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# MONOTONE APPROXIMATION

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**How close can one approximate a monotone function by a monotone polynomial of degree $\leqq n$, or a convex function by a convex polynomial of degree $\leqq n$? This leads to the following general question. Let $k$ and $n$ be given, and suppose a real fuction $f$ satisfies $f^{(k)}(x) \geqq 0$ throughout a closed, finite interval $[a, b]$. How close can one approximate $f$ on $[a, b]$ by a polynomial of degree $\leqq n$ whose $k$th derivative, too, is $\geqq 0$ there? We give an answer to the question.**

2. THEOREM 1. *Let $k$ and $p$ be integers, $1 \leqq k \leqq p$, and let a real function $f$ satisfy throughout* $[a, b]$

$$f^{(k)}(x) \geqq 0 ,$$
$$|f^{(p)}(x_2) - f^{(p)}(x_1)| \leqq \lambda |x_2 - x_1| ,$$

*$\lambda$ being a constant. Then for every integer $n(\geqq p)$ there exists a real polynomial $Q_n(x)$ of degree*[1] *$\leqq n$ such that*

(a) $Q_n^{(k)}(x) \geqq 0$ *throughout* $[a, b]$,

(b) $\operatorname*{Max}_{a \leqq x \leqq b} |f(x) - Q_n(x)| \leqq 2\lambda\left(\frac{\pi}{4}\right)^{p-k+1}(b-a)^{p+1}\left[k! \prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1}.$

3. To prove Theorem 1, we begin by quoting the following result of J. Favard [2] and N. Ahiezer and M. Krein [1] which strengthens a previous result of D. Jackson.

THEOREM 2. (*Favard, Ahiezer-Krein*) *Let $f$ (with period $2\pi$) map the reals into the reals, and satisfy for every real $x_1, x_2$*

$$(1) \qquad |f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1| ,$$

*$\lambda$ being a constant. Then for $n = 0, 1, 2, \cdots$, there exists a trigonometric polynomial $T_n(x) \equiv \sum_{\nu=0}^{n} a_\nu^{(n)} \cos \nu x + b_\nu^{(n)} \sin \nu x$ such that* $\max_{0 \leqq x \leqq 2\pi} |f(x) - T_n(x)| \leq \lambda(\pi/2)[1/(n+1)]$.

From Theorem 2 one obtains by the method of [3], pp. 13–14 the following

THEOREM 3. *Let $f$ be a real function satisfying* (1) *throughout $[a, b]$, $\lambda$ being a constant. Then for $n = 0, 1, 2, \cdots$, there exists a*

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[1] By degree of a polynomial we mean its exact degree. (The degree of the polynomial 0 is $-1$).

*polynomial* $P_n(x)$ *of degree* $\leqq n$ *such that*

$$\max_{a\leqq x\leqq b}|f(x)-P_n(x)|\leqq \lambda\frac{\pi}{4}\frac{b-a}{n+1}\,.$$

For future use, we make the following simple observation. (Compare [3], p. 16).

LEMMA. *Let* $f$ *be a real function, continuous in* $[a,b]$ *and differentiable in* $(a,b)$. *Let* $n$ *be an integer* ($\geqq 0$), $q_{n-1}(x)$ *a real polynomial of degree* $\leqq n-1$, *and let* $\varepsilon$ *be such that* $|f'(x)-q_{n-1}(x)|\leqq\varepsilon$ *throughout* $(a,b)$. *Then there exists a polynomial* $P_n(x)$ *of degree* $\leqq n$ *such that*

$$\max_{a\leqq x\leqq b}|f(x)-P_n(x)|\leqq \varepsilon\frac{\pi}{4}\frac{b-a}{n+1}\,. \tag{2}$$

To prove the lemma, set $r(x)\equiv f(x)-\int_a^x q_{n-1}(t)dt$. Throughout $(a,b)$, $|r'(x)|\leqq\varepsilon$, and therefore, throughout $[a,b]$, $|r(x_2)-r(x_1)|\leqq\varepsilon|x_2-x_1|$. By Theorem 3, there exists a polynomial $\pi_n(x)$ of degree $\leqq n$ such that $\max_{a\leqq x\leqq b}|r(x)-\pi_n(x)|\leqq\varepsilon(\pi/4)(b-a)/(n+1)$. Setting $P_n(x)\equiv\pi_n(x)+\int_a^x q_{n-1}(t)dt$, we obtain (2).

