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A MORE GENERAL PROPERTY THAN DOMINATION FOR SETS OF PROBABILITY MEASURES

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In posing a statistical problem one specifies a set X, a σ-field S of subsets of X, and a collection M of probability measures on (X, S). It is often convenient to impose some condition on M in order to avoid measure theoretic difficulties and the condition most often used is domination, i.e., the existence of a probability measure with respect to which each of the measures in M is absolutely continuous. In this paper we introduce a more general condition, which we call compactness, implying the existence of a best sufficient subfield and of certain estimates. It is also possible to characterize, under this condition, those functions on M admitting unbiased estimates of certain types.

The increased generality thus afforded should be useful in dealing with certain problems in stochastic estimation where M is not known a priori to be dominated. In any case it is hoped that the present exposition, which leans heavily on some of the more elementary parts of functional analysis, will appeal to those who are oriented toward that subject.

1. The compactness condition. We will assume throughout this paper that the field S is closed with respect to M, that is that S contains every set whose outer measure is 0 for each μ in M. Such sets will be referred to hereafter as M-null sets.

For each μ in M, S-measurable f and real number p with $1 \leqq p < \infty$ we will write $||f||_{p,\mu}$ for the (finite or infinite) number $\left[\int |f|^p d\mu\right]^{1/p}$ and $||f||_{\infty,\mu}$ for the μ-essential supremum of $|f|$. For all $p \geqq 1$ we define

$$||f||_{p,M} = \sup_{\mu \in M} ||f||_{p,\mu}$$

and write $E_p(X, S, M)$ for the set of f with $||f||_{p,M} < \infty$. In what follows, whenever no confusion can result, we will write E_p for $E_p(X, S, M)$ and $||f||_p$ for $||f||_{p,M}$. We will also use the same symbol for a measurable function as for its equivalence class in $E_p(X, S, M)$.

LEMMA 1.1. *E_p with $||f||_p$ as norm is a Banach space.*

Proof. Only the completeness of E_p needs to be proved. If (f_n)

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is a Cauchy sequence in E_p, we can choose a subsequence (f_{n_j}) satisfying $\sum_{j=1}^{\infty}||f_{n_{j+1}} - f_{n_j}||_p < \infty$. Since

$$\sum_{j=1} ||f_{n_{j+1}} - f_{n_j}||_{p,\mu} \leqq \sum_{j=1} ||f_{n_{j+1}} - f_{n_j}||_p < \infty$$

the sequence (f_{n_j}) converges almost everywhere with respect to each μ in M. Writing f for the limit of the f_{n_j}'s we have

$$||f - f_{n_j}||_p = \sup_{\mu \in M} ||f - f_{n_j}||_{p,\mu} \leqq \sup_{\mu \in M} \sum_{k=j}^{\infty} ||f_{n_{k+1}} - f_{n_k}||_{p,\mu}$$
$$\leqq \sum_{k=j}^{\infty} ||f_{n_{k+1}} - f_{n_k}||_p$$

which goes to 0 as j goes to ∞.

The spaces E_p are new as far as we know but they are related to spaces considered by other authors. In particular if M is a dominated set of Borel measures on a locally compact Hausdorff space then E_1 is a Kothe space (see reference [1]). The subset $\mathscr{E}_p$ of the dual space, introducted below, is closely related to the Kothe dual. On the other hand if $1 < p < \infty$ and E_p is reflexive it is an MT space (see reference [3]).

Each μ in M and h in $L_q(\mu)$ give rise to an element $l(h, \mu)$ in $E_p(X, S, M)^*$ through the formula $l(h, \mu)(f) = \int fhd\mu$. Clearly $||\, l(h, \mu)\, || \leqq ||\, h\, ||_{q,\mu}$. We will write $\mathscr{E}_p(X, S, M)$ for the set of all finite linear combinations of such elements. $\mathscr{E}_p(X, S, M)$ is a total subset of $E_p(X, S, M)^*$, i.e., if $l(f) = 0$ for some f in $E_p(X, S, M)$ and every l in $\mathscr{E}_p(X, S, M)$ then $f = 0$. Hence $\mathscr{E}_p(X, S, M)$ induces a Hausdorff topology on $E_p(X, S, M)$, namely the weakest topology in which the elements of $\mathscr{E}_p(X, S, M)$ are continuous. We will write $B_p(X, S, M)$ for the unit ball in $E_p(X, S, M)$ and will generally shorten $B_p(X, S, M)$ and $\mathscr{E}_p(X, S, M)$ to B_p and $\mathscr{E}_p$ respectively.

DEFINITION. (X, S, M) is compact if and only if $B_p(X, S, M)$ is compact in the $\mathscr{E}_p(X, S, M)$ topology for some p, $1 < p < \infty$. It will be seen later (Theorem 1.1) that if $B_p(X, S, M)$ is $\mathscr{E}_p(X, S, M)$ compact for some $1 < p < \infty$ it is compact for all such p.

We note before going on that M can always be replaced by the set $C(M) = [\sum_{i=1}^{n} \alpha_i \mu_i \mid \alpha_i \geqq 0,\ \sum_{i=1}^{n} \alpha_i = 1,\ \mu_i \in M]$ since $||f||_{p,M} = ||f||_{p,C(M)}$ and $\mathscr{E}_p(X, S, M) = \mathscr{E}_p(X, S, C(M))$.

$W_p(\mu)$, the weakly topologized unit ball in $L_p(\mu)$ is compact if $1 < p < \infty$ and hence so is the product space $\prod_{\mu \in M} W_p(\mu)$ with the usual Tychonoff topology. The diagonal mapping i_p which sends each f in B_p into the element of the product space whose value at $W_p(\mu)$ is f maps B_p in a one-to-one way into the product space and the topology thus induced on B_p is easily seen to be identical to the $\mathscr{E}_p$

topology. Thus (X, S, M) is compact if and only if $i_p(B_p)$ is closed. Elements of the product space will be written (f_μ). We will write $f = g[\mu]$ if f is equal to g almost everywhere with respect to μ.

