IDENTITY AND UNIQUENESS THEOREMS FOR AUTOMORPHIC FUNCTIONS

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1. Introduction. Let C and D denote the unit circle and the unit disk, respectively, and let $\rho(z, z')$ denote the non-Euclidean hyperbolic distance between the points z and z' in D [3, Chapter II]. Bagemihl and Seidel have proved the following identity theorem [2, Theorem 3, p. 13].

THEOREM A. *Let $f(z)$ be a meromorphic function of bounded characteristic in D, and let $\{z_n\}$ be a sequence of points in D with at least two limit points in C, such that $|z_n| \to 1$ and $\rho(z_n, z_{n+1}) < M$ for every n, where M is a positive constant. If $f(z_n) \to c$, then $f(z) \equiv c$.*

There is also a corresponding uniqueness theorem [2, Theorem 4, p. 14].

THEOREM B. *Let $f(z)$ and $g(z)$ be meromorphic functions of bounded characteristic in D, and let $\{z_n\}$ be a sequence of points in D with at least two limit points in C, such that $|z_n| \to 1$ and $\rho(z_n, z_{n+1}) < M$ for every n, where M is a positive constant. If $\{f(z_n) - g(z_n)\} \to 0$, then $f(z) \equiv g(n)$.*

Along the same lines, Bagemihl has proved an identity theorem for normal functions [1, Theorem 3, p. 4].

THEOREM C. *Let $f(z)$ be a normal meromorphic function and let $\{z_n\}$ be a sequence of points in D with at least two limit points in C, such that $|z_n| \to 1$ and $\rho(z_n, z_{n+1}) \to 0$. If $f(z_n) \to c$, then $f(z) \equiv c$.*

This paper will investigate such identity and uniqueness theorems for automorphic functions. It shall be shown that there is a result analoguous to Theorem C for automorphic functions with Fuchsian groups of the first kind. However, an example will show that there is no corresponding theorem for automorphic functions with Fuchsian groups of the second kind. In the case of automorphic functions with

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Fuchsian groups of the first kind, a corresponding uniqueness theorem holds.

2. Some notation and terminology.

We begin with some basic definitions about Fuchsian groups and automorphic functions.[1]

DEFINITION 1. *A group* $\mathscr{G}$ *of linear transformations of* D *onto itself is called a Fuchsian group if there exists a point* $z_0 \in D$ *and a neighborhood* $N(z_0)$ *of* z_0 *such that* $N(z_0) \cap S(z_0) = \phi$ *for each* $S \in \mathscr{G}$ *which is not the identity transformation.*

DEFINITION 2. *Let* $\mathscr{G}$ *be a Fuchsian group and let* $S(z) = (az + b)/(cz + d)$, $ad - bc = 1$, $c \neq 0$, *be an element of* $\mathscr{G}$. *The circle* $|cz + d| = 1$ *is called an isometric circle of* $\mathscr{G}$, *or simply an isometric circle, and is denoted by* $I(S)$. *The disk* $|cz + d| \leqq 1$ *is called an isometric disk and is denoted by* $K(S)$.

DEFINITION 3. *A limit point of the centers of the isometric circles of the transformations of a Fuchsian group* $\mathscr{G}$ *is called a limit point of* $\mathscr{G}$.

DEFINITION 4. *If every point of* C *is a limit point of* $\mathscr{G}$, *then* $\mathscr{G}$ *is called a Fuchsian group of the first kind. A Fuchsian group which is not of the first kind is said to be of the second kind.*

DEFINITION 5. *Let* $\mathscr{G}$ *be a Fuchsian group and let* R_0' *be the set of points in* D *exterior to every isometric disk of* $\mathscr{G}$. *If* R_0 *consists of* R_0' *plus a set of boundary points of* R_0' *such that every point of* D *is congruent under* $\mathscr{G}$ *to exactly one point of* R_0, *then* R_0 *is called the fundamental region of* $\mathscr{G}$.

We note that the fundamental region R_0 is not unique, but $\bar{R}_0$, the closure of R_0, is unique.

DEFINITION 6. *Let* $f(z)$ *be a meromorphic function in* D, *and let* $\mathscr{G}$ *be a Fuchsian group. If* $f(S(z)) = f(z)$ *for each* $S \in \mathscr{G}$ *and each* $z \in D$, *then* $f(z)$ *is called an automorphic function.*

In addition to the standard terminology, we shall make use of what will be called an L-set.

DEFINITION 7. *An arcwise connected*[2] *subset* A *of* D *is called an*

[1] For a more complete discussion of automorphic functions, the reader is referred to the book by Ford [**4**].

[2] Arcwise connected here means that any two points in A may be jointed be a Jordan arc contained in A.