From Theorem 3 and the Lemma one gets readily (cf. [3], pp. 16–17) the following

THEOREM 4. *Let* $f$ *be a real function satisfying throughout* $[a,b]$, *for some constant integer* $p(\geqq 0)$ *and some constant* $\lambda$,

$$|f^{(p)}(x_2)-f^{(p)}(x_1)|\leqq\lambda|x_2-x_1|\,.$$

*Then for every integer* $n(\geqq p)$ *there exists a polynomial* $P_n(x)$ *of degree* $\leqq n$ *such that*

$$\max_{a\leqq x\leqq b}|f(x)-P_n(x)|\leqq\lambda\left[\frac{\pi}{4}(b-a)\right]^{p+1}\left[\prod_{\nu=0}^{p}(n+1-\nu)\right]^{-1}.$$

3. *Proof of Theorem* 1. Let $n$ be an integer $\geqq p$. Set $f_n(x)\equiv f^{(k)}(x)+\lambda[(\pi/4)(b-a)]^{p-k+1}[\prod_{\nu=k}^{p}(n+1-\nu)]^{-1}$. Then throughout $[a,b]$, $|f_n^{(p-k)}(x_2)-f_n^{(p-k)}(x_1)|\leqq\lambda|x_2-x_1|$. By Theorem 4, there exists a real polynomial $P_{n-k}(x)$ of degree $\leqq n-k$ such that

$$\max_{a\leqq x\leqq b}|f_n(x)-P_{n-k}(x)|\leqq\lambda\left[\frac{\pi}{4}(b-a)\right]^{p-k+1}\left[\prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1}.$$

So, throughout $[a,b]$, $P_{n-k}(x)\geqq f^{(k)}(x)\geqq 0$. Let

$$Q_n(x) \equiv \left[\sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(a)}{\nu!}(x-a)^\nu\right] + \int_a^{t_{k+1}}\int_a^{t_k}\cdots\int_a^{t_2} P_{n-k}(t_1)dt_1dt_2\cdots dt_k$$

($t_{k+1}$ being here and below, $x$). Then $Q_n(x)$ is a real polynomial of degree $\leqq n$, and $Q_n^{(k)}(x) = P_{n-k}(x) \geqq 0$ throughout $[a, b]$. Furthermore, throughout that interval, we have

$$f(x) = \left[\sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(a)}{\nu!}(x-a)^\nu\right] + \int_a^{t_{k+1}}\int_a^{t_k}\cdots\int_a^{t_2} f^{(k)}(t_1)dt_1\cdots dt_k ,$$

and therefore

$$\begin{aligned} |f(x) - Q_n(x)| &\leqq \int_a^{t_{k+1}}\int_a^{t_k}\cdots\int_a^{t_2}|f^{(k)}(t_1) - P_{n-k}(t_1)|\, dt_1\cdots dt_k \\ &\leqq 2\lambda\left[\frac{\pi}{4}(b-a)\right]^{p-k+1}\left[\prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1}\frac{(x-a)^k}{k!} \\ &\leqq 2\lambda\left(\frac{\pi}{4}\right)^{p-k+1}(b-a)^{p+1}\left[k!\prod_{\nu=k}^{p}(n+1-\nu)\right]^{-1} . \end{aligned}$$

4. The following Theorem 5 deals with a somewhat more general situation than that of Theorem 1.

THEOREM 5. *Let $k$ and $p$ be integers, $1 \leqq k \leqq p$, and let a real function $f$ satisfy throughout $[a, b]$*

$$f^{(k)}(x) \geqq 0 ,$$
$$|f^{(p)}(x)| \leqq M ,$$

*$M$ being a constant. Let $\omega(x)$ be the modulus of continuity of $f^{(p)}$ in $[a, b]$. Then for every integer $n\,(\geqq p)$ there exists a real polynomial $Q_n(x)$ of degree $\leqq n$ such that*

(a) $$Q_n^{(k)}(x) \geqq 0 \quad \textit{throughout } [a, b] ,$$

(b) $$\max_{a\leqq x\leqq b}|f(x) - Q_n(x)| \leqq 2\left(1+\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right)^{p-k}(b-a)^p\left[k!\prod_{\nu=k}^{p-1}(n+1-\nu)\right]^{-1}\omega\left(\frac{b-a}{n-p+1}\right)$$

(an "empty" product means always 1).