LEMMA 1.2. *The following are equivalent:*

1. *(f_μ) is the closure of $i_p(B_p)$;*
2. *for every finite set $\mu_1, \cdots, \mu_n$ from M there is an f in B_p satisfying $f = f_{\mu_i}[\mu_i]$ for $i = 1, \cdots, n$;*
3. *for every countable set (μ_i) from M there is an f in B_p satisfying $f = f_{\mu_i}[\mu_i]$ for all i.*

Proof. Clearly the third condition implies the second which in turn implies the first. We will complete the proof by showing that the first condition implies the third. Let $\nu = \sum_{n=1}^{\infty} 2^{-n}\mu_n$, let A_n be the set where $d\mu_1/d\nu$ and $d\mu_n/d\nu$ are positive, and let g be the characteristic function of a measurable subset of A_n on which $d\mu_1/d\mu_n$ is bounded. Since (f_μ) is in the closure of $i_p(B_p)$ there exists, for every positive ε, an h in B_p for which $\left|\int (f_{\mu_1} - h) g d\mu_1\right| < \varepsilon$ and

$$\left|\int (f_{\mu_n} - h) g \frac{d\mu_1}{d\mu_n} d\mu_n \right| = \left|\int (f_{\mu_n} - h) g d\mu_1\right| < \varepsilon$$

form which it follows that $\left|\int (f_{\mu_1} - f_{\mu_n}) g d\mu_1\right| < 2\varepsilon$. Since ε is arbitrary, $\int (f_{\mu_1} - f_{\mu_n}) g d\mu_1 = 0$ and hence $f_{\mu_1} = f_{\mu_n}[\mu_1]$ on A_n. Thus, if we define g_n to be the characteristic function of the set where $(d\mu_n/d\nu) > 0$ and $(d\mu_j/d\nu) = 0$ for $j < n$ and set $f = \sum_{n=1}^{\infty} g_n f_{\mu_n}$ we have $f = f_{\mu_n}[\mu_n]$ for all n.

THEOREM 1.1. *$B_p(X, S, M)$ is compact in the $\mathscr{E}_p(X, S, M)$ topology for some p, $1 < p < \infty$ if and only if $B_\infty(X, S, M)$ is compact in the $\mathscr{E}_1(X, S, M)$ topology. The $\mathscr{E}_p(X, S, M)$ topology coincides with the $\mathscr{E}_1(X, S, M)$ topology on $B_\infty(X, S, M)$ for all p with $1 \leqq p < \infty$.*

Proof. We will write $f^{(n)}$ for the function whose value at x is $f(x)$ if $|f(x)| \leqq n$ and 0 otherwise. The last assertion follows from the fact that any function $l(f, \mu)$ from $\mathscr{E}_p$ is the uniform limit on B_∞ of the $\mathscr{E}_1$-continuous functions $l(f^{(n)}, \mu)$. If $i_p(B_p)$ is compact, so is its closed subset $i_p(B_\infty)$. Hence B_∞ is $\mathscr{E}_p$-compact and consequently $\mathscr{E}_1$-compact if B_p is $\mathscr{E}_p$-compact. Conversely, if B_∞ is $\mathscr{E}_1$-compact and (f_μ) is in the closure of $i_p(B_p)$, then $((1/n)f_\mu^{(n)})$ is in the $\mathscr{E}_1$-closure of $i_p(B_\infty)$ so there exists a b_n in B_∞ with $nb_n = f_\mu^{(n)}[\mu]$ for all μ and it is easily seen that nb_n converges almost everywhere with respect to each μ to a function f which is therefore S-measurable and satisfies $f = f_\mu[\mu]$ for all μ.

Another characterization of compactness is contained in Theorem 3.4.

THEOREM 1.2. *If (X, S, M) is compact, so is (X, S, M') for any $M' \subset M$. If (X, S, M) is compact, so is $(X, S, \bar{M})$ where $\bar{M}$ is the set of probability measures ν which are dominated by some countable subset of M. (X, S, M) is compact if M is dominated.*

Proof. The identity map from $B_\infty(X, S, M)$ to $B_\infty(X, S, M')$ is continuous so its image is a compact subset of $B_\infty(X, S, M')$. Since any equivalence class in $B_\infty(X, S, M')$ contains an element f with $|f| \leqq 1$ everywhere the image of $B_\infty(X, S, M)$ is all of $B_\infty(X, S, M')$ so the first assertion is proved. Any ν in $\bar{M}$ has an expansion $d\nu = \sum_{i=1}^\infty f_i d\mu_i$ where the μ_i are in M and the f_i are nonnegative functions with $\sum_{i=1}^\infty \int f_i d\mu_i = 1$. If h is bounded, the function $l(h, \nu)$ on B_∞ is the uniform limit of the $\mathscr{C}_1$-continuous functions $\sum_{i=1}^n l(\inf(f_i, n)h, \mu_1)$. Hence no new continuous functions are added and the topology is the same—in particular compactness is preserved. The last assertion follows from the fact that $(X, S, (\mu))$ is compact and that M is a subset of some $(\bar{\mu})$ if it is dominated.

Two unsolved problems should be mentioned at this point. First, if (X, S, M_i) are compact, is $(X, S, M_1 \cup M_2)$ compact, or, what is probably equivalent, is $(M, S, \bigcup_{i=1}^\infty M_i)$ compact? Second, if (X, S, M) and (Y, T, N) are compact and $X \times Y$, $S \times T$, and $M \times N$ are the product space, the field generated by the S and T cylinder sets and the set of product measures, is $(X \times Y, S \times T, M \times N)$ compact? The second problem corresponds to the case of independent trials.

We close this section with a list of examples.

