \sigma_n) f^2 = ghf^2$ and $\lim \rho_n f^2 = ghf^2$ so $\lim (\gamma_n \sigma_n - \rho_n) f^2 = 0$. By condition (3.1),

$$\lim \varphi(\gamma_n \sigma_n - \rho_n) = \lim \varphi(\gamma_n)\varphi(\sigma_n) - \lim \varphi(\rho_n) = 0 ,$$

so $\bar{\varphi}(gh) = \bar{\varphi}(g)\bar{\varphi}(h)$. Thus $\bar{\varphi}$ is multiplicative and determines a maximal regular ideal $M = \{g : \bar{\varphi}(g) = 0\}$. $\bar{\varphi}(g) = g(M)$ for all $g \in R$.

If $\gamma_n f \to f^2$ then $\gamma\gamma_n f \to \gamma f^2$ by (3, c). Thus $\bar{\varphi}(\gamma f) = \varphi(\gamma)\bar{\varphi}(f)$, and $\bar{\varphi}(f) \neq 0$ by (3.2). Thus

$$\varphi(\gamma) = \frac{\bar{\varphi}(\gamma f)}{\bar{\varphi}(f)} = \frac{(\gamma f)(M)}{f(M)} ,$$

for all $\gamma \in P(S)$.

Now suppose that $f' \in R'$ also satisfies the condition of the definition of the R-semicharacter φ. Let M' be the maximal regular ideal constructed as above using f' in place of f. Then also

$$\varphi(\gamma) = \frac{(\gamma f')(M')}{f'(M')}$$

for all $\gamma \in P(S)$. There exists an element $g \in R'$ such that

$$g(M) \neq 0 \neq g(M') ,$$

and it is easy to see that

$$\varphi(\gamma) = \frac{(\gamma g)(M)}{g(M)} = \frac{(\gamma g)(M')}{g(M')}$$

for all γ. Let $\lim \rho_n g = hg$, then by the above we see that

$$\lim \varphi(\rho_n) = h(M) = h(M')$$

for all $h \in R$. Thus $M = M'$ and M is independent of the choice of the function f.

Now let M be a maximal regular ideal. Since R' is dense in R there exists an $f \in R'$ such that $f(M) \neq 0$. Define

$$\varphi(\chi) = \frac{(\chi f)(M)}{f(M)}$$

for all $\chi \in S$. It is clear that φ is independent of the choice of f with the above properties. φ is a homomorphism of S, for

$$\varphi(\chi_1\chi_2) = \frac{(\chi_1\chi_2 f)(M)}{f(M)} = \frac{(\chi_1\chi_2 f^2)(M)}{f^2(M)}$$

$$= \frac{(\chi_1 f)(M)(\chi_2 f)(M)}{f^2(M)} = \varphi(\chi_1)\varphi(\chi_2) .$$

If we choose f so that $f(M) = 1$ then, by the obvious linearity, $|\varphi(\gamma)| = |(\gamma f)(M)| \leqq ||\gamma f||$ for all $\gamma \in P(S)$. Moreover, if $\gamma_n f \to f^2$ then $\lim \varphi(\gamma_n) = f(M) \neq 0$. Thus φ is an R-semicharacter of S.

It is easily seen that this correspondence between maximal regular ideal and R-semicharacters is one-to-one.

COROLLARY. *If R satisfies the conditions of Lemma* 3 *and if, for each $f \in R'$, the mapping $\chi \to \chi f$ is continuous on S in the $\hat{G}$-topology, then the maximal regular ideals of R correspond to the continuous R-semicharacters of S.*

4. For both $R(p, S)$ and homogeneous algebras we can sharpen somewhat the above characterization of the maximal regular ideals.

In the case of $R(\mathrm{p}, S)$ suppose that φ is an R-semicharacter of S. Then $|\varphi(\chi)| \leqq ||\chi f||$ where f is any element of R not in the corresponding maximal regular ideal. But in $R(p, S)$ it is easy to see that $||\chi f|| \leqq p(\chi) ||f||$ so we conclude that

$$(4.1) \qquad |\varphi(\chi)| \leqq p(\chi) \text{ for all } \chi \in S .$$

Conversely, suppose $\varphi \in \mathrm{Hom}(S, C)$ satisfies (4.1). Then, if φ is not a. e. 0, we can find in R a function f which is the Fourier transform of a function $\hat{f}$ for which $\int \hat{f} * \hat{f}(\chi)\varphi(\chi)d\chi = \left[\int \hat{f}(\chi)\varphi(\chi)d\chi\right]^2 = 1$. If $\gamma = \sum \alpha_i \chi_i$ then

$$| \varphi(\gamma) | = \left| \varphi(\gamma) \cdot \int \hat{f}(\chi)\varphi(\chi)d\chi \right| = \left| \int \sum \alpha_i \hat{f}_{\chi_i}(\chi)\varphi(\chi)d\chi \right|$$
$$\leqq \int \left| \sum \alpha_i \hat{f}_{\chi_i}(\chi) \right| p(\chi)d\chi = \| \chi f \| .$$

Moreover,

$$| \varphi(\gamma) - 1 | = \left| \int \sum \alpha_i \hat{f}_{\chi_i}(\chi)\varphi(\chi)d\chi - \int \hat{f} * \hat{f}(\chi)\varphi(\chi)d\chi \right| \leqq \| \gamma f - f^2 \| ,$$

so if $\lim \gamma_n f = f^2$ then $\lim \varphi(\gamma_n) = 1$. Thus φ is an R-semicharacter of S.

If we call an element of Hom (S, C) which satisfies (4.1) a *p-semicharacter* of R then we have shown that the R-semicharacters of S are precisely the p-semicharacters of S. Thus we have.

