BOUNDED GENERALIZED ANALYTIC FUNCTIONS ON THE TORUS

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1. Introduction. We shall operate in Euclidean k-space, E_k, $k \geqq 2$, and use the following notation:

$$x = (x_1, \cdots, x_k) ; \qquad y = (y_1, \cdots, y_k) ;$$
$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \cdots, \alpha x_k + \beta y_k) ;$$
$$(x, y) = x_1 y_1 + \cdots + x_k y_k ; \qquad |x| = (x, x)^{1/2} .$$

T_k will designate the k-dimensional torus $\{x; -\pi < x_j \leqq \pi, j = 1, \cdots, k\}$, v will always designate a point a distance one from the origin, i.e., $|v| = 1$, and m will always designate an integral lattice point. If f is in L^1 on T_k, then $\hat{f}(m)$ will designate the mth Fourier coefficient of f, i.e., $(2\pi)^{-k}\int_{T_k} f(x)e^{-i(m,x)}dx$.

We shall say that $f(x)$ in L^1 on T_k is a generalized analytic function on T_k if there exists v such that f is in A_v, where $A_v = A_v^+ \cup A_{-v}^+$, and A_v^+ is defined as follows:

f is in A_v^+ if there exists an m_0 such that if $m \neq m_0$ and $(m - m_0, v) \leqq 0$, then $\hat{f}(m) = 0$.

We shall say that $f(x)$ in L^1 on T_k is a strictly generalized analytic function on T_k if there exists a v such that f is in B_v, where $B_v = B_v^+ \cup B_{-v}^+$, and B_v^+ is defined as follows:

f is in B_v^+ if there exists an m_0 and a γ with $0 < \gamma < 1$ such that if $(m - m_0, v) < \gamma |m - m_0|$, then $\hat{f}(m) = 0$.

It is quite clear that $B_v \subset A_v$. In this paper, we shall obtain a result which is false for bounded functions in A_v but which is true for bounded functions in B_v. It is primarily with the class B_v and its extension to finite complex measures that the classical paper of Bochner [2, p. 718] is concerned. On T_k, it is essentially with the class A_v that the papers of Helson and Lowdenslager [5], [6], and de Leeuw and Glicksberg [4] are concerned.

We shall be concerned in this paper with a class of functions C_v which for bounded functions is intermediate between the two classes B_v and A_v.

We first note that if f is in B_v^+, then $\sum_m |\hat{f}(m)| e^{(m,v)\sigma} < \infty$ for every $\sigma < 0$. For with $||f||_p$, $1 \leqq p \leqq \infty$, designating the L^p-norm of f on T_k, we see that there exists a γ with $0 < \gamma < 1$ and an m_0 such that that

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$$\sum_m |\hat{f}(m)| e^{(m,v)\sigma} \leqq ||f||_1 \sum_{\gamma|m-m_0| \leqq (m-m_0,v)} e^{(m,v)\sigma} ,$$

and

$$\sum_{\gamma|m-m_0| \leqq (m-m_0,v)} e^{(m,v)\sigma} \leqq e^{(m_0,v)\sigma} \sum_m e^{\gamma|m-m_0|\sigma} < \infty .$$

Next, we note that if $\sum_m |\hat{f}(m)| e^{(m,v)\sigma_0} < \infty$, then

(1) there exists a function $g(x)$ in L^1 on T_k which is continuous in an open subset of T_k and which furthermore has $\sum_m \hat{f}(m) e^{(m,v)\sigma_0} e^{i(m,x)}$ as its Fourier series.

We use (1) to define the class $C_v = C_v^+ \cup C_{-v}^+$. In particular we say that f is in C_v^+ if the following three conditions are met:

(i) f is in L^∞ on T_k,

(ii) f is in A_v^+,

(iii) there exists a $\sigma_0 < 0$ such that (1) holds.

We note once again that if (ii) is replaced by

(ii′) f is in B_v^+,

then (iii) follows automatically.