EXAMPLE 1. Let (α) be a parameter set and let $(X_\alpha, M_\alpha, S_\alpha)$ be compact with disjoint X_α. Let $X = \bigcup_\alpha X_\alpha$, $S = [A \mid A \cap X_\alpha \in S_\alpha]$ and extend M_α to S be defining $\mu(A) = \mu(A \cap X_\alpha)$ for μ in M_α. Then $(X, S, \bigcup_\alpha M_\alpha)$ is compact for if (f_μ) is in the closure of $i_p(B_p)$ there is, for each α, an f_α with $f_\alpha = f_\mu[\mu]$ for $\mu \in M_\alpha$ and the f obtained by setting f equal to f_α on X_α is S-measurable and $i_p(f) = (f_\mu)$. Note that $\bigcup_\alpha M_\alpha$ cannot be dominated if the parameter set is not countable so that the compactness condition is really more general than domination.

EXAMPLE 2. Let X be the closed interval $[0, 1]$, S the Borel sets and M all the measures which are either concentrated at a point or else are absolutely continuous with respect to Lebesgue measure, Every subset of $[0, 1]$ gives rise to an element in the closure of $i_p(B_p)$ on setting $f_\mu = 1$ if μ is concentated at a point x in A and 0 otherwise. It is easily seen that $(f_\mu) = i_p(b)$ implies that b is the characteristic function of A which is impossible if A is not in S so (X, S, M) is not compact. If only the point measures were involved we could replace S by the set T of all subsets of X in which case (X, T, M) would be compact, but Lebesgue measure, of course, cannot be extended to T.

EXAMPLE 3. Let ω be a probability measure on (X, S) and for some $C \geqq 1$ set $M = [\mu \mid \mu$ is absolutely continuous with respect to ω and $(d\mu/d\omega) \leqq C]$. Then $\int |f|^p \, d\omega \leqq \sup_{\mu \in M} \int |f|^p \, d\mu \leqq C \int |f|^p \, d\omega$ so $E_p(X, S, M)$ is equivalent to $L_p(\omega)$. Thus $E_p(X, S, M)$ is reflexive if $1 < p < \infty$. Reflexive E_p's are discussed in § 4.

EXAMPLE 4. Let ν be a probability measure on (X, S) and let $M = [\mu \mid \mu$ is absolutely continuous with respect to $\nu]$. It is easily seen that E_p is isometrically equivalent to $L_\infty(\nu)$ for all p, $1 \leqq p \leqq \infty$.

2. Sufficient subfields of S. We will need the following extension of S.

DEFINITION. $\hat{S} = [A \mid$ for every μ in M, A is equal almost everywhere to an element of $S]$.

It is clear that $S \subset \hat{S}$ and that every μ in M can be extended to $\hat{S}$. A function b is $\hat{S}$-measurable if and only if, for each μ, it is almost everywhere equal to an S-measurable function. $\hat{S}$ may properly contain S, in fact, if M is the set of all point measures on X and S is any field, then there are no M-null sets but $\hat{S}$ is the field of all subsets of X.

THEOREM 2.1. *If (X, S, M) is compact, then $S = \hat{S}$.*

Proof. As previously noted we can replace M by $C(M)$, the convex set spanned by M. Let b be an $\hat{S}$-measurable function of absolute bound 1. For each μ there is a b_μ in B_∞ equal μ-almost everywhere to b. (b_μ) is in the closure of $i_p(B_\infty)$ since for any $\mu_1, \cdots, \mu_n$ if $\nu = (1/n) \sum_{i=1}^n \mu_i$, then $b_\nu = b_{\mu_i}[\mu_i]$ for each i. Hence there is an S-measurable b_1 with $b_1 = b_\mu = b[\mu]$ and b and b_1 clearly differ only on an M-null set.

THEOREM 2.2. *If (X, S, M) is compact and T is a subfield of S, then $(X, \hat{T}, M)$ is compact.*

Proof. If (b_μ) is in the closure of $i_p(B_p(X, \hat{T}, M))$, then for every $\mu_1, \cdots, \mu_n$ there is a $\hat{T}$ measurable b' with $b' = b_{\mu_i}[\mu_1]$. Since $\hat{T} \subset \hat{S} = S$, b' can be replaced by an S measurable b'' so (b_μ) is in the closure of $i_p(B_p(X, S, M))$. Hence there is an S-measurable function b with $b = b_\mu[\mu]$ for all μ and b is clearly $\hat{T}$-measurable.

If T is a subfield of S, μ is a probability measure on S, and f is in $L_p(\mu)$, then the conditional expectation[1] of f on T with respect to μ written $E(f \mid T, \mu)$ is the unique T-measurable element of $L_p(\mu)$ satisfying $\int gE(f \mid T, \mu)d\mu = \int gfd\mu$ for every T-measurable element of $L_q(\mu)$. If $a \leqq f \leqq b$ then $a \leqq E(f \mid T, \mu) \leqq b$. If there exists a T-measurable function satisfying the above equation for all μ in M, we will write it $E(f \mid T, M)$. If $E(b \mid T, M)$ exists for each bounded S-measurable b, the subfield T is said to be sufficient.

THEOREM 2.3. *If T is a sufficient subfield for $(X, \hat{S}, M)$, then $T = \hat{T}$.*

Proof. Let b be a bounded $\hat{T}$-measurable function and $b' = E(b \mid T, M)$. $b - b'$ is $\hat{T}$-measurable and if c is any other bounded $\hat{T}$-measurable function, there is for each μ a T-measurable function c_μ with $c = c_\mu[\mu]$ so $\int (b - b')cd\mu = \int (b - b')c_\mu d\mu = 0$. Hence b differs from the T-measurable function b' only on an M-null set.

THEOREM 2.4. *If $\hat{T}$ is a sufficient subfield for $(X, \hat{S}, M)$ then $(X, \hat{S}, M)$ is compact if and only if $(X, \hat{T}, M)$ is compact and $[b \mid b \in B_\infty(X, \hat{S}, M)$ and $E(b \mid \hat{T}, M) = 0]$ is compact in the $\mathscr{E}_1(X, \hat{S}, M)$* topology.

