BOUNDARY KERNEL FUNCTIONS FOR DOMAINS ON COMPLEX MANIFOLDS

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1. Introduction. Let D be a domain with piecewise differentiable boundary on a complex manifold X on which the holomorphic functions separate points. $L^2(d\sigma)$ is the space of square integrable functions on the boundary ∂D of D with respect to a surface measure $d\sigma$ on ∂D associated with a given riemannian metric on X. We can consider the space $H(\bar{D})$ of holomorphic functions on $\bar{D}$ as a subspace of $L^2(d\sigma)$. Let H^2 be the closure of $H(\bar{D})$ in $L^2(d\sigma)$.

The restriction mapping from $H(\bar{D})$ into the space $H(D)$ of holomorphic functions on D is shown to extend to a continuous mapping $i: H^2 \to H(D)$ (Lemma 4.1). A kernel $k: D \to H^2$ is associated with this mapping; k is conjugate holomorphic, and $\tilde{k} = i \circ k$ is a holomorphic kernel function on $D \times D^*$ where D^* denotes the space D with the conjugate structure (Theorem 5.1). In § 6 we discuss the special case of Reinhardt domains in C^n, and in § 7 an attempt is made to generalize Theorem 5.1 to domains on analytic spaces.

The author would like to thank E. Bishop for the hint that the results of this paper could as well be proven for complex manifolds on which the holomorphic functions separate points rather than only for Stein manifolds as was originally done, with only minor changes in the proofs.

2. Nowhere degenerate mappings. In the following X will always be an analytic space of pure dimension n. We assume that X is "countable at infinity" i.e. that it can be covered by a countable number of compact sets. We also assume that the holomorphic functions on X separate points.

Under these hypotheses there are nowhere degenerate holomorphic mappings from X into n-dimensional complex affine space C^n; a nowhere degenerate mapping is a map $f: X \to C^n$ such that for any $p \in C^n$, $\{f(x) = p\}$ is a discrete set on X. In fact it is proved in [1] that *the set of all nowhere degenerate holomorphic mappings from X into C^n is dense in the Frechet space of all holomorphic mappings from X into C^n* (Theorem 1 in [1]).

If $f: X \to C^n$ is a holomorphic nowhere degenerate mapping then each point $x \in X$ has a neighborhood U_x with the following property:

$f(U_x)$ is a polycylinder in C^n with center $f(x)$; f is a proper

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mapping from U_x onto $f(U_x)$, and there is a proper subvariety Δ_x of $f(U_x)$ such that $U_x - f^{-1}(\Delta_x) \to f(U_x) - \Delta_x$ is an s_x-sheeted covering map; the set $f^{-1}(\Delta_x) \cap U_x$ is closed and nowhere dense in U_x

This can for instance be verified as follows. Let j be an embedding of a neighborhood U'_x of x into some C^k; (f, j) gives an embedding of U'_x into $C^n \times C^k$ and $(f(x), j(x))$ is an isolated point of $(f, j)(U'_x) \cap \{f(x)\} \times C^k$ so that the assertion is a well known fact (see for instance the discussion in [5, section 2]).

LEMMA 2.1. *Let U be a relatively compact open subset of X and $y \in U$. There is a nowhere degenerate holomorphic mapping $\pi: X \to C^n$ such that $\{x \in X: \pi(x) = \pi(y)\}$ does not contain any point of the boundary ∂U of U.*

Proof. The set of all holomorphic mappings $g: X \to C^n$ with $\{x \in X: g(x) = g(y)\} \cap \partial U = \phi$ is clearly open in the space of all holomorphic mappings $X \to C^n$. Thus, by the previous remarks, it suffices to prove that this set is not empty.

Let Q be a relatively compact open subset of X containing the closure $\bar{U}$ of U. Let f be any nowhere degenerate holomorphic mapping from X into C^n. For simplicity we assume that $f(y) = 0$. Let $h_1, \cdots, h_k$ be holomorphic functions on X vanishing on $f^{-1}(0) \cap U$ such that $\{h_1 = \cdots = h_k = 0\} \cap \partial U = \phi$; this can be done since the holomorphic functions on X separate points. Notice that the difficulty lies in proving that one can choose n such functions.

Let $S = Q \times C^{nk}$ and denote the projection of S on Q by q and the projection onto C^{nk} by p. Define the functions

$$F_j(x, t) = f_j(x) + \sum_{i=1}^{k} t_{ij} h_i(x)$$

holomorphic on S. We want to show that we can choose t in such a way that $\{F_j(x, t) = 0, 1 \leqq j \leqq n\}$ does not meet ∂U.

There is a neighborhood N of 0 in C^{nk} such that for each $t \in N$

$$\{F_1(x, t) = \cdots = F_n(x, t) = 0\} \cap Q$$

is finite because there are no compact subvarieties on X.

Let $B = \partial U \times C^{nk} \subset S$ and $V = \{(x, t): F_j(x, t) = 0, 1 \leqq j \leqq n\}$. We show that 0 is not an interior point of $p(B \cap V)$. We prove in fact that $N \cap p(B \cap V)$ is of first category.