Theorem 5 is proved by means of the following Theorem 6, in the same way that Theorem 1 was proved by means of Theorem 4.

THEOREM 6. *Let $f$ be a real function having a bounded* $p$th *($p \geqq 0$) derivative throughout $[a, b]$. Let $\omega(x)$ be as in Theorem 5. Then for every integer $n$ ($\geqq p$) there exists a polynomial $P_n(x)$ of degree $\leqq n$ such that throughout $[a, b]$*

$$|f(x) - P_n(x)| \leqq \left(1 + \frac{\pi}{4}\right)\left[\frac{\pi}{4}(b-a)\right]^p \left[\prod_{\nu=0}^{p-1}(n+1-\nu)\right]^{-1} \omega\left(\frac{b-a}{n-p+1}\right).$$

5. Theorem 6 follows from Theorem 3 by Jackson's method ([3], pp. 15–18). For the reader's convenience we hereby prove Theorem 6 in full. We do it by induction on $p$. Suppose first $p = 0$. Let $n$ be an integer ($\geqq 0$). Let $\phi(x)$ be the function whose graph is obtained by joining successively the points $(\xi_\nu, f(\xi_\nu))$ ($\nu = 0, 1, \cdots, n+1$) of the $x, y$ plane, where $\xi_\nu = a + [(b-a)/(n+1)]\nu$. For $\nu = 1, 2, \cdots, n+1$ we have $|\phi(\xi_\nu) - \phi(\xi_{\nu-1})| \leqq \omega[(b-a)/(n+1)]$. Hence, if $a \leqq x_1 < x_2 \leqq b$, then

$$\frac{|\phi(x_2) - \phi(x_1)|}{x_2 - x_1} \leqq \frac{n+1}{b-a}\omega\left(\frac{b-a}{n+1}\right).$$

By Theorem 3, there exists a polynomial $P_n(x)$ of degree $\leqq n$ such that throughout $[a, b]$

$$|\phi(x) - P_n(x)| \leqq \frac{n+1}{b-a}\omega\left(\frac{b-a}{n+1}\right)\frac{\pi}{4}\frac{b-a}{n+1} = \frac{\pi}{4}\omega\left(\frac{b-a}{n+1}\right).$$

Clearly, for every $x \in [a, b]$, $|f(x) - \phi(x)| \leqq \omega[(b-a)/(n+1)]$. Therefore, throughout $[a, b]$, $|f(x) - P_n(x)| \leqq [1 + (\pi/4)]\omega[(b-a)/(n+1)]$. This proves Theorem 6 when $p = 0$. Suppose the theorem was proved for some $p - 1$ ($\geqq 0$). We shall prove it for $p$. Let $n$ be an integer ($\geqq p$). By our hypothesis there exists a polynomial $P_{n-1}(x)$ of degree $\leqq n - 1$ such that throughout $[a, b]$

$$|f'(x) - P_{n-1}(x)| \leqq \left(1 + \frac{\pi}{4}\right)\left[\frac{\pi}{4}(b-a)\right]^{p-1}\left[\prod_{\nu=1}^{p-1}(n+1-\nu)\right]^{-1}\omega\left(\frac{b-a}{n-p+1}\right).$$

By the lemma, there exists a polynomial $P_n(x)$ of degree $\leqq n$, such that

$$\max_{a \leqq x \leqq b}|f(x) - P_n(x)| \leqq \left(1 + \frac{\pi}{4}\right)\left[\frac{\pi}{4}(\mathrm{b}-a)\right]^p\left[\prod_{\nu=0}^{p-1}(n+1-\nu)\right]^{-1} w\left(\frac{b-a}{n-p+1}\right).$$

This completes the proof.

## References

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