With every unit point $v = (v_1, \cdots, v_k)$ there is also associated a one-parameter subgroup of T_k which we shall call G_v where

$$G_v = \{x; -\pi < x_j \leqq \pi, x_j \equiv tv_j \bmod 2\pi, -\infty < t < \infty\} .$$

If v is linearly independent with respect to rational coefficients, then G_v is dense on T_k. If v is linearly dependent with respect to rational coefficients, G_v is not dense on T_k. (We say $v = (v_1, \cdots, v_k)$ is linearly dependent with respect to rational coefficients if there exist rational numbers $r_1, \cdots, r_k$ with $r_1^2 + \cdots + r_k^2 \neq 0$ such that $\sum_{j=1}^k r_j v_j = 0$.) In either case, however, the statement that a set $E \subset G_v$ is of positive linear measure is well-defined. In particular, we set $E^* = \{t;$ there exists an x in E such that $x_j \equiv tv_j \bmod 2\pi$ for $j = 1, \cdots, k\}$. Then E^* is a set on the real line $-\infty < t < \infty$. We say that E is of positive linear measure if E^* is a set with positive 1-dimensional Lebesgue measure.

In the sequel, we shall work primarily with functions f in L^∞ on T_k. Also, all functions initially defined in T_k will be understood to be extended to all of E_k by periodicity of period 2π in each variable.

Given a function f in L^∞ on T_k, we shall set

$$f(x, h) = \sum_m \hat{f}(m) e^{i(m,x)} e^{-|m|h} \quad \text{for } h > 0 . \tag{2}$$

We shall say that f vanishes at x_0 if

$$\lim_{h \to 0+} f(x_0, h) = 0 . \tag{3}$$

We note that the changing of f on a set of k-dimensional measure zero does not affect its vanishing at the point x_0. (In classical terminology, (3) says that the Fourier series of f is Abel summable to zero at x_0.)

We shall say that f vanishes on a set E if f vanishes at all points of E.

With $B(x, h)$ representing the open k-ball with center x and radius h and $|B(x, h)|$ representing the k-dimensional volume of $B(x, h)$, we set

$$f_h(x) = |B(x, h)|^{-1}\int_{B(x,h)} f(y)dy \tag{4}$$

and note that if $\lim_{h\to 0} f_h(x_0) = 0$, then f vanishes at x_0, i.e., $\lim_{h\to 0+} f(x_0, h) = 0$ (See [10, p. 55]).

The theorem that we shall prove is the following:

THEOREM. *A necessary and sufficient condition that every f in C_v which vanishes on a subset of G_v of positive linear measure be zero almost everywhere on T_k is that v be linearly independent with respect to rational coefficients.*

We first note that the sufficiency of the above theorem is false for bounded functions in A_v. This fact will be established in § 4.

We next note that if $f(x)$ is in C_v, so is $f(x + x_0)$. Consequently, the above theorem implies that if f is in C_v, v linearly independent with respect to rational coefficients, and f vanishes on a subset of $x_0 + G_v$ of positive linear measure, then f is zero almost everywhere on T_k.

We finally note that for $k = 1$ the above theorem reduces to the well-known theorem of F. and M. Riesz for holomorphic functions on the unit disc in the form that they first proved it, i.e., for bounded functions, [9]. There have been other extensions of the F. and M. Riesz Theorem to higher dimensions (see [5, p. 176] and [4, p. 188]), but these always involve the vanishing of f on sets of positive k-dimensional measure. Here, we only ask that f vanish on particular sets of positive 1-dimensional measure, but on the other hand, we deal with a more restricted class of functions.

2. Proof of sufficiency.

Since $C_v = C_{-v}$ and $G_v = G_{-v}$ with no loss in generality, we can assume from the start that f is in C_v^+.

Since f is in C_v^+, it is in A_v^+. Consequently there exists an m_0 such that $\hat{f}(m) = 0$ if $m \neq m_0$ and $(m - m_0, v) \leqq 0$. If we set $a(x) = e^{-i(m_0,x)}f(x)$, then $a(x)$ is in A_v^+ with $m_0 = 0$. Furthermore, it is clear that since $f(x)$ satisfies (1), $a(x)$ does also. If we can show that

$$\text{if } \lim_{h\to 0+} f(x_0, h) = 0, \text{ then } \lim_{h\to 0+} a(x_0, h) = 0 , \tag{5}$$

it will be sufficient to prove the theorem for $a(x)$.