Proof. Suppose first that $(X, \hat{S}, M)$ is compact. Then $(X, \hat{T}, M)$ is compact by Theorem 2.2. $B_2(X, \hat{S}, M)$ is $\mathscr{E}_1$ compact and hence so is its closed subset $B_\infty(X, \hat{S}, M)$. Thus it only remains to show that $K = [b \mid E(b \mid \hat{T}, M) = 0]$ is $\mathscr{E}_1$ closed. But if c is in the closure of K, μ is in M and f is a bounded $\hat{T}$-measurable function then there is a sequence (b_n) from K with

$$\int cfd\mu = \lim_{n\to\infty} \int b_n fd\mu = \lim_{n\to\infty} \int E(b_n \mid \hat{T}, M)fd\mu = 0$$

and it follows that $E(c \mid \hat{T}, M) = 0$, i.e., that K is closed.

Suppose conversely that $(X, \hat{T}, M)$ is compact. If (b_μ) is in the closure of $i_2(B_\infty(X, \hat{S}, M))$ then $(E(b_\mu \mid \hat{T}, \mu))$ is in the closure of $i_2(B_\infty(X, \hat{T}, M))$ since $b_{\mu_i} = b[\mu_i]$ for $i = 1, \cdots, n$ implies $E(b_{\mu_i} \mid \hat{T}, M) =$

[1] For definitions and properties of sufficient and pairwise sufficient subfields and conditional expectations, see [2].

$E(b \mid \hat{T}, M)[\mu_i]$ for $i = 1, \cdots, n$. Hence these is a $\hat{T}$-measurable c with $c = E(b_\mu \mid \hat{T}, M)[\mu]$ for all μ in M. $((1/2)_\mu - c)$ is thus in the closure of $\varphi^{-1}(0) \cap B_\infty(X, \hat{S}, M)$ so if this set is compact and hence closed there is a b in $B_\infty(X, \hat{S}, M)$ with $b = (1/2)(b_\mu - c)[\mu]$ for all μ. Thus $(b)_\mu = i_2(c + 2b)$ is in $B_\infty(X, \hat{S}, M)$.

THEOREM 2.5. *If (X, S, M) is compact, then there exists a best sufficient subfield of S, i.e., a sufficient subfield T such that $T \subset T_1$ for any other sufficient subfield T_1.*

Proof. Let T_0 be the subfield generated by all the functions $[d\mu/d(\mu + \nu)]$ for μ and ν in M. T_0 is pairwise sufficient,[2] i.e., for any μ and ν in M and b in $B_\infty(X, S, M)$ there is a T_0-measurable b' with $b' = E(b \mid T_0, \mu)[\mu]$ and $b' = E(b \mid T_0, \nu)[\nu]$. This property is easily extended to finite subsets of M[3] and it follows that $(E(b \mid T_0, \mu))$ is the closure of $i_p(B_\infty(X, S, M))$ so there is a b'' with $b'' = E(b \mid T_0, \mu)[\mu]$ for all μ. b'' is $\hat{T}_0$-measurable and $b'' = E(b \mid \hat{T}_0, M)$ so $\hat{T}_0 = T$ is a sufficient subfield. By Theorem 2.1 $S = \hat{S}$ and thus by Theorem 2.3 any sufficient subfield T_1 of S has $T_1 = \hat{T}_1$. It is known[4] that T_1 contains T_0 if it is sufficient so $T_1 = \hat{T}_1 \supset \hat{T}_0 = T$.

3. Estimation. If F is a real-valued function on M and f is an estimate of F, that is, an S-measurable function, then one measure of the error to be expected from f is $e_p(f) = \sup_{\mu \in M} \| f - F(\mu) \|_{p,\mu}$.

THEOREM 3.1. *If (X, S, M) is compact, F is a bounded function on M, and $1 < p \leqq \infty$, then there is an f in $E_p(X, S, M)$ which minimizes $e_p(f)$.*

Proof. Replacing F by aF we can assume that $\sup_{\mu \in M} | F(\mu) | \leqq (1/3)$. If $\alpha = \inf_{f \in E_p} e_p(f)$, then $\alpha \leqq e_p(0) = \sup_{\mu \in M} | F(\mu) | \leqq (1/3)$. Let (f_n) be a sequence from E_p with $e_p(f_n)$ converging to α. For large enough n, $\| f_n \|_{p,\mu} \leqq \| f_n - F(\mu) \|_{p,\mu} + | F(\mu) | \leqq e_p(f_n) + | F(\mu) | \leqq 1$ so f_n is in B_p. The sequence has a point of accumulation f in B_p and for any μ in M and h in $L_q(\mu)$,

$$\left| \int (f - F(\mu)) h d\mu \right| = \lim_{j \to \infty} \left| \int (f_{n_j} - F(\mu)) h d\mu \right|$$
$$\leqq \limsup_{j \to \infty} \| f_{n_j} - F(\mu) \|_{p,\mu} \| h \|_{q,\mu} \leqq \alpha \| h \|_{q,\mu}$$

so $\| f - F(\mu) \|_{p,\mu} \leqq \alpha$ and hence $e_p(f) = \alpha$.

[2] Ibid.
[3] Ibid.
[4] Ibid.

An estimate is said to be unbiased if $\int f d\mu = F(\mu)$ for all μ in M.

THEOREM 3.2. *If (X, S, M) is compact, F is a bounded function on M, and $1 < p \leqq \infty$, then if there is an unbiased estimate of F in E_p, there is one which minimizes $e_p(f)$ among all the unbiased estimates of F in $E_p(X, S, M)$.*

Proof. $C_p = \left[f \mid f \in B_p \text{ and } \int f d\mu = F(\mu) \text{ for all } \mu\right]$ is an $\mathscr{E}_p$-closed and hence compact, subset of B_p. The proof is essentially the same as the proof of Theorem 3.1 with B_p replaced by C_p.