$q: V \to Q$ is certainly locally open at every point (x, t) with $h_i(x) \neq 0$ for some i, thus in particular at every point of $B \cap V$. Since $q(B \cap V) \subset \partial U$, $B \cap V$ cannot contain interior points. Hence $B \cap V$

is a closed nowhere dense subset of V. $p^{-1}(t) \cap V$ is finite for all $t \in N$. Therefore $p^{-1}(N) \cap V$ is of pure dimension nk and every point $z \in p^{-1}(N) \cap V$ has a neighborhood W_z such that $p: W_z \to p(W_z)$ is a proper mapping onto a neighborhood of $p(z)$ which is a covering map off some proper subvariety of $p(W_z)$.

Thus $p(B \cap W_z)$ is closed and nowhere dense in $p(W_z)$. Therefore $p(B \cap V) \cap N$ is of second category since $p^{-1}(N) \cap V$ is second countable. This completes the proof.

3. The martinelli integral formula.

For $z, \zeta \in C^n$, $z \neq \zeta$, let

$$\alpha_\zeta = \frac{n!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} \sigma_j \tag{3.1}$$

where

$$\sigma_j = dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_j \wedge [d\bar{z}_j] \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

the term in brackets being left out

The Martinelli formula ([2], [12]) asserts that

$$f(\zeta) = \int_{\partial D} f\alpha_\zeta , \qquad \zeta \in D \tag{3.2}$$

for every function f holomorphic in a neighborhood of the closure $\bar{D}$ of the bounded domain D in C^n with piecewise differentiable boundary ∂D.

That D has piecewise differentiable boundary shall mean the following:

There is a finite simplicial complex K in C^n with these properties:

(1) Every simplex $s \in K$ is a C^∞ mapping from a neighborhood of a standard simplex Δ^k in some R^k, $k \leq 2n$, into C^n which yields a C^∞ embedding of the closure of Δ^k into C^n.

(2) The support of K is $\bar{D}$.

(3) $\partial D = \partial \bar{D}$ and there is a subcomplex K_0 of K whose support is ∂D.

We consider the $2n$-dimensional simplices in K with the orientation induced by C^n and the $(2n-1)$-dimensional simplices in K_0 with the natural orientation that they carry as boundaries of $2n$-dimensional simplices in K i.e., they are oriented in such a way that the positive normal points into the domain D.

The integration in (3.2) is then to be interpreted as integration over the chain ΣC_ν where C_ν are the $(2n-1)$-dimensional simplices in K_0, and the integral is independent of the particular choice of the complex K.

Suppose now X is a complex manifold of pure dimension n and D a relatively compact domain in X with piecewise differentiable boundary

∂D. Suppose that the holomorphic functions on X separate points. Let π be a nowhere degenerate mapping from X into C^n. Let α_ζ, $\zeta \in C^n$ be the $(n-1)$-form on $C^n - \{\zeta\}$ defined by (3.1). For $y \in C$ we denote the form $\pi^*\alpha_{\pi(y)}$ on $C - \{\pi^{-1}\pi(y)\}$ by β_y.

LEMMA 3.1. *Suppose $\pi^{-1}\pi(y) \cap \partial D = \emptyset$. Then for every function f holomorphic in a relatively compact neighborhood U of $\bar{D}$,*

$$\sum_{t \in \pi^{-1}\pi(y) \cap D} f(t) = \int_{\partial D} f\beta_y \tag{3.3}$$

counting multiplicities in the summation.

Proof. Choose neighborhoods $P_1, \cdots, P_s$ of the points $y = t_1, \cdots, t_s$ in $\pi^{-1}\pi(y) \cap D$ such that $P_i \cap P_j = \emptyset$ for $i \neq j$, $P_j \subset D$, $\pi(P_j) = \pi(P_1)$ and $\pi^{-1}\pi(P_1) \cap D = P_1 \cup \cdots \cup P_s$; shrinking the P_j if necessary we may assume that there is a nowhere dense subvariety Δ of $\pi(P_1)$ such that π is biholomorphic at every point of $P_j - \pi^{-1}(\Delta)$, $1 \leqq j \leqq s$.

Let y' be any point in P_1 with $\pi(y') \notin \Delta$. Choose neighborhoods $U_1, \cdots, U_\ell$ of the points $y' = t'_1, \cdots, t'_\ell$ in $\pi^{-1}\pi(y') \cap D$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$, $U_j \subset D$, and π is biholomorphic on U_j, $1 \leqq i$, $j \leqq \ell$. We can modify a triangulation of $\bar{D}$ in such a way that there are triangles $D_1, \cdots, D_\ell$ such that t'_j is an interior point of D_j and $D_j \subset U_j$, $1 \leqq j \leqq \ell$. Let $D' = D - D_1 \cup \cdots \cup D_\ell$.