To establish (5), set $b(x) = a(x) - e^{-i(m_0,x_0)}f(x)$. Then $a(x, h) = b(x, h) + e^{i(m_0,x_0)}f(x, h)$, and by the remark after (4), (5) will follow once it is shown that $b_h(x_0) \to 0$ as $h \to 0$. But

$$\begin{aligned} | b_h(x_0) | &\leqq 0(h^{-k}) \|f\|_\infty \int_{B(x_0,h)} | e^{-i(m_0,x)} - e^{-i(m_0,x_0)} | dx \\ &\leqq 0(h^{-k}) \|f\|_\infty | m_0 | \int_{B(x_0,h)} | x - x_0 | dx \\ &\leqq o(1) \quad \text{as } h \to 0 , \end{aligned}$$

and (5) is established.

We now replace $a(x)$ by $f(x)$ and proceed, i.e., we set

$$M = \{m; (m, v) \geqq 0\} \tag{6}$$

and assume

$$\text{if } m \text{ is not in } M, \text{ then } \hat{f}(m) = 0 . \tag{7}$$

Setting $P(x,h) = \sum_m e^{i(m,x)-|m|h}$ for $h > 0$ and noticing that $P(x, h) > 0$ for x on T_k and $h > 0$, [3, p. 32], and that $(2\pi)^{-k}\int_{T_k} P(x, h)dx = 1$ we see that $f(x, h)$ defined in (2) is given by

$$f(x, h) = (2\pi)^{-k} \int_{T_k} f(x - y)P(y, h)dy .$$

Consequently,

$$|f(x, h)| \leqq \|f\|_\infty \quad \text{for } h > 0 \text{ and } x \text{ on } T_k . \tag{8}$$

Next, with $z = \sigma + it$ and $\sigma \leqq 0$, we set

$$\begin{aligned} F(z, h) &= \sum_m \hat{f}(m)e^{i(tv,m)}e^{\sigma(v,m)}e^{-|m|h} \\ &= \sum_{m \text{ in } M} \hat{f}(m)e^{\lambda_m z}e^{-|m|h} \end{aligned} \tag{9}$$

where

$$\lambda_m = (m, v) \text{ for } m \text{ in } M . \tag{10}$$

By (6), (7), (9), and (10), $F(z, h)$ is, for fixed $h > 0$, analytic in the left half-plane $\sigma < 0$ and continuous in the closed half-plane $\sigma \leqq 0$. Furthermore, since $F(it, h) = f(tv, h)$, we have by (8) that

$$\sup_{-\infty<t<\infty} | F(it, h) | \leqq \|f\|_\infty \quad \text{for } h > 0 . \tag{11}$$

Also, it is clear that for $\sigma \leqq 0$, $| F(\sigma+it, h) | \leqq \sum_{m \text{ in } M} |\hat{f}(m) | e^{-|m|h} < \infty$ and therefore that

$$\lim_{\sigma\to-\infty} \sup_{-\infty<t<\infty} |F(\sigma+it, h)| \leqq |\hat{f}(0)| \leqq ||f||_\infty .$$

Consequently, it follows from the Phragmen-Lindelof theorem, [1, p. 137], that

$$(12) \qquad ||F(z, h)|| \leqq ||f||_\infty \quad \text{for } \sigma \leqq 0 \text{ and } h > 0 .$$

But then by Montel's theorem]1, p. 132],

(13) there exists a function $F(z)$, analytic for $\sigma < 0$, and a sequence $h_1 > h_2 > \cdots > h_j > \cdots \to 0$ such that $\lim_{j\to\infty} F(z, h_j) = F(z)$ uniformly on any compact subset of the open left half-plane $\sigma < 0$.

We propose to show that $F(z)$ is identically zero. To do this we look at $F(it, h_j)$. By (11), $\{F(it, h_j)\}_{j=1}^\infty$ is a bounded sequence of continuous functions on the interval $-\infty < t < \infty$. Consequently, it follows from the notion of weak* convergence that there exists a function $q(t)$ in L^∞ on $-\infty < t < \infty$, with $|q(t)| \leqq ||f||_\infty$ for almost every t and a subsequence $\{h_{j_n}\}_{n=1}^\infty$ of $\{h_j\}_{j=1}^\infty$ with $\lim_{n\to\infty} h_{j_n} = 0$ such that for every $\xi(t)$ in $L^\infty \cap L^1$ on $-\infty < t < \infty$,

$$(14) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} \xi(t)^F(it, h_{j_n})dt = \int_{-\infty}^{\infty} \xi(t)q(t)dt .$$