We will say that an estimate f of F is *p-admissible* if f is in $E_p(X, S, M)$ and there is no g in $E_p(X, S, M)$ with $\| g - F(\mu) \|_{p,\mu} \leqq \| f - F(\mu) \|_{p,\mu}$ for all μ in M and $\| g - F(\nu) \|_{p,\nu} < \| f - F(\nu) \|_{p,\nu}$ for some ν in M. We will say that f is a *p-admissible unbiased estimate* of F if f is an unbiased estimate of F in $E_p(X, S, M)$ and there is no unbiased estimate g of F in $E_p(X, S, M)$ with $\| g - F(\mu) \|_{p,\mu} \leqq \| f - F(\mu) \|_{p,\mu}$ for all $\mu \in M$ and $\| g - F(\nu) \|_{p,\nu} < \| f - F(\nu) \|_{p,\nu}$ for some ν in M.

THEOREM 3.3. *Suppose (X, S, M) is compact, F is a bounded function on M and $1 < p < \infty$. Then for every estimate f of F in $E_p(X, S, M)$ there is a p-admissible estimate f_0 of F with $\| f_0 - F(\mu) \|_{p,\mu} \leqq \| f - F(\mu) \|_{p,\mu}$ for all μ in M and for every unbiased estimate g of F in $E_p(X, S, M)$ there is a p-admissible unbiased estimate g_0 of F with $\| g_0 - F(\mu) \|_{p,\mu} \leqq \| g - F(\mu) \|_{p,\mu}$ for all μ in M.*

Proof. We will write $g < h$ if $\| g - F(\mu) \|_{p,\mu} \leqq \| h - F(\mu) \|_{p,\mu}$ for all μ in M, and D_g for the set $[h \mid h < g]$. D_g is $\mathscr{E}_p$ closed and if h is in D_g then

$$\begin{aligned} \| h \|_{p,\mu} &\leqq \| h - F(\mu) \|_{p,\mu} + | F(\mu) | \\ &\leqq \| g - F(\mu) \|_{p,\mu} + | F(\mu) | \\ &\leqq \| g \|_p + 2 \sup_{\mu \in M} | F(\mu) | = K \end{aligned}$$

Hence all the D_g for $g < f$ are compact subsets of KB_p. Thus if D_{g_α} is a linearly ordered set of such sets, i.e., $\alpha_1 < \alpha_2$ implies $D_{g_{\alpha_1}} \subset D_{g_{\alpha_2}}$, their intersection is nonempty. Clearly $D_g \subset D_{g_\alpha}$ for any g in the intersection and any α. By Zorn's lemma then there is a minimal such D_g and any element of D_g satisfies the conditions for f_0. The proof for the unbiased case is similar.

Theorem 3.3 does not hold without the assumption of compactness. If in Example 2 we set $F(\mu) = 1$ for μ which are concentrated on a point x in some fixed nonmeasurable set A and $F(\mu) = 0$ for all other

μ in M it is clear that any estimator of F can be improved upon.

The compactness of (X, S, M) does not imply that $E_p(X, S, M)$ is reflexive (see Example 4) but the next theorem shows that $E_p^{**}(X, S, M)$ is the direct sum of the image of $E_p(X, S, M)$ under the natural map and the annihilator of $\mathscr{E}_p(X, S, M)$ if (X, S, M) is compact and $1 < p < \infty$.

THEOREM 3.4. *(X, S, M) is compact if and only if for each $1 < p < \infty$ and L in $E_p^{**}(X, S, M)$ with $\|L\| \leqq A$ there is an f in $E_p(X, S, M)$ with $\|f\|_{p,M} \leqq A$ and $L(l(h, \mu)) = \int hfd\mu$ for all μ in M and h in $L_q(\mu)$.*

Proof. Suppose the condition of the theorem is satisfied and (f_μ) is in the closure of $i_p(B_p)$. The functional L on $\mathscr{E}_p$ given by $L(l(h, \mu)) = \int hf_\mu d\mu$ is well defined for if $l(h, \mu) = l(g, \nu)$ and f is an element of B_p satisfying $f = f_\mu[\mu]$ and $f = f_\nu[\nu]$ then $L(l(h, \mu)) = l(h, \mu)(f) = l(g, \nu)(f) = L(l(g, \nu))$. L is also bounded on $\mathscr{E}_p(X, S, M)$ since, for some f in $B_p(X, S, M)$ with $f = f_\mu[\mu]$ $|L(l(h, \mu)| = |l(h, \mu)(f)| \leqq \|l\| \|f\| \leqq \|l\|$. By the Hahn-Banach theorem L has an extension $\bar{L}$ to $E_p^{**}(X, S, M)$ so there is an f in $E_p(X, S, M)$ with $\bar{L}(l(h, \mu)) = \int hfd\mu = \int hf_\mu d\mu$ for all μ in M and h in $L_q(\mu)$. Clearly $f = f_\mu[\mu]$ for all μ in M, i.e., $i_p(B_p)$ is closed and hence (X, S, M) is compact. Suppose conversely that (X, S, M) is compact and L is an element of $E_p^{**}(X, S, M)$. It will be sufficient to do the case $\|L\| \leqq 1$. For each μ we can define a linear functional L_μ on $L_q(\mu)$ by setting $L_\mu(h) = L(l(h, \mu))$. Since $|L_\mu(h)| \leqq \|l(h, \mu)\| \leqq \|h\|_{q,\mu}$ there is an f_μ in $L_p(\mu)$ with $\|f_\mu\|_{p,\mu} \leqq 1$ and $L_\mu(h) = \int hf_\mu d\mu$. The proof will be completed if we can show that (f_μ) is in the closure of $i_p(B_p)$ for then there will be an f with $f = f_\mu[\mu]$ and $L(l(h, \mu)) = L_\mu(h) = \int hfd\mu = l(h, \mu)(f)$. For any $\mu_1, \cdots, \mu_n$ let $\nu = (1/n) \sum_{i=1}^n \mu_i$. By the argument above there is an f_ν satisfying $\int f_\nu hd\nu = L(l(h, \nu))$ for all h in $L_q(d\nu)$. If h_j is in $L_q(\mu_j)$, then $h_j(d\mu_j/d\nu) \leqq nh_j$ is in $L_q(d\nu)$ so

$$\int f_\nu h_j d\mu_j = \int f_\nu h_j \frac{d\mu_j}{d\nu}\, d\nu = L\Big(l\Big(h_j \frac{d\mu_j}{d\nu}, \nu\Big)\Big)$$

$$= L(l(h_j, \mu_j)) = \int f_{\mu_j} h_j d\mu_j$$

and hence $f_\nu = f_{\mu_j}[\mu_j]$ for $j = 1, \cdots, n$.