L-set of D if $\bar{A} \cap C \neq \phi$.

3. The main theorems. The proofs of the main results require some preliminary lemmas.

LEMMA 1. *Let $\mathscr{G}$ be a Fuchsian group, and let S be a non-elliptic transformation in $\mathscr{G}$. Let B be the straight line segment from 0 to $S(0)$, and let $A_S = \bigcup_{n=0}^{\infty} S^n(B)$. Then A_S is an L-set of D.*

Proof. It is clear that A_S is arcwise connected. Since S is non-elliptic, $\{S^n\}$ is an infinite sequence of distinct elements of $\mathscr{G}$, and $\{S^n(0)\}$ may have no limit point in D. Hence A_S has a limit point in C.

In fact, it is easily seen that A_S has but a single limit point in C. This follows from the geometry of the isometric disks.

LEMMA 2. *Let $\mathscr{G}$ be a Fuchsian group of the first kind, and let $\widehat{\alpha\beta}$ be an arc of C. Then there exists an L-set A of D whose limit points on C are interior points of $\widehat{\alpha\beta}$ and each point of A is congruent under $\mathscr{G}$ to a point of a compact subset K of D.*

Proof. It is well known that if a Fuchsian group contains only elliptic elements, then it is is a finite group of the second kind. Let S be any nonelliptic element of $\mathscr{G}$ and let A_S be as described in Lemma 1. Let z_1 be the limit point of A_S in C. If z_1 is an interior point of the arc $\widehat{\alpha\beta}$, we set $A = A_S$. If z_1 is not an interior point of $\widehat{\alpha\beta}$, let z_2 be any interior point of $\widehat{\alpha\beta}$. Let $\{T_n\}$ be a sequence of elements of $\mathscr{G}$ such that $T_n(0) \rightarrow z_2$. Let W be any element of $\mathscr{G}$ for which z_1 is not a fixed point. Then either z_1 or $W(z_1)$ (or perhaps both) is not in infinitely many of the $K(T_n)$ and, by the geometry of isometric circles (see [4, p. 26]), there exists an integer N such that either $T_N(z_1)$ or $T_N(W(z_1))$ is an interior point of $\widehat{\alpha\beta}$. Then we set A equal to the corresponding $T_N(A_S)$ or $T_N(W(A_S))$. Every point of A is congruent under $\mathscr{G}$ to a point of B. Setting $K = B$, we see that A is the desired L-set.

LEMMA 3. *Let $\mathscr{S}$ be a Fuchsian group of the first kind, and let $\widehat{\alpha\beta}$ be an arc of C. Let $\{z_n\}$ be a sequence of points in D such that $\rho(z_n, z_{n+1}) \rightarrow 0$ and the set of limit points of $\{z_n\}$ consists of the arc $\widehat{\alpha\beta}$. Let L be the set of limit points of the set $\{S(z_m) : S \in \mathscr{G}, n = 1, 2, 3, \cdots\}$. Then there exists a point of L in D.*

Proof. By Lemma 2, there exists an L-set A of D all of whose points are congruent under $\mathscr{G}$ to points on a compact subset K of D, and whose limit points in C are interior points of $\widehat{\alpha\beta}$. Since $\rho(z_n, z_{n+1}) \to 0$, there exists a subsequence $\{z_{n_k}\}$ such that $\rho(z_{n_k}, A) \to 0$. Then there exist transformations $S_k \in \mathscr{G}$ such that $\{S_k(z_{n_k})\}$ has a limit point $z_0 \in K$. But $z_0 \in L$ and $z_0 \in D$.

LEMMA 4. *Let $\mathscr{G}$ be a Fuchsian group of the first kind and let $\widehat{\alpha\beta}$ be an arc of C. Let $\{z_n\}$ be a sequence of points in D such that $\rho(z_n, z_{n+1}) \to 0$ and the set of limit points of $\{z_n\}$ cansists of the arc $\widehat{\alpha\beta}$. Let L be the set of limit points of the set $\{S(z_n) : S \in \mathscr{G}, n = 1, 2, 3, \cdots\}$. Then any point of $L \cap D$ is a limit point of L.*

Proof. If $z_0 \in L \cap D$, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a sequence $\{S_k\}$ of elements of such that $S_k(z_{n_k}) \to z_0$. Since $\{z_n\}$ has more than one limit point in C, the n_k's can be chosen so that there exists a sequence of integers $\{p_k\}$ such that $p_k < n_{k+1} - n_k$ and $\rho(z_{n_k}, z_{n_k+p_k}) \to \infty$. Let $r > 0$ be given. Then for k sufficiently large, we have $\rho(z_0, S_k(z_{n_k})) < r$ and $\rho(z_0, S_k(z_{n_k+p_k})) > r$. If, for each k, q_k is the least positive integer such that $\rho(z_0, S_k(z_{n_k+q_k})) > r$, then $\rho(z_0, S_k(z_{n_k+q_k-1})) \leqq r$, and it is easy to see that the sequence $\{S_k(z_{n_k+q_k})\}$ has a limit point w_r on the circle $\rho(z, z_0) = r$. Hence $w_r \in L$. Thus for each n there exists a point $w_n \in L$ such that $\rho(w_n, z_0) = 1/n$. This implies that $w_n \to z_0$ and that z_0 is a limit point of $L \cap D$.

