ON HIGH SUBGROUPS

CHARLES K. MEGIBBEN

One of the purposes of this paper is to answer the following three questions:

(1) What groups G with $G^1 = 0$ are direct summands of all groups containing them as high subgroups?

(2) If G is a Σ-group, are all high subgroups of G endomorphic images of G (see [3] and [4])?

(3) If G is a torsion Σ-group, is every subgroup of G a Σ-group (see [3])?

The answer is affirmative to (2) and negative to (3). However an affirmative answer can be given to (3) when $|G^1| \leqq \aleph_0$.

All groups in this paper will be assumed to be additively written abelian groups. For the most part, the notation and terminology of [2] will be followed. If G is a group, G_t will denote the torsion subgroup of G and G^1 the subgroup of elements of infinite height, that is, $G^1 = \bigcap_{n=1}^{\infty} nG$. A torsion group is said to be closed if each p-primary component is a closed p-group (see [2], pp. 114–117). A mixed abelian group is said to split if it decomposes into a direct sum of a torsion and torsion free group. By the n-adic topology on the group G, we shall mean the topology defined by taking as neighborhoods of 0 the subgroups nG for each positive integer n. A subgroup H of G is said to be a high subgroup if H is maximal in G with respect to $H \cap G^1 = 0$. If H is a high subgroup of G, then H is pure in G and G/H is divisible (see [3]). If all high subgroups of G are direct sums of cyclic groups, then G is said to be a Σ-group. If one high subgroup of G is a direct sum of cyclic groups, then all high subgroups of G are isomorphic and G is a Σ-group (see [4]).

1. High subgroups. Let G be an arbitrary abelian group and let D be a minimal divisible group containing G^1. Then let K be the amalgamated sum of G and D over G^1, that is, K is the abelian group generated by the elements of G and D subject only to $G \cap D = G^1$. (K can be realized as $(G + D)/L$ where L is the subgroup of $G + D$ consisting of all elements of the form $(x, -x)$ with $x \in G^1$.) It then follows that $K/G = \{G, D\}/G \cong D/(G \cap D) = D/G^1$, and similarly that $K/D \cong G/G^1$.

LEMMA 1. *If D is minimal divisible containing G^1 and if K is the amalgamated sum of G and D over G^1, then*

Received September 26, 1963, and in revised form December 3, 1963.

(i) *G is a pure subgroup of K;*
(ii) *$K = H + D$ with $H \cong G/G^1$;*
(iii) *$H \cap G$ is a high subgroup of G;*
(iv) *$K = \{H, G\}$; and*
(v) *G is a subdirect sum of H and D.*

Proof. If $g \in G$ and $nk = g$ for some $k \in K$, then we write $k = g_1 + d$ with $g_1 \in G$ and $d \in D$. Then $nd = g - ng_1$ is an element of $G \cap D = G^1$ and hence there is a $g_2 \in G$ such that $ng_2 = g - ng_1$, that is, $g = n(g_1 + g_2)$ and we conclude that G is pure in K.

(ii) is immediate since divisible subgroups are always direct summands and, as observed above, $K/D \cong G/G^1$.

In order to show that $H \cap G$ is high in G we need only prove for every $g \in G\backslash H$ that $\{H \cap G, g\} \cap G^1 \neq 0$. If $g \in G\backslash H$, we write $g = h + d$ with $h \in H$, $d \in D$ and $d \neq 0$. Since D is minimal divisible containing G^1, for some integer n, nd is a nonzero element in G^1. Then $nh = n(g - d) \in H \cap G$ and $nd = ng - nh$ is a nonzero element of $\{H \cap G, g\} \cap G^1$.

Let p be an arbitrary prime. Since $G^1[p] = D[p]$, $K[p] \subseteqq \{H, G\}$. Assume that we have established that $K[p^n]$ is contained in $\{H, G\}$. In order to show that $K[p^{n+1}]$ is contained in $\{H, G\}$, we need only consider elements in $D[p^{n+1}]$. If $d \in D[p^{n+1}]$, then $p^n d \in G^1$ and therefore there is a $g \in G$ such that $d - g \in K[p^n] \subseteqq \{H, G\}$, that is, $d \in \{H, G\}$ and we conclude that $K[p^{n+1}] \subseteqq \{H, G\}$. Clearly then the torsion subgroup of K is contained in $\{H, G\}$. To complete the proof that $K = \{H, G\}$, we show that $D \subseteqq \{H, G\}$. Indeed if $d \in D$ with $d \neq 0$, then nd is a nonzero element of G^1 for some integer n. Then there is a $g \in G$ such that $d - g \in K[n] \subseteqq \{H, G\}$ and therefore $d \in \{H, G\}$.