THEOREM 3.5. *If (X, S, M) is compact and $1 < p < \infty$, then a bounded function F on M has an unbiased estimator in $E_p(X, S, M)$ of norm not greater than A if and only if*

$$\left|\sum_{i=1}^{n} c_i F(\mu_i)\right| \leqq A \sup_{\|f\|_p \leqq 1} \left|\sum_{i=1}^{n} c_i \int f d\mu_i\right|$$

for every finite set of real numbers (c_i) *and elements* (μ_i) *from* M.

Proof. The linear functional L_0 given by: $L_0(\sum_{i=1}^{n} c_i l(1, \mu)) = \sum_{i=1}^{n} c_i F(\mu_i)$ has bound not greater than A on its domain, hence, by the Hahn-Banach theorem, it has an extension L in E_p^{**} of norm not greater than A. By the preceding theorem there is an f in E_p with $\|f\|_p \leqq A$ and $\int f d\mu = L(l(1, \mu)) = L_0(l(1, \mu)) = F(\mu)$. The converse is trivial since if f is the assumed estimate,

$$\left|\sum_{=1}^{n} c_i F(\mu_i)\right| = \left|\sum_{i=1}^{n} c_i \int f d\mu_i\right| \leqq A \left\|\sum_{i=1}^{n} c_i l(1, \mu_i)\right\|$$

$$= A \sup_{\|f\|_p \leqq 1} \left|\sum_{i=1}^{n} c_i \int f d\mu_i\right| .$$

4. **Reflexivity of $E_p(X, S, M)$.** We have already given (Example 3 of § 1) an example in which $E_p(X, S, M)$ is reflexive for all $1 < p < \infty$. It is clear that the set M used there could be chosen considerably smaller while still retaining the property that $E_p(X, S, M)$ is equivalent to $L_p(\omega)$ for each $1 < p < \infty$. The following example shows that this is by no means the more general case of a reflexive $E_p(X, S, M)$.

EXAMPLE 5. Let μ be a nonatomic probability measure on (X, S) and y a point in X such that the set (y) is in S. Choose p and s with $1 \leqq p < s < \infty$. For each g in $L_s(\mu)$ let μ_g be the measure defined by

$$\int f d\mu_g = \frac{\int f |g|^{s-p} d\mu}{\left[\int |g|^s d\mu\right]^{1-p/s}} + c_g f(y)$$

where

$$c_g = 1 - \frac{\int |g|^{s-p} d\mu}{\left[\int |g|^s d\mu\right]^{1-p/s}} .$$

An application of Hölders inequality shows that $c_g \geqq 0$ so μ_g is a positive measure and since $\int d\mu_g = 1$ it is a probability measure. We will write μ_0 for the probability measure concentrated at y and set $M = [\mu_g \mid g \in L_s(\mu)]$. We have, using Hölders inequality for the pair $(s/p, (s/s - p))$,

$$\begin{aligned}\int |f|^p \, d\mu_g &\leqq \frac{\left[\int |f|^s d\mu\right]^{p/s}\left[\int |g|^s \, d\mu\right]^{1-(p/s)}}{\left[\int |g|^s \, d\mu\right]^{1-(p/s)}} + c_g |f(y)|^p \\ &\leqq \left\{\left[\int |f|^s \, d\mu\right]^{1/s}\right\}^p + |f(y)|^p \\ &\leqq \left\{\left[\int |f|^s \, d\mu\right]^{1/s} + |f(y)|\right\}^p \\ &\leqq \left\{2\left[\int |f|^s \, d(\mu + \mu_0)\right]^{1/s}\right\}^p\end{aligned}$$

so $||f||_{p,M} \leqq C\,||f||_{s,\mu+\mu_0}$. Setting $g = \inf(|f|, n)$ for f in $E_p(X, S, M)$ we have

$$||f||^p_{p,M} \geqq \int |f|^p \, d\mu_g \geqq \left[\int |g|^s \, d\mu\right]^{p/s}$$

so f is in $L_s(\mu)$. Finally,

$$\begin{aligned}||f||^p_{p,M} &\geqq \frac{1}{2}\int |f|^p \, d(\mu_f + \mu_0) \\ &\geqq \frac{1}{2}\left\{\left[\int |f|^s \, d\mu\right]^{1/s} + |f(y)|\right\}^p \\ &\geqq c\left\{\left[\int |f|^s \, d\mu\right]^{1/s} + |f(y)|\right\}^p \\ &\geqq c\left[\int |f|^s \, d(\mu + \mu_0)\right]^{p/s}\end{aligned}$$

so $c\,||f||_{s,\mu+\mu_0} \leqq ||f||_{p,M} \leqq C\,||f||_{s,\mu+\mu_0}$ and $E_p(X, S, M)$ is reflexive.

In this example M is unbounded, that is no σ-finite measure ω exists with $(d\nu/d\omega) \leqq 1$ for all ν in M. Choosing $p = 1$ also gives a case where $E_1(X, S, M)$ is reflexive.

LEMMA 4.1. *If $E_p(X, S, M)$ is reflexive then $\mathscr{E}_p(X, S, M)$ is norm dense in $E_p(X, S, M)^*$ and all l in $E_p(X, S, M)^*$ are countably additive, i.e., if (f_n) is a nonincreasing sequence of functions in $E_p(X, S, M)$ converging to 0 except on an M-null set then $l(f_n)$ converges to 0.*

Proof. For any l outside the closure of the convex set $\mathscr{E}_p$ there is an L in E_p^{**} with $L(l) = 1$ and $L(\mathscr{E}_p) = 0$ by the Hahn-Banach theorem. By reflexivity L is the image of some f in E_p, but $L(\mathscr{E}_p) = \mathscr{E}_p(f) = 0$ implies that $f = 0$ which is a contradiction. The countable additivity of elements of E_p^* now follows directly from the fact that they can be approximated in norm by elements of $\mathscr{E}_p$.

