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THE NILPOTENT PART OF A SPECTRAL OPERATOR, II

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Let T be a spectral operator on a Banach space, such that its resolvent satisfies a mth order rate of growth condition. If N be the nilpotent part of T, it is known that $N^m = 0$ on Hilbert space. We show that $N^m = 0$ on an L_p space $(1 < p < \infty)$. Known examples show that N^m need not be zero even on an uniformly convex space.

We will consider a bounded spectral operator $T = \int \lambda E(d\lambda) + N$ which operates on an L_p space $(1 < p < \infty)$. $E(\circ)$ is the resolution of the identity and N is the nilpotent part of T [1; pp. 333–334]. We will denote by M a finite constant for which $M^{-1} \operatorname{ess}_\xi \cdot \inf \cdot | a(\xi) | \leqq \left| \int a(\xi) E(d\xi) \right| \leqq M \operatorname{ess}_\xi \cdot \sup \cdot | a(\xi) |$ is true for all bounded Borel functions $a(\xi)$, [1; Theorem 7, p. 330].

Suppose that T satisfies an mth order rate of growth condition on its resolvent: given any Borel subset σ of the spectrum of T, its restriction T_σ to the range of $E(\sigma)$ has $\bar{\sigma}$ as spectrum and we assume that for $| \zeta | \leqq | T | + 1$,

$$| (\zeta - T_\sigma)^{-1} | \leqq K[\text{distance } (\zeta, \sigma)]^{-m}$$

where K and m are constants independent of σ and ζ.

It is known that in Hilbert space, this implies $N^m = 0$ [1; Theorem 11, p. 337], and that in a reflexive Banach space $N^{m+1} = 0$, but in general no more [2; Theorem 3.1, p. 1226; Examples 4.4, p. 1230]. However, in the case of a reflexive L_p space, we will show that in fact $N^m = 0$. It is immaterial whether we show $N^m = 0$ or $N^{*m} = 0$, so that we may assume that $p \geqq 2$. We will dispense with the continual remarks that our L_p functions $x(s)$ are defined for only almost every s.

It is known that for any complex numbers $\lambda_1, \cdots, \lambda_n$ and $p \geqq 2$ we have

$$(1)\quad \begin{aligned} \left(\sum_{\nu=1}^{n} | \lambda_\nu |^2 \right)^{p/2} &\leqq (2\pi)^{-n} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \, | e^{i\theta_1}\lambda_1 + \cdots + e^{i\theta_n}\lambda_n |^p \\ &\leqq C(p) \left(\sum_{\nu=0}^{n} | \lambda_\nu |^2 \right)^{p/2} \end{aligned}$$

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where $C(p)$ is independent of n and the choice of the λ's [3; Proposition 1].

Given $\varepsilon > 0$, let the spectrum of T be decomposed into $n = O(\varepsilon^{-2})$ Borel subsets $\sigma_1, \cdots, \sigma_n$ with each σ_ν contained in the disc $|\zeta - \zeta_\nu| \leqq \varepsilon$, and let $E_\nu = E(\sigma_\nu)$. For a given function $x(s)$ in L_p, let $\lambda_\nu(s) = (E_\nu x)(s)$. For each s, apply (1) to these $\lambda_\nu(s)$ and then integrate over all s:

$$\begin{aligned} &\int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x(s)]|^2\Big)^{p/2} \\ (2)\qquad &\leqq (2\pi)^{-n}\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \int ds\, |[(e^{i\theta_1}E_1 + \cdots + e^{i\theta_n}E_n)x](s)|^p \\ &\leqq C(p)\int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2}. \end{aligned}$$

For each choice of θ_ν we have (since $\Sigma E_\nu = I$)

$$M^{-1}|x| \leqq |(e^{i\theta_1}E_1 + \cdots + e^{i\theta_n}E_n)x| \leqq M|x|,$$

so that upon performing the integrations in the middle of (2) we have

$$(3a)\qquad \int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2} \leqq M^p |x|^p$$

and

$$(3b)\qquad M^{-p}|x|^p \leqq C(p)\int ds\Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2}.$$

Now in (3b), replace x by $N^m x$ and apply the Holder inequality to the sum on the right hand side to obtain

$$\begin{aligned} (4)\qquad |N^m x|^p &\leqq C(p)M^p \int ds \sum_{\nu=1}^{n} |[E_\nu N^m x](s)|^p \cdot n^{(p/2)-1} \\ &= C(p)M^p n^{(p/2)-1} \sum_{\nu=1}^{n} |N^m E_\nu x|^p. \end{aligned}$$

It is a standard computation that

$$|N^m E_\nu x| \leqq 2\cdot 3^m KM\varepsilon |E_\nu x|.$$

For completeness, we digress for a moment to include a proof: Let Γ $(=\Gamma_\nu)$ be the contour $|\zeta - \zeta_\nu| = 2\varepsilon$, so that any point of Γ is at least ε away from σ_ν, but no point of σ_ν is further than 3ε from any point in Γ. Then we have

$$N^m E_\nu = \frac{1}{2\pi i}\int_\Gamma d\zeta(\zeta - T_{\sigma_\nu})^{-1}\int_{\sigma_\nu}(\zeta - \xi)^m E(d\xi)$$

and thus

$$|N^m E_\nu| \leqq \frac{1}{2\pi}\int_\Gamma |d\zeta|\, K\varepsilon^{-m} M(3\varepsilon)^m$$
$$= 2\cdot 3^m KM\varepsilon\ .$$

We now insert this estimate in (4) to obtain (with lumping all inessential constants together)

$$|N^m x|^p \leqq C(p) M^p n^{p/2-1} \sum_{\nu=1}^{n} (3^m KM\varepsilon)^p |E_\nu x|^p$$
$$= C n^{p/2-1}\varepsilon^p \int ds \sum_{\nu=1}^{n} |[E_\nu x](s)|^p \qquad \text{(since } p \geqq 2)$$
$$\leqq C n^{p/2-1}\varepsilon^p \int ds \Big(\sum_{\nu=1}^{n} |[E_\nu x](s)|^2\Big)^{p/2} \qquad \text{(by 3a)}$$
$$\leqq C n^{p/2-1}\varepsilon^p \cdot M^p |x|^p\ .$$

Now we need only remember that $n = O(\varepsilon^{-2})$ to see that

$$|N^m x|^p = O(\varepsilon^2)\,|x|^p\ .$$

Since ε may be arbitrarily small, $N^m x = 0$ for all x, so $N^m = 0$ as was to be proved.

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