Choosing ξ in (14) to be the function

$$\xi(u) = -\sigma[\sigma^2 + (u-t)^2]^{-1}\pi^{-1} \quad \text{where } \sigma < 0 ,$$

we see from (13) that

$$(15) \qquad \begin{aligned} F(\sigma+it) &= \lim_{n\to\infty} F(\sigma+it, h_{j_n}) \\ &= \lim_{n\to\infty} -\pi^{-1}\sigma \int_{-\infty}^{\infty} F(iu, h_{j_n})[\sigma^2 + (u-t)^2]^{-1}du \\ &= -\pi^{-1}\sigma \int_{-\infty}^{\infty} q(u)[\sigma^2 + (u-t)^2]^{-1}du . \end{aligned}$$

Since $|F(\sigma+it, h)| \leqq ||f||_\infty$ for $h > 0$ and $\sigma \leqq 0$, it follows from (13) that $|F(\sigma+it)| \leqq ||f||_\infty$ for $\sigma < 0$, and consequently from (15) and [7, p. 447] that

$$(16) \qquad \lim_{\sigma\to0-} F(\sigma+it) = q(t) \text{ for almost every } t .$$

If we can show that $q(t) = 0$ on a set of positive measure, then it will follow from (16) and the F. and M. Riesz Theorem for a half-plane, [7, p. 449], that $F(\sigma+it)$ is identically zero for $\sigma < 0$.

To show that $q(t) = 0$ on a set of positive measure we set

$$E^* = \left\{t, \lim_{h\to0} f(tv, h) = 0\right\} .$$

By hypothesis, E^* is a set of positive linear measure in the infinite interval $-\infty < t < \infty$. Let B^* be any measurable subset of E^* of finite measure and let $\xi_{B^*}(t)$ be the indicator function of B^*. Then by (14)

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\xi_{B^*}(t)F(it, h_{j_n})dt = \int_{B^*}q(t)dt\,. \tag{17}$$

However, $F(it, h_{i_n}) = f(tv, h_{j_n})$, $f(tv, h_{j_n}) \to 0$ as $n \to \infty$ for t in B^*, and $|f(tv, h_{j_n})| \leqq \|f\|_\infty$. We conclude from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\xi_{B^*}(t)F(it, h_{j_n})dt = 0\,. \tag{18}$$

From (17) and (18), we obtain that $\int_{B^*}q(t)dt = 0$. B^*, however, is an arbitrary subset of E^* of finite measure. Therefore $q(t)$ must equal zero almost everywhere in E^*. Consequently, $q(t) = 0$ on a set of positive measure, and we have that

$$F(\sigma + it) = 0 \quad \text{for } \sigma < 0\,. \tag{19}$$

By hypothesis, there exist a $\sigma_0 < 0$, an open set $U \subset T_k$ and a function $g(x)$ in L^1 on T_k such that the following facts prevail:

$$\hat{g}(m) = \hat{f}(m)e^{(v,m)\sigma_0} \quad \text{for every } m\,: \tag{21}$$

$$g \text{ is continuous in } U\,. \tag{22}$$

From (9), (13), and (19), it follows that

$$\lim_{j\to\infty}\sum_m \hat{f}(m)e^{(v,m)\sigma_0}e^{i(tv,m)}e^{-|m|h_j} = 0 \quad \text{for } -\infty < t < \infty\,. \tag{23}$$

On the other hand, as is well-known (see [10, p. 55]), (21) and (22) imply

$$\lim_{j\to\infty}\sum \hat{f}(m)e^{(m,v)\sigma_0}e^{i(m,x)}e^{-|m|h_j} = g(x) \quad \text{for } x \text{ in } U\,. \tag{24}$$

We conclude from (23) and (24) that $g(x) = 0$ for x in $U \cap G_v$. However, since G_v is dense in T_k and U is open, $U \cap G_v$ is dense in U, and consequently, $g(x) = 0$ in all of U.