LEMMA 4.2. *If $E_p(X, S, M)$ is reflexive for some $1 < p < \infty$, f is in $E_1(X, S, M)$ and f_n is equal to f on the set where $|f(x)| \leqq n$ and vanishes elsewhere then $\sup_{\mu \in M} \int |f - f_n| d\mu \to 0$. If f is in $E_p(X, S, M)$ then $\|f - f_n\|_{p,M} \to 0$.*

Proof. We may suppose that $f \geqq 0$. If the first assertion is false for f then there is a $\theta > 0$ and a sequence (μ_n) from M with $\int (f - f_n) d\mu_n \geqq \theta$. The equation $l_n(h) = \int h(f - f_n)^{1/q} d\mu_n$ defines an element l_n in E_p^* with $\|l_n\|^q \leqq \int (f - f_n) d\mu_n \leqq 2\|f\|_{1,M}$. Since the unit ball in E_p^* is weakly compact the sequence l_n has a point of accumulation l. $f^{1/p}$ is in E_p and

$$l(f^{1/p}) = \lim_{j\to\infty} \int f^{1/p}(f - f_{n_j})^{1/q} d\mu_{n_j}$$
$$= \lim_{j\to\infty} \int (f - f_{n_j}) d\mu_{n_j} \geqq \theta > 0$$

while

$$l(f_n^{1/p}) = \lim_{j\to\infty} \int f_n^{1/p}(f - f_{m_j})^{1/q} d\mu_{m_j} = 0$$

which contradicts the countable additivity of l. If f is in E_p then, since $|f - f_n|^p \leqq ||f|^p - |f_n|^p|$,

$$\sup_{\mu \in M} \int |f - f_n|^p d\mu \leqq \sup_{\mu \in M} \int ||f|^p - |f_n|^p| d\mu$$

which goes to 0.

It is easy to construct examples, nonreflexive of course, for which the bounded functions are not dense in $E_p(X, S, M)$. If we take M to be $[\mu_n \mid n = 1, 2, \cdots]$ where μ_n is defined by: $\int f d\mu_n = \int_n^{n+1} f(x) dx$ and set $f(x) = n^{1/p}$ for $n \leqq x \leqq n + (1/n)$ and 0 elsewhere, $\|f - b\|_{p,M} = 1$ for any bounded b.

We can replace M by $C(M)$, the set of finite convex combinations of elements of M, as already noted. Let K_p be the weak closure in $E_p(X, S, M)^*$ of the set $[l(1, \mu) \mid \mu \in C(M)]$. K_p is weakly compact if $E_p(X, S, M)$ is reflexive.

LEMMA 4.3. *If $E_p(X, S, M)$ is reflexive every element l of K_p can be represented in the form $l(f) = \int f d\nu$ for some probability measure ν. Let $M_p' = [\nu \mid l(1, \nu) \in K_p]$. Then $E_p(X, S, M) = E_p(X, S, M_p')$, in fact $\|f\|_{p,M} = \|f\|_{p,M_p'}$ for all f in $E_p(X, S, M)$.*

Proof. Any l in K_p is positive and countably additive and has

$l(1) = 1$ so can be represented as a probability integral, i.e., $l(f) = \int f d\nu$. For any f in $E_p(X, S, M)$ if f_n is the function whose value is $f(x)$ or 0 depending on whether $|f(x)| \leqq n$ or not and (μ_j) is a sequence from $C(M)$ with $l(1, \mu_j)$ converging to l we have

$$\begin{aligned}\int |f|^p d\nu &= \lim_n \int |f_n|^p d\nu \\ &= \lim_n \lim_j \int |f_n|^p d\mu_j \leqq \lim_n ||f_n||_p^p .\end{aligned}$$

In the reflexive case this latter limit is $||f||_{p,M}^p$ which completes the proof.

THEOREM 4.1. *If $E_p(X, S, M)$ is reflexive M'_p is dominated.*

Proof. We define measures μ_n in M'_p, sets A_n in S, and numbers α_n inductively as follows: $\alpha_1 = 1$, $A_1 = X$, μ_1 is arbitrary, α_{n+1} is the supremum of the numbers $\mu(A)$ for μ in M'_p and A such that $\mu_1(A) = \mu_2(A) = \cdots = \mu_n(A) = 0$, and μ_{n+1} and A_{n+1} are chosen to satisfy

$$\mu_{n+1}(A_{n+1}) \geqq (\alpha_{n+1/2}) \quad \text{and} \quad \mu_1(A_{n+1}) = \cdots = \mu_n(A_{n+1}) = 0 .$$

(α_n) is a decreasing sequence and if $\lim \alpha_n = \alpha$ and $B_n = A_n - \bigcup_{k=n+1}^{\infty} A_k$ then the B_n are disjoint, $\mu_n(B_n) \geqq \alpha/2$, and $\mu_n(B_m) = 0$ if $m > n$. Let l be a point of accumulation of $l(1, \mu_n)$ in $E_p(X, S, M)^*$ and let f_n be the characteristic function of the set $\bigcup_{k=n}^{\infty} B_k$. Then (f_n) decreases to 0 but

$$l(f_n) = \lim_{j\to\infty} \int g_n d\mu_{m_j} \geqq \underline{\lim}_{j\to\infty} \int f_{m_j} d\mu_{m_j} \geqq \alpha/2$$

so that $\alpha \leqq 2 \lim_{n\to\infty} l(f_n) = 0$. Now if $\mu_i(A) = 0$ for all i and μ is in M'_p then $\mu(A) \leqq 2\alpha_i$ for each i so M'_p is dominated by $\sum_{n=1}^{\infty} 2^{-n}\mu_n$.