Since $K = \{H, G\} = \{D, G\}$, for each $h \in H$ there is a $g \in G$ and a $d \in D$ such that $g = h + d$; and similarly, for each $d \in D$ there is a $g \in G$ and an $h \in H$ such that $g = h + d$. Thus, G is a subdirect sum of H and D and the kernels are obviously $H \cap G$ and $D \cap G = G^1$.

Lemma 1 suggests a useful method for constructing groups with certain properties. Indeed, it can be shown without difficulty that if D is minimal divisible containing the group A and if H is a group without elements of infinite height having a pure subgroup B such that $H/B \cong D/A$, then any subdirect sum G of H and D with kernels B and A is a pure subgroup of $H + D$ such that

(i) $G^1 = A$,
(ii) $G/G^1 \cong H$, and
(iii) B is a high group of G.

THEOREM 1. *Let M be an abelian group without elements of* *infinite height. Then M is a direct summand of every group containing it as a high subgroup if and only if M_t is closed.*

Proof. Suppose that M_t is closed and that G contains M as a high subgroup with $G/M = D$. The G can be represented as a subdirect sum of H and D where $H \cong G/G^1$ and $M = H \cap G$. Therefore $D/G^1 \cong H/M \cong (H/M_t)/(M/M_t)$ and since M/M_t is a torsion free pure subgroup of H/M_t and D/G^1 is torsion, $H/M_t = J/M_t + M/M_t$. But since M_t is closed and is pure in the torsion group J, $J = L + M_t$ with $L \cong D/G^1$. Thus $H = L + M$ and since M is a direct summand of H, M is necessarily a direct summand of G.

Suppose new that M_t is not closed. Let M^* be the n-adic completion of M. Then M is a pure subgroup of M^* with M^*/M divisible. Since M_t^* is closed, $\{M, M_t^*\}/M$ is nonzero and is moreover the torsion subgroup of M^*/M. Let $H = \{M, M_t^*\}$ and choose a direct sum A of cyclic groups such that if D is minimal divisible containing A, then $D/A \cong H/M$. If G is a subdirect sum of H and D with kernels M and A, then G will be a reduced group having M as a high subgroup.

THEOREM 2. *If some high subgroup of G splits, then G/G^1 splits.*

Proof. Let $T + F$ be a high subgroup of G where T is torsion and F is torsion free. Let D be minimal divisible containing G^1 and let K be the amalgamated sum of G and D over G^1. Then if $K = H + D$, we may assume that $G \cap H = T + F$. Then $H/(T + F) = H/(H \cap G) \cong \{H, G\}/G = K/G \cong D/G^1$. Therefore since D/G^1 is torsion and $(T + F)/T$ is a torsion free pure subgroup of H/T, $H/T = M/T + (T + F)/T$ for some subgroup M of H. Hence $H = M + F$ where M is necessarily a torsion group since M/T is torsion, that is, H splits and $H \cong G/G^1$.

COROLLARY 1. *If some high subgroup of G is torsion, then G/G^1 is torsion and therefore all high subgroups of G are torsion.*

COROLLARY 2. (Irwin, Peercy and Walker [4]) *If A is a high subgroup of G and $A = T + F$ where T is torsion and F is torsion free, then $G = L + F$ with L/T divisible.*

Proof. Let D and K be as in the proof of Theorem 2 and suppose that $K = H + D$ with $H \cap G = A$. Then $K = (M + F) + D$ and therefore $G = L + F$ where $L = G \cap (M + D)$. Finally, we observe that $L/T \cong G/(T + F) = G/(H \cap G) \cong D$.

REMARK. From an example in [4], G need not split if G/G^1 splits.

2. Σ-groups.

THEOREM 3. *If G is a Σ-group, then every high subgroup of G is an endomorphic image of G. More generally, if the high subgroup H of G splits and the torsion subgroup of H is a direct sum of cyclic groups, then H is an endomorphic image of G.*

Proof. If $H = T + F$ where T is torsion and F is torsion free, then $G/G^1 \cong M + F$ where M is torsion and contains T as a pure subgroup. If T is a direct sum of cyclic groups, then T is a basic subgroup of M and therefore an endomorphic image of M. Clearly then if T is a direct sum of cyclic groups, H is an endomorphic image of G.

Requiring a Σ-group to have at most countably many elements of infinite height imposes severe restrictions on the structure of the group.