We are now in a position to prove an identity theorem for automorphic functions with Fuchsian groups of the first kind.

IDENTITY THEOREM. *Let $\mathscr{G}$ be a Fuchsian group of the first kind and let $f(z)$ be a meromorphic function automorphic with respect to $\mathscr{G}$. Let $\{z_n\}$ be a sequence of points in D with at least two limit points on C, such that $|z_n| \to 1$ and $\rho(z_n, z_{n+1}) \to 0$. If $f(z_n) \to c$, then $f(z) \equiv c$.*

Proof. Let L be the set of limit points of the set $\{S(z_n) : S \in \mathscr{G}, n = 1, 2, 3, \cdots\}$. By Lemma 3 there exists a point $z_0 \in L \cap D$, and by Lemma 4 there exists a sequence $\{w_n\}$ of points in $L \cap D$ such that $w_n \to z_0$. But $f(w_n) = c$ for each n and $f(z_0) = c$. Therefore, since $f(z)$ assumes the same value on a sequence of points in D with a limit point in D, $f(z) \equiv c$.

Since the sum of two automorphic functions with the same Fuchsian group is itself automorphic, we have a corresponding uniqueness theorem.

UNIQUENESS THEOREM. *Let $\mathscr{G}$ be a Fuchsian group of the first kind, and let $f(z)$ and $g(z)$ be two meromorphic functions automorphic with respect to $\mathscr{G}$. Let $\{z_n\}$ be a sequence of points in D with at* *least two limit points on C such that $|z| \to 1$ and $\rho(z_n, z_{n+1}) \to 0$. If $\{f(z_n) - g(z_n)\} \to 0$, then $f(z) \equiv g(z)$.*

4. Groups of the second kind. The identity and uniqueness theorems stated in § 3 are not true if $\mathscr{G}$ is a Fuchsian group of the second kind.

EXAMPLE. *There exists a nonconstant meromorphic function $f(z)$ automorphic with respect to a Fuchsian group $\mathscr{G}$ of the second kind, and a set of points $\{z_n\}$ in D with at least two limit points in C such that $|z_n| \to 1$, $\rho(z_n, z_{n+1}) \to 0$, and $f(z_n) \to 0$.*

Proof. Let $\mathscr{G}$ be a Fuchsian group of the second kind. Without loss of generality, we may assume that there exists a number k, $0 < k < 1$, such that the set $\{z : z \in D, \mathscr{R}(z) > 1 - k\}$ is a subset of R_0. Let F be the union of the half-plane $\mathscr{R}(z) < 1 - k/2$ and D. Let $\{z_n\}$ be a sequence of points in D with at least two limit points on C such that $|z_n| \to 1$, $\mathscr{R}(z_n) > 1 - k/2$, and $\rho(z_n, z_{n+1}) \to 0$. By a well-known theorem of Mittag-Leffler together with the Riemann Mapping Theorem, there exists a meromorphic function $M(z)$ in F whose only poles are at the points $\{z_n\}$, each pole being of order one. Then $M(z)$ is holomorphic and uniformly bounded in $D - R_0$. If the elements of $\mathscr{G}$ are denoted by $\{S_n\}$, where $S_n(z) = (a_n z + b_n)/(c_n z + d_n)$, then for $m \geqq 2$,

$$H_1(z) = \sum_{n=1}^{\infty} M(S_n(z)) \cdot \frac{1}{(c_n z + d_n)^{2m}}$$

and

$$H_2(z) = \sum_{n=1}^{\infty} \frac{1}{S_n(z)} \cdot \frac{1}{(c_n z + d_n)^{2m}}$$

define meromorphic functions in D, and

$$f(z) = H_2(z)/H_1(z)$$

is automorphic with respect to $\mathscr{G}$ (The argument that $H_1(z)$ is meromorphic is identical to that of the proof of [4, Theorem 2, p. 105]). Further, for each n, $f(z_n) = 0$. But $f(z)$ has a pole at $z = 0$, and thus $f(z)$ is nonconstant function.

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