LEMMA 4.4. *For each g in $E_p(X, S, M)$ there is a μ in M'_p with $\int |g|^p d\mu = ||g||_{p,M}^p$ if $E_p(X, S, M)$ is reflexive.*

Proof. Let $l(1, \mu)$ be a point of accumulation of a sequence $l(1, \mu_n)$ with $\int |g|^p d\mu_n \to ||g||_{p,M}^p$. Setting $g_k(x)$ equal to $g(x)$ or 0 depending on whether $|g(x)| \leqq k$ or not we have

$$\begin{aligned}\int |g|^p d\mu &= \lim_k \int |g_k|^p d\mu = \lim_k \lim_j \int |g_k|^p d\mu_{n_j} \\ &\geqq \underline{\lim}_k \underline{\lim}_j \left(\int |g|^p d\mu_{n_j} - ||g - g_k||_{p,M}^p \right) \\ &= ||g||_{p,M}^p .\end{aligned}$$

THEOREM 4.2. *If $E_p(X, S, M)$ is reflexive and $1 < p < \infty$ then for every l in $E_p(X, S, M)^*$ there is a g in $E_p(X, S, M)$ and a μ in M'_p with*

$$\int |g|^p \, d\mu = \|g\|^p_{p,M} = 1 ,$$
$$l(g) = \|l\|$$

and

$$l = \|l\| \, l(|g|^{p-1} \operatorname{sign}(g), \mu) .$$

Proof. We will write g^{p-1} for $|g|^{p-1} \operatorname{sign}(g)$ throughout this proof and will assume that $\|l\| = 1$. Since the unit ball in $E_p(X, S, M)$ is compact it contains a g with $l(g) = \|l\|$ and clearly $\|g\|_{p,M} = 1$. By the preceding lemma the convex set of μ's in M' with $\int |g|^p \, d\mu = 1$ is nonempty. We wish to show that the set $C = \left[l(g^{p-1}, \mu) \,\middle|\, \int |g|^p \, d\mu = 1 \right]$ is weakly closed and hence compact and since C is convex it will be sufficient to show that it is strongly closed. If $l(1, \mu_n)$ converges to l' and μ is an accumulation point of the μ_n then for any bounded h

$$l(g^{p-1}, \mu)(h) = \lim_j l(g^{p-1}, \mu_{k_j})(h) = l'(h)$$

so $l(g^{p-1}, \mu) = l'$. A straightforward argument similar to the proof of Lemma 4.4 shows that $\int |g|^p \, d\mu = 1$ and completes the proof that C is closed.

If l is not in C then by reflexivity and the Hahn-Banach theorem there is an h_1 in $E_p(X, S, M)$ with $c(h_1) \leqq \alpha < \beta = l(h_1)$ for all c in C. Replacing h_1 by $h = h_1 - \alpha g$ and setting $\gamma = \beta - \alpha$ we have $c(h) \leqq 0 < \gamma = l(h)$. For every $\varepsilon \geqq 0$

$$|l(g + \varepsilon h)|^p = (1 + \varepsilon\gamma)^p \leqq \sup_{\nu \in M'} \int |g + \varepsilon h|^p \, d\nu$$

so there exists ν_ε in M' with

$$1 + p\varepsilon\gamma \leqq \int (|g|^p + p\varepsilon g^{p-1} h) d\nu_\varepsilon + o(\varepsilon)$$

or

$$p\gamma \leqq \frac{1}{\varepsilon} \left(\int |g|^p \, d\nu_\varepsilon - 1 \right) + p l(g^{p-1}, \nu_\varepsilon)(h) + o(1) .$$

It follows that $\int |g|^p \, d\nu_\varepsilon \to 1$ and then, by using bounded approximations to $|g|^p$ and applying Lemma 4.2, that $\int |g|^p \, d\mu = 1$ and hence $l(g^{p-1}, \mu)(h) \leqq 0$ whenever $l(1, \mu)$ is a point of accumulation of the

$l(1, \nu_\varepsilon)$. But, setting $h_n(x)$ equal to $h(x)$ or 0 depending on whether $|h(x)| \leqq n$ or not, we have

$$\begin{aligned} l(g^{p-1}, \mu)(h) &= \lim_n l(g^{p-1}, \mu(h_n) \\ &= \lim_n \lim_j l(g^{p-1}, \nu_{\varepsilon_j})(h_n) \\ &\geqq \underline{\lim}_n (\gamma - \| h - h_n \|_{p,M}) = \gamma > 0 \end{aligned}$$

which is a contradiction.

THEOREM 4.3. *If* $1 < p < \infty$, $E_p(X, S, M)$ *is reflexive and* L *is a linear subset dense in each* $L_p(\mu)$ *for* μ *in* M'_p *then* L *is dense in* $E_p(X, S, M)$.

Proof. If L is not dense there is an element l in E_p^* with $l(L) = 0$ but $l \neq 0$. But $l = l(h, \mu)$ for some μ in M'_p and h in $L_q(\mu)$ and h must be 0 since $\int hfd\mu$ vanishes for f in a dense subset of $L_p(\mu)$.

The above theorem does not hold if we only require L to be dense in $L_p(\mu)$ for μ in M. For example let $X = [0, 2]$, S be the Borel sets and $M = [\mu_a \mid 0 < a \leqq 1]$ where $\int fd\mu_a = \int_a^{a+1} f(x)dx$. Then $E_p(X, \hat{S}, M)$ is equivalent to $L_p(dx)$ for all p and μ_0 is in M'_p for every $1 < p < \infty$. The set $L = \left[f \mid f \in E_p \text{ and } \int_0^1 f(x)dx = 0\right]$ is dense in each $L_p(\mu_a)$ since it contains, for each g in $L_p(\mu_a)$ the function $\bar{g}$,

$$\bar{g}(x) = \begin{cases} -\dfrac{1}{a}\displaystyle\int_a^{a+1} g(x)dx & \text{if } 0 \leqq x \leqq a \\ g(x) & \text{if } a < x \leqq a+1 \\ 0 & \text{if } a+1 < x \leqq 2\,. \end{cases}$$

L is not dense in $E_p(X, S, M)$ for any p since $l(1, \mu_0)$ is in every $E_p(X, S, M)^*$ and $l(1, \mu_0)(L) = 0$.

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