Suppose that $B(x_0, h_0) \subset U$. Then for $0 < h < h_0$ and $g_h(x)$ defined by (4), we have that $g_h(x)$ is a continuous periodic function which for each fixed h is zero on an open set. In particular, $g_h(x + x_0)$ is zero on a subset of G_v of positive linear measure. Since

$$\hat{g}_h(m) = \hat{f}(m)e^{(m,v)\sigma_0}|B(0, h)|^{-1}\int_{B(0,h)}e^{i(m,x)}dx\,,$$

we conclude from the argument previously given that $g_h(tv + x_0) = 0$ for $-\infty < t < \infty$ and $0 < h < h_0$. But then the continuous function $g_h(x)$ is zero on a dense subset of T_k, and therefore for $0 < h < h_0$, $g_h(x) = 0$ for all x on T_k. Consequently, $g(x) = 0$ almost everywhere on T_k. We conclude from (21) that $\hat{f}(m) = 0$ for every m. Therefore $f(x) = 0$ almost everywhere, and the proof of the sufficiency is complete.

3. Proof of necessity. Let $v = (v_1, \cdots, v_k)$ be linearly dependent over the rationals with $v_1^2 + \cdots + v_k^2 = 1$. We shall show that there exists a nonzero trigonometric polynomial $f(x)$ in B_v^+ (and therefore in C_v^+) such that $f(x) = 0$ for x in G_v.

Two cases present themselves. Either there exists a coordinate v_{j_0} of v which is zero or all the coordinates of v are different from zero. We handle the former case first.

Since $|v| = 1$, there exists a coordinate v_{j_1} of v which is different from zero. Let m' be the integral lattice point with 1 in the j_0-coordinate, $\operatorname{sgn} v_{j_1}$ in the j_1-coordinate, and zero at all other coordinates. Similarly define m'' to be the integral lattice point with 2 in the j_0-coordinate, $\operatorname{sgn} v_{j_1}$ in the j_1-coordinate, and zero at all other coordinates. Then $(m', v) = (m'', v) = |v_{j_1}| > 0$, and the trigonometric polynomial $f(x) = e^{i(m',x)} - e^{i(m'',x)}$ is clearly in B_v^+. Also, $f(tv) = e^{it(m',v)} - e^{it(m'',v)} = 0$ for $-\infty < t < \infty$; $f(x)$ is zero on G_v, and the first case is settled.

Next, suppose that all the coordinates of v are different from zero. Since by assumption v is linearly dependent with respect to rational coefficients, there exists a nonzero integral lattice point m such that $(m, v) = 0$. Let m_{j_0} be the first coordinate of m which is different from zero. We can assume $\operatorname{sgn} m_{j_0} = \operatorname{sgn} v_{j_0}$ for otherwise we can replace m by $-m$. Let m' be the integral lattice point with $\operatorname{sgn} v_{j_0}$ in the j_0-coordinate and zero elsewhere. Set $m'' = m + m'$. Then

$$(m'', v) = (m + m', v) = (m', v) = |v_{j_0}| > 0 \,,$$

and the trigonometric polynomial $f(x) = e^{i(m',x)} - e^{i(m'',x)}$ is in B_v^+ and is zero on G_v. The second case is settled, and the proof of the theorem is complete.

4. Counter-example for A_v. Given v linearly independent with respect to rational coefficients, we shall exhibit a function $f(x)$ in L^∞ on T_k and in A_v^+ such that

$$\lim_{h \to 0} f_h(x) = 0 \quad \text{for every } x \text{ in } G_v \tag{25}$$

and such that $f(x) \neq 0$ in a set of positive measure on T_k.

We note once again that (25) implies that f vanishes on all of G_v.

We start in the classical manner (see [11, p. 276 and p. 105]). Observing that G_v is of k-dimensional measure zero, we see that there exists a sequence of sets $\{G_n\}_{n=1}^{\infty}$ each open in the torus sense on T_k with the following properties:

$$T_k \supset G_1 \supset G_2 \supset \cdots \supset G_n \cdots \supset G_v ; \tag{26}$$

$$\text{the } k\text{-dimensional measure of } G_n \text{ is } \leqq n^{-4} . \tag{27}$$

We set

$$\begin{aligned} g_n(x) &= n^2 \quad \text{for } x \text{ in } G_n , \\ &= 0 \quad \text{for } x \text{ in } T_k - G_n , \end{aligned} \tag{28}$$

and

$$g(x) = \sum_{n=1}^{\infty} g_n(x) . \tag{29}$$

Now $\int_{T_k} g(x)dx \leqq \sum_{n=1}^{\infty} n^{-2}$. Consequently, $g(x)$ is a nonnegative function on T_k, and the set $\{x; g(x) = +\infty\}$ is of k-dimensional measure zero.