THEOREM 4. *If G is a Σ-group such that $|G^1| \leqq \aleph_0$, then G/G^1 is a direct sum of cyclic groups.*

Proof. If $H = T + F$ is a high subgroup of G and if F is free and T is a torsion direct sum of cyclic groups, then $G/G^1 \cong M + F$ where M is a torsion direct sum of cyclic groups provided $|G^1| \leqq \aleph_0$. Indeed, $M/T \cong D/G^1$, where D is minimal divisible containing G^1, and since D/G^1 is necessarily at most countable and M is without elements of infinite height, Theorem 33.4 in [2] implies that M is a direct sum of cyclic groups.

REMARK. A glance at the proof of Theorem 4 should suggest an extremely simple proof of the fact that if one high subgroup of a group is a direct sum of cyclic groups then all high subgroups of the group are isomorphic.

THEOREM 5. *If G is a torsion Σ-group and G^1 has an at most countable basic subgroup, then G/G^1 is a direct sum of cyclic groups.*

Proof. We need only observe that if G^1 has an at most countable basic subgroup and if D is minimal divisible containing G^1, then $|D/G^1| \leqq \aleph_0$ (see [2], p. 110).

EXAMPLE 1. The restrictions in Theorems 3 and 4 are necessary. Indeed, let $B = \sum_{n=1}^{\infty} C(p^n)$ where p is a prime and let $\bar{B}$ be the torsion subgroup of $\sum_{n=1}^{\infty *} C(p^n)$. Then B is pure in $\bar{B}$ and $\bar{B}/B$ is isomorphic to $2^{\aleph_0}$ copies of $C(p^\infty)$. Next set $A = \sum_{\lambda \in \Lambda} \{a_\lambda\}$, where $\{a_\lambda\} \cong C(p)$ for each λ and $|\Lambda| = 2^{\aleph_0}$. Then if D is minimal divisible containing

A, $D/A \cong \bar{B}/B$. If G is a subdirect sum of $\bar{B}$ and D with kernels $\bar{B}$ and A, then G is a Σ-group such that $G/G^1 \cong \bar{B}$.

The proof of the following lemma follows immediately from results in [1].

LEMMA 2. *If A is an at most countable subgroup of the torsion group G such that G/A is a direct sum of cyclic groups, then G is the direct sum of an at most countable group and a direct sum of cyclic groups.*

THEOREM 6. *If G is a torsion group such that $|G^1| \leqq \aleph_0$, then the following three conditions are equivalent:*

(i) *G is a Σ-group.*

(ii) *G/G^1 is a direct sum of cyclic groups.*

(iii) *$G = H + C$ where $|H| \leqq \aleph_0$ and C is a direct sum of cyclic groups.*

Proof. (i) implies (ii) by Theorem 4. (ii) implies (iii) by Lemma 2. And, finally, it is easy to see that (iii) always implies (i).

THEOREM 7. *If G is a torsion Σ-group such that $|G^1| \leqq \aleph_0$, then every subgroup of G is a Σ-group.*

Proof. Let G be a torsion Σ-group such that $|G^1| \leqq \aleph_0$ and let H be a subgroup of G. In order to show that H is a Σ-group, it suffices to show that H/H^1 is a direct sum of cyclic groups. Since $H/H \cap G^1 \cong \{H, G^1\}/G^1$, $H/H \cap G^1$ is a direct sum of cyclic groups. But this group is isomorphic to $(H/H^1)/(H \cap G^1/H^1)$ and since $H \cap G^1/H^1$ is at most countable and H/H^1 is without elements of infinite height, we conclude from Lemma 2 that H/H^1 is a direct sum of cyclic groups.

COROLLARY 3. *If the torsion group G is the direct sum of a countable group and a direct sum of cyclic groups, then every subgroup of G has a similar direct decomposition.*

EXAMPLE 2. The restriction that $|G^1| \leqq \aleph_0$ in Theorem 7 is necessary. Indeed, if $\bar{B}$ is as in Example 1, it is then easy to construct by methods we have used above a primary Σ-group G with $G^1 = \bar{B}$. Then G^1 is itself a subgroup of G which is not a Σ-group.

REFERENCES

1. B. Charles, *Note sur la structure des groupes abéliens primaires*, C. R. Acad. Sci. Paris, **252** (1961), 1547–1548.

2. L. Fuchs, *Abelian groups*, Budapest, 1958.
3. J. Irwin and E. Walker, *On N-high subgroups of abelian groups*, Pacific J. Math., **11** (1960), 1363–1374.
4. J. Irwin, C. Peercy and E. Walker, *Splitting properties of high subgroups*, Bull. Soc. Math. France, **90** (1962), 185–192.

TEXAS TECHNOLOGIAL COLLEGE, LUBBOCK