For any function g holomorphic in an open set W in C^n, $d(g\alpha_\zeta) = 0$ on $W - \{\zeta\}$. Therefore $d(f\beta_{y'}) = 0$ in a neighborhood of any point in $U - \pi^{-1}\pi(y')$ at which π is biholomorphic. Since the set of such points is open and nowhere dense in $U - \pi^{-1}\pi(y')$, $d(f\beta_{y'}) = 0$ on $U - \pi^{-1}\pi(y')$. Thus by Stokes' theorem

$$\int_{\partial D} f\beta_{y'} = \sum_1^\ell \int_{\partial D_j} f\beta_{y'} = \sum_1^\ell f(t'_j)\,. \tag{3.4}$$

The left hand side of (3.4) is a continuous function of $y' \in P_1$ and so is the right hand side. Since the equality (3.4) holds on the dense subset $P_1 - \pi^{-1}(\Delta)$, it holds on all of P_1. This completes the proof.

Now we make use of a device of [1] (Theorem 7) to write Lemma 3.1 in a more general form.

Let w be a holomorphic function on U. Define

$$\tilde{w}(x, y) = (w(x_2) - w(y)) \cdots (w(x_s) - w(y))$$

where s is the number of sheets of D over $\pi(x)$ and $\pi^{-1}\pi(x) \cap D = \{x = x_1, x_2, \cdots, x_s\}$ counting multiplicities. $\tilde{w}(x, y)$ is clearly holomorphic in y. It is also holomorphic in (x, y) in a neighborhood of every point (x^0, y^0) with $\pi^{-1}\pi(x^0) \cap \partial D = \emptyset$. One writes

$$\tilde{w}(x, y) = \Sigma \pm a_i(x)\,(w(y))^i$$

where the a_i are elementary symmetric polynomials of $\{w(x_2), \cdots, w(x_s)\}$. Every such polynomial is a linear combination of an elementary symmetric polynomial ω of $\{w(x_1), \cdots, w(x_s)\}$ and the product of $w(x) = w(x_1)$ with a polynomial $a_i(x)$ of lower order. Thus if each ω is holomorphic, one can prove by induction that the a_i are holomorphic. As before let $P = P_1 \cup \cdots \cup P_s$ be a neighborhood of $\pi^{-1}\pi(x^0) \cap D$ such that π is a covering map off some proper subvariety Δ in $\pi(P)$ and $\pi^{-1}\pi(P) \cap \partial D = \emptyset$. ω can then be considered as a function on $\pi(P)$. ω is holomorphic on $\pi(P) - \Delta$, hence ω is holomorphic on all of $\pi(P)$ by a well known theorem on removable singularities (see for instance [13]).

LEMMA 3.2. *With the notation of Lemma* 1.1

$$\int_{\partial D} f\tilde{w}(y, \cdot)\beta_y = \hat{w}(y) f(y)$$

where $\hat{w}(y) = \tilde{w}(y, y)$ *is holomorphic in a neighborhood of* y.

Proof. By Lemma 3.1

$$\int_{\partial D} f\tilde{w}(y, \cdot)\beta_y = \sum_1^{\ell} f(y_j)\, \tilde{w}(y, y_j)$$

where $\pi^{-1}\pi(y) \cap D = \{y_1, \cdots, y_\ell\}$ (counting multiplicities). Since $\tilde{w}(y, y_j) = 0$ for $y_j \neq y$, the lemma is proved.

4. H^2-spaces. Let X be a complex manifold of pure dimension n such that the holomorphic functions on X separate points. Let ds^2 be a riemannian metric on X. Assume D is a domain on X with piecewise differentiable boundary. Let $d\sigma$ be the volume element on ∂D associated with the metric induced by ds^2. $L^2 = L^2(d\sigma)$ is the Hilbert space of square integrable functions on ∂D. The space $H(\bar{D})$ of functions holomorphic in a neighborhood of $\bar{D}$ is in a natural way a subspace of L^2, the natural map $H(\bar{D}) \rightarrow L^2$ being injective by the maximum principle. Let $H^2 = H^2(\partial D)$ be the closure of $H(D)$ in L^2.

Consider now the restriction map r: $H(\bar{D}) \rightarrow H(D)$ from $H(\bar{D})$ into the space $H(D)$ of holomorphic functions on D. $H(D)$ is a Frechet space in the topology of uniform convergence on compact sets. We would like to extend r to a continuous mapping from H^2 into $H(D)$. For this it suffices to prove the following lemma.

LEMMA 4.1. *The restriction mapping* r: $H(\bar{D}) \rightarrow H(D)$ *is continuous if we consider* $H(\bar{D})$ *with the topology induced by* L^2.

Proof. Suppose f converges to 0 in $H(\bar{D})$. Let $x \in D$. We show that f converges to 0 uniformly in some neighborhood of x.

According to Lemma 2.1 we can find a nowhere degenerate mapping π from X into C^n such that $\pi^{-1}\pi(U) \cap \partial D = \varnothing$ for some neighborhood $U \subset D$ of x. By Lemma 3.2,

$$\hat{w}(y)\,[f(y)]^2 = \int_{\partial D} f^2 \tilde{w}(y, \cdot)\beta_y \quad \text{for } y \in U\,. \tag{4.1}$$

If we can prove that the integral on the right of (4.1) converges uniformly to 0 in a neighborhood of x, then f will also converge uniformly to 0 in a neighborhood of x. This can be proved as follows. By the closure of modules theorem [8] [15], $\hat{w}$ generates a closed ideal $\hat{w}(H(W))$ in $H(W)$ for every open subset W of X. Thus by the open mapping theorem, the mapping $H(W) \to \hat{w}(H(W))$ which is multiplication by $\hat{w}$ is a homeomorphism whenever it is one-to-one; but for every point in U there is such a neighborhood and a suitable $\hat{w}$ since the functions on X separate points. In fact, we can choose w in such a way that $\hat{w}$ is different from zero in some neighborhood of a given point in U and thus avoid the use of the closure of modules theorem. However, the above argument using the closure of modules theorem will generalize to analytic spaces that we are going to discuss in § 7.