Next, we set $a(x) = e^{-g(x)}$ and observe that $a(x)$ is a Borel measurable function on T_k with the following properties:

$$0 \leqq a(x) \leqq 1 \quad \text{for } x \text{ in } T_k , \tag{30}$$

$$\{x; a(x) = 0\} \text{ is of } k\text{-dimensional measure zero.} \tag{31}$$

Observing that $G_v \subset G_n$ for every n by (27) and that by (29), $a(x) \leqq e^{-g_n(x)}$, we see from (28) that for fixed n and a fixed x_0 in G_v, $a_h(x_0) \leqq e^{-n^2}$ for h sufficiently small. We conclude that

$$\lim_{h\to 0} a_h(x) = 0 \quad \text{for } x \text{ in } G_v . \tag{32}$$

From (31) and (32), we see that there is no constant such that $a(x)$ is equal to it almost everywhere on T_k. Consequently there exists an $m_0 \neq 0$ such that $\hat{a}(m_0) \neq 0$. Since $a(-x)$ satisfies (30), (31), and (32), with no loss in generality, we can also assume that $(m_0, v) > 0$. Thus we have

$$\hat{a}(m_0) \neq 0 \quad \text{and } (m_0, v) > 0 . \tag{33}$$

Next, as in [8, p. 60], we introduce the complex Borel measure μ on T_k defined by

$$\int_{T_k} b(x)d\mu(x) = \int_{-\infty}^{\infty} b(tv)(1 - it)^{-2}dt \tag{34}$$

for every bounded Borel measurable function on T_k.

From the fact that

$$\int_{-\infty}^{\infty} e^{i\lambda t}(1-it)^{-2}dt = 0 \qquad \text{for } \lambda \geqq 0$$
$$= -(2\pi)\lambda e^{\lambda} \qquad \text{for } \lambda < 0\,,$$

we see that $\hat{\mu}(m) = (2\pi)^{-k}\int_{T_k} e^{-i(m,x)}d\mu(x)$ is such that

$$\hat{\mu}(m) \neq 0 \quad \text{for } (m, v) > 0$$
$$= 0 \quad \text{for } (m, v) \leqq 0\,. \tag{35}$$

We set

$$f(x) = (2\pi)^{-k}\int_{T_k} a(x-y)d\mu(y) \tag{36}$$

and shall show that f has the requisite properties set forth at the beginning of this section.

In the first placce, we see from (30), (34), and (36)

$$|f(x)| \leqq (2\pi)^{-k}\int_{-\infty}^{\infty}(1+t^2)^{-1}dt \quad \text{for } x \text{ in } T_k\,,$$

and consequently $f(x)$ in L^∞ on T_k.

In the second place, we observe from (36) that $\hat{f}(m) = \hat{a}(m)\hat{\mu}(m)$ and consequently by (35) that $f(x)$ is in A_v^+. Furthermore, by (33) and (35), $\hat{f}(m_0) \neq 0$. Consequently, $f(x) \neq 0$ on a set of positive measure on T_k.

All that remains to establish is (25). Let x_0 be a fixed point in G_v. Then by (36) and Fubini's theorem,

$$(2\pi)^k f_h(x_0) = \int_{T_k} a_h(x_0-y)d\mu(y)$$
$$= \int_{-\infty}^{\infty} a_h(x_0-tv)(1-it)^{-2}dt\,. \tag{37}$$

By (30), $|a_h(x)| \leqq 1$ for all x on T_k. Furthermore, since x_0 is in G_v, so is $x_0 - tv$ for $-\infty < t < \infty$. Therefore, by (32), $\lim_{h\to 0} a_h(x_0-tv) = 0$ for $-\infty < t < \infty$. We consequently conclude from the Lebesgue dominated convergence theorem and (37) that $\lim_{h\to 0} f_h(x_0) = 0$, and (25) is established.

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