THEOREM 4. *The maximal regular ideals of $R(p, S)$ are in one-to-one correspondence with the p-semicharacters of S.*

Observe that this result is indeed a sharpening of the earlier characterization of the maximal resular ideals of $R(p, S)$. For one thing, the conditions for a p-semicharacter involve only the elements of S while those for an R-semicharacter involve all of $P(S)$. For another, the conditions for a p-semicharacter do not involve one in the actual structure of $R(p, S)$.

Theorem 4 is a natural generalization of the well-known results for $R(p, \hat{G})$. If, for instance, $S = \hat{G}$ and p satisfies (1.3) is it not hard to see that p must be identically 1. Thus the p-semicharacters of S become the continuous characters of $\hat{G}$, that is, by the duality theorem, the elements of G.

Now suppose that R is a homogeous algebra of functions on G, which we again assume to be connected as well as locally compact and abelian. If S is the semigroup of characters whose existence is asserted in Theorem 2, suppose that φ is an R-semicharacter of S and that M is the corresponding maximal regular ideal in R. Choose $f \in R'$ with $f(M) \neq 0$ and let C be the compact supporting set for f. If $\gamma \in P(S)$ and $g \in R$ is such that $g(t) = \gamma(t)$ for all $t \in C$ then $\gamma f = gf$. Thus $\varphi(\gamma) = (\gamma f)(M)/f(M) = g(M)$ and $| \varphi(\gamma) | \leqq \| g \|$. Thus

$$| \varphi(\gamma) | \leqq \inf \{ \| g \| : g(t) = \gamma(t) \text{on } C \} . \tag{4.2}$$

Conversely, if $\varphi \in$ Hom (S, C) satisfies (4.2) for some compact set C then let $e \in R'$ be a unit modulo C and $e' \in R'$ be a unit modulo the support of e. Then γe and $\gamma e'$ are both identically γ on C, so

$$| \varphi(\gamma) | \leqq \| \gamma e \| \text{ and } | \varphi(\gamma) | \leqq \| \gamma e' \| . \tag{4.3}$$

If $\lim \gamma_n e' = (e')^2$ then, since $e'e = (e')^2 e = e$, $\lim \gamma_n e = e$. Thus $\lim (\gamma_n - 1)e = 0$, so, by (4.3), $\lim \varphi(\gamma_n - 1) = \lim \varphi(\gamma_n) - 1 = 0$. Thus φ is an R-semicharacter of S.

Now suppose that φ satisfies the formally weaker condition

$$(4.4) \qquad |\varphi(\chi)| \leqq \inf \{\| g \| : g(t) = \chi(t) \text{ on } C\} \text{ for all } \chi \in S .$$

The let C_α be a compact set of the type employed in the proof of Theorem 2 with $C \subset C_\alpha$. Then, certainly, φ satisfies (4.4) for all $\chi \in S_\alpha$, the corresponding subset of S. The whole problem may then be transfered to the algebra $\bar{R}_\alpha$ in the proof of Theorem 2 which is an algebra of functions on the compact abelian group G/D_α. Methods of deLeeuw [2] may then be used to show that φ satisfies (4.2) for all $\gamma \in P(S_\alpha)$.

The proof of Theorem 2 shows that $S_\alpha f$ generates Rf topologically if f is supported by C_α, and it is easily seen that this, together with (4.2) for all $\gamma \in P(S_\alpha)$, is enough in the proof of Theorem 3 to show that φ determines a maximal regular ideal. Hence φ is an R-semicharacter of S, and the R-semicharacters of S may therefore be described as the elements of Hom (S, C) which satisfy condition (4.4) for some compact set in G.

Given a compact C define

$$p_C(\chi) = \inf \{\| g \| : g(t) = \chi(t) \text{ on } C\} .$$

Since $p_C(\chi)$ is just the norm of the element "χ" in the difference algebra $R/k(C)$, it is clear that p_C has properties (1.1) and (1.2) on S. Let $C_1 \subset C_2 \subset \cdots$ be a σ-covering of G by compact sets and let p_n be the function p_{C_n}. Then $\{p_n\}$ is a nondecreasing sequence of functions satisfying (1.1) and (1.2) on S, and $\varphi \in$ Hom (S, C) is an R-semicharacter of S if and only if there exists an n such that

$$(4.5) \qquad |\varphi(\chi)| \leqq p_n(\chi) \text{ for all } \chi \in S ,$$

that is, if φ is a p_n-semicharacter of S.

To summarize what we have proved:

THEOREM 5. *If R is a homogeneous algebra over the connected locally compact abelian group G then there exists a generating subsemigroup S of $\hat{G}$ containing the identity and a nondecreasing sequence of real valued functions p_n each satisfying conditions* (1.1) *and* (1.2) *on S such that the maximal regular ideals of R are in one-to-one correspondence with the p_n-semicharacters of S for $n = 1, 2, \cdots$.*

We conclude with two rather obvious corollaries of Theorem 5.

COROLLARY. *If, in the homogenous algebra R, convergence is uniform convergence then the maximal regular ideals correspond to the continuous p_n-semicharacters of S.*

This follows from the fact that in this case the mapping $\chi \to \chi f$ is continuous for each $f \in R'$.

COROLLARY. *If the homogeneous algebra R contains a bounded (in the norm) sequence of units modulo a σ-covering $\{C_n\}$ of G and if the mapping $f \to \chi f$ is bounded on R' then, for each $x \in S$, $p_n(\chi)$ is bounded. If $p(\chi) = \sup p_n(\chi)$ then p is also a real function satisfying* (1.1) *and* (1.2) *on S and every R-semicharacter of S is a p-semicharacter of S.*

Whether, in the setting of this corollary, every p-semicharacter of S is an R-semicharacter of S remains a matter of conjecture at the present.

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