Thus we have to concentrate on the integral on the right of (4.1). $\tilde{w}(y, \cdot)$ is uniformly bounded on ∂D for y in a neighborhood of x, so we need only show that

$$\int_{\partial D} |f|^2\, |\beta_y| \tag{4.2}$$

converges to 0 uniformly on some neighborhood of x. Or it suffices to prove that for any triangle Δ in the triangulation of ∂D,

$$\int_{\Delta} |f|^2\, |\beta_y| \tag{4.3}$$

converges to 0 uniformly in a neighborhood of x. There is a submanifold $M \supset \bar{\Delta}$ containing Δ as an open subset, and M has global coordinates $t_1, \cdots, t_{2n-1}$ (M is the diffeomorphic image of a neighborhood of a standard $(2n-1)$-dimensional simplex). Since β_y depends continuously on y if y is restricted to a suitable neighborhood of x we have an estimate

$$\int_{\Delta} |f|^2\, |\beta_y| \leqq c \int_{\Delta} |f|^2\, dt_1 \wedge \cdots \wedge dt_{2n-1} \tag{4.4}$$

for y in a neighborhood $U' \subset U$ of x. Let $d\sigma = g(t)\,dt_1 \wedge \cdots \wedge dt_{2n-1}$. There is a constant $c' > 0$ such that $g(t) > c'$ for $t \in \bar{\Delta}$. Thus

$$(4.5)\qquad c' \int_{\Delta} |f|^2\, dt_1 \wedge \cdots \wedge dt_{2n-1} \leqq \int_{\Delta} |f|^2 g\, dt_1 \wedge \cdots \wedge dt_{2n-1} \leqq \|f\|_2^2$$

where $\|f\|_2$ is the norm of f in L^2. This shows that the right hand side of (4.4) converges to 0 as f tends to 0. This completes the proof.

We say that $f \in H^2$ possesses a "holomorphic continuation" $\tilde{f} = i(f)$ into D. Though it seems plausible that two different elements of H^2 have different holomorphic continuations into D (as is the case if D is the unit disc in C^1), I see no way of proving this in this general setting.

5. The boundary kernel function. Let η_w be the continuous linear functional on $H(D)$ which is evaluation at $w \in H(D)$, $\eta_w(f) = f(w)$. $\eta_w \circ i$ is a continuous linear functional on H^2, i.e. η_w is an element of the dual $\bar{H}^2$ of H^2. We do not identify H^2 with $\bar{H}^2$ but think of $\bar{H}^2$ as the space $H^2(\partial D^*)$ where D^* is the domain D considered as a subset of the space X^*, which is the complex manifold X with the conjugate analytic structure. Thus $\bar{H}^2$ are the conjugates of the functions in H^2, and the pairing between elements in H^2 and $\bar{H}^2$ is given by

$$\langle f, g\rangle = \int_{\partial D} f g \, d\sigma\ , \quad f \in H^2, g \in \bar{H}^2\ .$$

Let $k_w = \eta_w \circ i \in \bar{H}^2$. For $f \in H^2$ we have $\langle f, k_w\rangle = i(f)(w)$; thus $\langle f, k_w\rangle$ is holomorphic in w, or k_w defines an $\bar{H}^2 =$ valued holomorphic function k on D (cf. for instance [10] or [4, Theorem 4.1]).

Suppose now $h_1, h_2, \cdots$ is an orthonormal base for H^2, then $\bar{h}_1, \bar{h}_2$ is the dual orthonormal base for $\bar{H}^2$ and

$$k_w = \Sigma \langle h_m, k_w\rangle \overline{h_m}\ ,$$

where the sum converges uniformly on compact sets in D because of the continuity of k. Let $\tilde{k}$ be the function $(z, w) \rightarrow i(\overline{k_w})(z)$ defined on $D \times D^*$, $\tilde{k} = \langle \bar{k}_w, k_z\rangle$. Then

$$\tilde{k}(z, w) = \Sigma\, i(h_m)(z) \cdot \overline{i(h_m)(w)}\ ,$$

where the sum converges uniformly on compact subsets of $D \times D^*$ because of the continuity of i. Hence $\tilde{k}$ is holomorphic on $D \times D^*$. We recollect:

THEOREM 5.1. *Let D be a domain with piecewise differentiable boundary on a complex manifold X. We assume that the holomorphic functions on X separate points. Let $d\sigma$ be the surface element on ∂D associated with a riemannian metric on X. H^2 is the closure of $H(\bar{D})$ in $L^2(d\sigma)$. There is a natural continuous mapping $i\colon H^2 \rightarrow H(D)$ that extends the restriction mapping $H(\bar{D}) \rightarrow H(D)$. There is a conjugate holomorphic H^2-valued function $\bar{k}$ on D such that*

$$i(f)(w) = \int_{\partial D} f k_w \, d\sigma \,, \quad w \in D \,.$$

$i(\bar{k}) = \tilde{k}$ is a holomorphic function on $D \times D^*$ where D^* is the space D with the conjugate structure. If $\{h_m\}$ is an orthonormal base for H^2 then

$$\tilde{k} = \Sigma\, i(h_m)(z) \cdot \overline{i(h_m)(w)} \,,$$

and the sum converges uniformly on compact subsets of $D \times D^*$.

Similar considerations have been made by various authors, especially by S. Bergman and G. Szegö (cf. the introduction in [11]). However, Theorem 5.1 seems to be the first general result concerning the convergence of the kernel function. It has been obtained before in the thesis [11] for the special case of holomorphically convex complete Reinhardt domains with smooth boundary in C^2. We are now able to extend some of the results of [11] to arbitrary dimensions.

6. Boundary kernel functions for certain Reinhardt domains.

Recall that a Reinhardt domain in C^n is a relatively compact connected subdomain D of C^n such that for every $z \in D, t \cdot z = (t_1 \cdot z_1, \cdots, t_n \cdot z_n)$ belongs to D for t an element of the torus $T^n = (e^{i\varphi_1}, \cdots, e^{i\varphi_n}), 0 \leqq \varphi_j < 2\pi$.

In the following we assume always that D is a Reinhardt domain with piecewise smooth boundary satisfying

$$(6.1) \qquad \begin{cases} z^0 \in \partial D \text{ and } z^0_{j_1} = \cdots = z^0_{j_k} = 0 \text{ for certain } j_i \text{ implies} \\ \text{that there are } z \in D \text{ arbitrarily close to } z^0 \text{ with} \\ z_{j_1} = \cdots = z_{j_k} = 0 \,. \end{cases}$$

Let $e_1, \cdots, e_m$ be those coordinate functions z_j for which $\{z_j = 0\} \cap D = \emptyset$, and $e_{m+1}, \cdots, e_n$ those for which $\{z_j = 0\} \cap D \neq \emptyset$. It has been shown [7, 13] that the polynomials in $e_1, \cdots, e_n, e_1^{-1}, \cdots, e_m^{-1}$ are dense in the closure $A(\bar{D})$ of $H(\bar{D})$ with respect to the sup norm.

Let H^2 be defined as before; H^2 is the closure of $A(\bar{D})$ in the space L^2 of square-integrable functions on ∂D (with respect to Lebesgue measure).

LEMMA 6.1. *The monomials $\{e_1^{\alpha_1} \cdot \cdots \cdot e_n^{\alpha_n}, \alpha_j \geqq 0$ for $j > m\}$ are orthogonal in H^2.*

Proof. Let T^n be the n-dimensional torus $\{t : |t_1| = \cdots = |t_n| = 1\} \subset C^n$. The elements of T^n act on ∂D by $t \circ z = (t_1 z_1, \cdots, t_n z_n)$ for $t \in T^n, z \in \partial D$. We consider ∂D with this structure. ∂D is essentially the disjoint union of a countable number of open subsets U_k isomorphic to $V_k \times T^n$ under a nonsingular C^2 map, where V_k is an open set in $\boldsymbol{R}^{n-1}$ i.e., we can find a disjoint union of such U_k having a complement of measure zero on ∂D. Namely, let U be the open set of regular points on ∂D; these are the points that have a neighborhood on ∂D which is a C^2 $(2n-1)$-dimensional real submanifold of $\boldsymbol{C}^n$. Notice that $T^n \circ U = U$. $\partial D - U$ has of course measure zero on ∂D. The set

$$U \cap \bigcup \{z_j = 0\}$$

has measure zero on U since the set $\bigcup \{z_j = 0\}$ has $(2n-1)$-dimensional Hausdorff measure zero. (For the definition of Hausdorff measure see: St. Saks, Theory of the Integral, Chapter II, § 8). $|z_1|, \cdots, |z_n|$, arg $z_1, \cdots,$ arg z_n have rank $2n$ at each point of $\boldsymbol{C}^n - \bigcup \{z_i = 0\}$. Thus at each point

$$z \in U - U \cap \bigcup \{z_j = 0\}$$

$|z_1|, \cdots, |z_n|$, arg $z_1, \cdots,$ arg z_n have rank $2n-1$. But

$$T^n \circ z \subset U - U \cap \bigcup \{z_j = 0\}$$

so that $|z_1|, \cdots, |z_n|$ must have rank $n-1$ at z; hence z has a neighborhood U_z in which $n-1$ of the $|z_j|$ and arg $z_1, \cdots,$ arg z_n are coordinates. These are then also coordinates for each point in $T^n \circ U_z$. Thus we can write $U - U \cap \bigcup \{z_j = 0\}$ essentially as the disjoint union of a countable number of open subsets U_k isomorphic to $V_k \times T^n$ as described above. Now let $x = (x_1, \cdots, x_{n-1})$ be a coordinate system for V_k and $\varphi = (\varphi_1, \cdots, \varphi_n)$ a coordinate system for T^n such that $t_j = e^{i\varphi_j}$ for $t \in T^n$. Let g_{ij} be the coefficients of the metric tensor in the coordinate system (x, φ). Since T^n acts isometrically on $\boldsymbol{C}^n$, it acts isometrically on $V_k \times T^n$, hence the g_{ij} are independent of the coordinates φ. For $d\sigma$ the Lebesgue measure on ∂D we have

$$\int_D e_1^{\alpha_1} \cdot \cdots \cdot e_n^{\alpha_n} \cdot \bar{e}_1^{\beta_1} \cdot \cdots \cdot \bar{e}_n^{\beta_n}\, d\sigma$$
$$= \Sigma_k \int_{V_k} \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_{n-1} \int_{T^n} e_1^{\alpha_1} \cdot \cdots \cdot \bar{e}_n^{\beta_n}\, d\varphi_1 \cdots d\varphi_n\,.$$

The integral over T^n is nonzero if and only if $\alpha_j = \beta_j$, $1 \leqq j \leqq n$.

We define

$$h_\alpha = \frac{e_1^{\alpha_1} \cdot \cdots \cdot e_n^{\alpha_n}}{\| e_1^{\alpha_1} \cdot \cdots \cdot e_n^{\alpha_n} \|_2} \quad (\alpha_j \geqq 0 \text{ for } j > m)\,. \tag{6.2}$$

COROLLARY 6.2. *The* $\{h_\alpha\}$ *form an orthonormal base for* H^2. *Now by Theorem* 5.1,

$$\tilde{k}(z, w) = \Sigma\, h_\alpha(z)\, \overline{h_\alpha(w)}$$

converges uniformly on compact subsets of $D \times D$. *But we can do better in this special case:*

THEOREM 6.3. *For each compact set* $K \subset D$ *there is a neighborhood* U *of* $\bar{D}$ *such that*

$$\tilde{k}(z, w) = \Sigma\, h_\alpha(z)\, \overline{h_\alpha(w)}$$

converges uniformly on $U \times K$. *Thus* $\tilde{k}(z, \bar{w})$ *can actually be considered as a holomorphic function on* D^* *with values in* $A(\bar{D})$. *For each* $w_0 \in D$, $\tilde{k}(z, w_0)$ *belongs to* $H(\bar{D})$.

Proof. Let $K \subset D$ be compact and choose a compact set $K_0 \subset D$ containing K in its interior. Let $z^0 \in \partial D$ and find $\eta \in C^n$, $\eta \neq 0$ such that $\eta \cdot z^0 = (\eta_1 z_1, \cdots, \eta_n z_n) \in D$. We can choose the η_j arbitrarily close to 1 (condition (6.1)). Now

$$\Sigma\, h_\alpha(\eta z^0)\, \overline{h_\alpha(y)}$$

converges uniformly in $y \in K_0$. Let $w \in K$ and consider $(z_j^0 + \varepsilon_j)w_j$. If $z_j^0 \neq 0$ define $y_j \in C$ by

$$(z_j^0 + \varepsilon_j)\bar{w}_j = \eta_j z_j^0\Big(\eta_j^{-1} + \frac{\varepsilon_j}{\eta_j z_j^0}\Big)\bar{w}_j = (\eta_j z_j^0)\bar{y}_j\ ,$$

and $y_j = w_j$ otherwise. For η close enough to 1, $y = (y_j) \in K_0$ as w runs through K and $\varepsilon = (\varepsilon_j)$ stays in a small neighborhood U of 0 in C^n. Thus if $z_j^0 \neq 0$ for all j,

$$\Sigma\, h_\alpha(z^0 + \varepsilon)\, \overline{h_\alpha(w)} \tag{6.3}$$

converges uniformly in $\varepsilon \in U$, $w \in K$. In general define $x_\varepsilon \in D$ by $(x_\varepsilon)_j = \eta_j z_j^0$ if $z_j^0 \neq 0$, $(x_\varepsilon)_j = \varepsilon_j$ if $z_j^0 = 0$. Then if U is chosen small enough,

$$\Sigma\, h_\alpha(x_\varepsilon)\, \overline{h_\alpha(y)}$$

converges uniformly in $\varepsilon \in U$, $y \in K_0$, and we can conclude as before that (6.3) converges uniformly in $\varepsilon \in U$, $w \in K$.

This proves the theorem since $\bar{D}$ is compact.

COROLLARY 6.4. *The map* $i\colon H^2 \to H(D)$ *constructed in Theorem* 5.1 *is injective.*

Proof. Suppose $f \in H^2$ and $i(f) = 0$. Then

$$i(f)(w) = \Sigma \langle f, \bar{h}_\alpha \rangle h_\alpha(w) = 0\ . \tag{6.4}$$

But (6.4) is a Laurent expansion; thus by the uniqueness theorem for

Laurent series, $\langle f, \bar{k}_\alpha \rangle = 0$ for all α, hence $f = 0$.

For instance, the boundary kernel function for the unit ball in C^n is

$$\tilde{k}(z, w) = \frac{2^{n-1}}{(2\pi)^n} \frac{(n-1)!}{(1 - z \cdot \bar{w})^n},$$

where $z \cdot \bar{w} = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. For $n = 2$, this formula has been calculated in [11]. However, I do not know whether anybody has established this formula for a general n before. Here is the calculation, a joint effort of H. Rossi and mine. Let v_x be the volume of the torus $x \cdot T^n$. Then the square of the L^2-norm of $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is

$$\int_{x \geqq 0} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} v_x d\eta = (2\pi)^n \int_{x \geqq 0} x_1^{2\alpha_1+1} \cdots x_n^{2\alpha_n+1} d\eta$$

where $d\eta$ is the Lebesgue measure on the unit sphere in R^n. Denote the last integral by a_α.
We have

$$\begin{aligned} a_\alpha & \int_0^\infty e^{-r^2} r^{2(\Sigma\alpha_j+n)-1} dr \\ &= \int_0^\infty \cdots \int_0^\infty e^{-r^2} x_1^{2\alpha_1+1} \cdots x_n^{2\alpha_n+1} dx_1 \cdots dx_n \\ &= \prod_{k=1}^n \int_0^\infty e^{-x_k^2} x_k^{2\alpha_k+1} dx_k , \end{aligned}$$

thus

$$a_\alpha = \frac{1}{2^{n-1}} \frac{\alpha_1! \cdots \alpha_n!}{(\Sigma\alpha_j + n - 1)!} .$$

For $\tilde{k}$ we get therefore

$$\begin{aligned} \tilde{k}(z, w) &= (2\pi)^{-n} \sum_{\alpha \geqq 0} a_\alpha^{-1} z^\alpha \bar{w}^\alpha \\ &= 2^{n-1} \cdot (2\pi)^{-1} \sum_{k=0}^\infty (k + n - 1) \cdots (k + 1) (\Sigma z_j \bar{w}_j)^k \\ &= \frac{2^{n-1}}{(2\pi)^n} \cdot \frac{(n-1)!}{(1 - z\bar{w})^n} . \end{aligned}$$

7. **Generalization to analytic spaces.** We are now going to discuss some ways of generalizing Theorem 5.1 to Stein analytic spaces.

To generalize the definition of a domain with piecewise differentiable boundary to domains on a complex analytic spaces we have only to define what we mean by differentiable and diffeomorphic mappings from a differentiable manifold M into an analytic space X. A mapping $f: M \to X$ is called differentiable if f maps the sheaf of germs of differentiable functions on X (as defined in [9]) into the sheaf of germs of differentiable functions on M. If a neighborhood V of $0 \in X$ is realized as a closed subvariety of a neighborhood U of 0 in some $\boldsymbol{C}^k$ then the germs of C^∞ functions on V are the restrictions of germs of C^∞ functions on U. The tangent space T_x to a point $x \in X$ is the space of point derivations on the ring of germs of real C^∞ functions at x. If $x \in V$, T_x is naturally embedded in the tangent space ${}_{2k}T_x$ to the C^∞ manifold $\boldsymbol{C}^k$ at x; the tangents in T_x are exactly those tangents in ${}_{2k}T_x$ that vanish on the germs of C^∞ functions at x vanishing on V. The C^∞ mapping $f: M \rightarrow X$ is called diffeomorphic at $t \in M$ if it induces an injection from the tangent space to M at t into $T_{f(t)}$.

In order to establish Lemma 3.2 for analytic spaces we have to say what we mean by a C^∞ differential form on an analytic space; or it suffices to define the sheaf Ω_k of germs of C^∞ differential k-forms on an analytic space X. Let $0 \in X$ and V be a neighborhood of 0 that can be embedded as a closed subvariety of a neighborhood U of 0 in $\boldsymbol{C}^n$. Let ${}_n\Omega_k$ be the sheaf of germs of C^∞ differential k-forms on U and ${}_n\Omega_k^0$ the subsheaf of ${}_n\Omega_k$ of germs ω which are finite sums of germs $h_1\alpha + dh_2 \wedge \beta$, the h_i being germs of C^∞ functions vanishing on V. $\Omega_k = {}_n\Omega_k/{}_n\Omega_k^0$ (restricted to V) is called the sheaf of germs of C^∞ differential k-forms on V.

If V' is another neighborhood of 0 on X that can be embedded as a closed subvariety of a neighborhood U' of 0 in $\boldsymbol{C}^m$, and Ω'_k is the sheaf on V' associated with this embedding, then by applying Lemmas 2.4c and 2.5b of [14] we find easily that Ω_k and Ω'_k must coincide in some neighborhood of 0 on X. Thus the sheaf Ω_k is well defined on X.

If Y is another analytic space and $f: X \rightarrow Y$ is a holomorphic map then there is a cannonical map of sheaves $f^*: \Omega'_k \rightarrow \Omega_k$, Ω'_k being the sheaf of germs on C^∞ differential k-forms on Y. This map can be locally defined as follows.

Suppose a neighborhood $V(V')$ of $0 \in X(f(0) = 0 \in Y)$ is embedded as a closed subvariety of a polycylinder $U(U')$ in $\boldsymbol{C}^n(\boldsymbol{C}^m)$. We may assume that f maps V into V'. We extend f to a holomorphic mapping F from U into $\boldsymbol{C}^m$, and we may as well assume that $F(U) \subset U'$ (shrinking U if necessary). Then F^* maps ${}_m\Omega_k^0$ (the sheaf of germs $\Sigma(h_1^j\alpha_j + dh_2^j \wedge \beta_j)$ where h_i^j vanish on V') into ${}_n\Omega_k^0$, and thus induces a map $f^*: \Omega'_{kf(0)} \rightarrow \Omega_{k0}$. It is easy to check that f^* does not depend on the extension F of f, nor does it depend on the choice of the embeddings as can be seen by applying Lemmas 2.4c and 2.5b of [14] as above.

The operator $d: {}_n\Omega_k \rightarrow {}_n\Omega_{k+1}$ passes to the quotient and yields an operator $d: \Omega_k \rightarrow \Omega_{k+1}$ which is independent of the local embedding of X. Also Stokes' theorem will hold for C^∞ chains on X (finite formal sums of C^∞ mappings from standard simplices into X) since that theorem is of formal nature.

With these definitions Lemmas 3.1 and 3.2 hold on any analytic space on which the holomorphic functions separate points.

Also the proof of Lemma 4.1 is valid for such an analytic space X with the following definition of a Riemannian metric ds^2 on X. ds^2 is an inner product on the real tangent spaces T_x to X with the following regularity property. If V is a neighborhood of $0 \in X$ which can be embedded as a closed subvariety of a neighborhood U of 0 in some C^k then there is a neighborhood W of 0 contained in U and a C^∞ Riemannian metric on W that induces ds^2 on each T_x, $x \in V \cap W$.

THEOREM 7.1. *Theorem* 5.1 *is valid also for an analytic space* X *of pure dimension on which the holomorphic functions separate points.*

Unfortunately, there may not exist many domains with piece-wise differentiable boundary on a given analytic space X. For instance, a relatively compact domain D on X which is the closure of its interior and whose boundary is a real analytic variety, is not necessarily a domain with piecewise differentiable boundary. We will call these domains with real analytic boundary. Though it is true that a domain with real analytic boundary admits some kind of triangulation [3], such triangulations are not suitable for our estimates in §§ 3 and 4. However, we have shown in [5] that we can integrate differential forms on ∂D if D is a domain with real analytic boundary. (Note that ∂D is orientable). Now Lemmas 3.1 and 3.2 will again follow as before provided we konw that Stokes' theorem holds for domains with real analytic boundaries.

CONJECTURE 7.2. *Stokes' theorem holds for domains with analytic boundaries.*

We will deal with this problem in a later paper.

Now let D be a domain with analytic boundary on the analytic space X of pure dimension n on which the holomorphic functions separate points. Suppose we are given a Riemannian metric on X. Let $d\sigma$ be the surface element on ∂D ($d\sigma$ is actually only defined on the set of regular points of ∂D). We have proved in [5] that every function holomorphic on $\bar{D}$ is in $L^2(d\sigma)$. As before let H^2 be the closure of $H(\bar{D})$ in L^2. Then Lemma 4.1 is valid if conjecture 7.2 holds. The proof is similar to the one given in § 4; we embed a sufficiently small neighborhood V in X of a point $x \in \partial D$ into some C^k. We can find a coordinate system $t_1, \cdots, t_{2k}$ for C^k such that for any permutation π of $(1, \cdots, 2k)$, $t_{\pi 1}, \cdots, t_{\pi(2n-1)}$ are coordinates for a subset Δ of $\partial D \cap V$ that differs from the set of regular points of $\partial D \cap V$ by a set of local

Lebesgue measure zero (see the first two paragraphs of § 3 in [5]). Then estimate (4.4) takes the form

$$\int_{\Delta} |f|^2 |\beta_y| \leqq c\, \Sigma_\pi \int_{\Delta} |f|^2\, dt_{\pi 1} \cdots dt_{\pi(2n-1)} \tag{4.4'}$$

where the sum runs over all permutations π. Let $d\sigma = g_\pi dt_{\pi 1} \cdots dt_{\pi(2n-1)}$; then by Lemma 4.4 in [5],

$$\begin{aligned} & c'\, \Sigma_\pi \int_{\Delta} |f|^2\, dt_{\pi 1} \cdots dt_{\pi(2n-1)} \\ & \leqq \Sigma_\pi \int_{\Delta} |f|^2\, g_\pi dt_{\pi 1} \wedge \cdots \wedge dt_{\pi(2n-1)} \\ & \leqq c'' \, ||f||^2 , \end{aligned} \tag{4.5}$$

where $||\cdot||$ is the norm in $L^2(d\sigma)$. Thus

THEOREM 7.3. *If conjecture* 7.2 *holds then Theorem* 5.1 *holds for domains with real analytic boundary on an analytic space of pure dimension on which the holomorphic functions separate points.*

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