PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

THE GENERALIZED GIBBS PHENOMENON FOR REGULAR HAUSDORFF MEANS

JONAH MANN AND DONALD J. NEWMAN

One says that the means $\sigma_n(x)$, of the Fourier series of a function $f(x)$, exhibit the (generalized) Gibbs phenomenon at the point $x = x_0$ if the interval between the upper and lower limit of $\sigma_n(x)$, as $n \to \infty$ and $x \to x_0$ independently, contains points outside the interval between the upper and lower limits of $f(x)$ as $x \to x_0$. *Theorem.* **In order that the Hausdorff summability method given by $g(t)$ not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that $1 - g(t)$ be positive definite. A new inequality which must be satisfied by $g(t)$, whenever $1 - g(t)$ is positive definite, is Re $z\int_0^1 (1 - zt)^n dg(t) \geqq 0$ where $z = 1 - e^{ix}$.**

This generalized definition of Gibbs phenomenon is an extension of the classical one, and is due to Kuttner [4].

Whereas originally the phenomenon was investigated for functions having a simple discontinuity at the point considered, he includes any Lebesgue integrable function. Kuttner proved the following:

THEOREM. *In order that a given K-method* [3, P. 56] *not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that the kernel $K_n(x)$ be bounded below.*

Here $K_n(x)$ are the means of the series $1/2 + \cos x + \cos 2x + \cdots$. For regular Hausdorff means [10] (which, being triangular, are K-methods) the kernel takes the form

$$K_n(x) = Im \frac{e^{ix/2}}{2 \sin x/2} \int_0^1 (1 - t + te^{ix})^n dg(t)$$

where $g(t)$ is of bounded variation in $0 \leq t \leqq 1$, $g(0+) = g(0) = 0$, and $g(1) = 1$. We find it useful to let $g(t)$ be normalized in $0 \leq t \leq 1$, and to define it outside this interval by $g(t) = 1$ for $t > 1$ and $g(-t) = g(t)$.

THEOREM. *In order that the Hausdorff summability method given by $g(t)$ not display the Gibbs phenomenon for any Lebesgue integrable function, it is necessary and sufficient that $1 - g(t)$ be positive definite.*

Received February 24, 1964.

(For the status of the corresponding problem for the classical Gibbs phenomenon, see [8], [7], and [6].)

Proof of necessity. We shall show that $K_n(x)$ is not bounded below if $1 - g(t)$ is not positive definite. Since $|1 - t + te^{ix}| \leqq 1$, Im $i\int_0^1 (1 - t + te^{ix})^n dg(t)$ is bounded, and it suffices to consider

$$h_n(x) = \operatorname{Im} \cot x/2 \int_0^1 (1 - t + te^{ix})^n dg(t) .$$

Let $1 - t + te^{ix} = Re^{i\alpha}$. Therefore $R \cos \alpha = 1 - t + t \cos x$, $R \sin \alpha = t \sin x$, $R^2 = 1 - 2t(1 - t)(1 - \cos x)$, and

$$\tan (x/2) h_n(x) = \operatorname{Im} \int_0^1 R^n e^{in\alpha} dg(t) = \int_0^1 R^n \sin n\alpha \, dg(t) .$$

We now choose a sequence of n and x so that $nx \to A < \infty$, A to be specified later, as $n \to \infty$ and $x \to 0$. Szász [8] shows that $1 - R^n = \lambda n(1 - R^2)$ where $0 < \lambda < 1$, and

$$\sin n\alpha - \sin ntx = 2 \cos n(\alpha + tx)/2 \cdot \sin 0(ntx^3) .$$

Since $1 - R^2 < x^2$ and $nx \to A$, it follows that

$$\begin{aligned} \tan (x/2) \cdot h_n(x) &= \int_0^1 \sin ntx \, dg(t) + 0(x) \\ &= nx \int_0^1 \cos ntx \cdot (1 - g(t)) dt + 0(x) . \end{aligned}$$

The last equality is obtained by integrating by parts.

According to the way the definition of $g(t)$ was extended,

$$\tan (x/2) \cdot h_n(x) = (nx/2) \int_{-\infty}^{\infty} e^{inxt} (1 - g(t)) dt + 0(x) .$$

Since $1 - g(t)$ belongs to $L^1(-\infty, \infty)$ and is of bounded variation in $(-\infty, \infty)$, it follows from Bochner's theorem [2] that its Fourier transform is not always nonnegative. Consequently, there is an $A_0 > 0$ for which

$$\int_{-\infty}^{\infty} e^{iA_0 t} (1 - g(t)) dt = -B < 0 .$$

Then let $A = A_0$ and obtain $\tan (x/2) \cdot h_n(x) \to -A_0 B/2$. (Taking the limit under the integral sign is permitted by "bounded convergence".) This implies that $h_n(x) \to -\infty$, and completes this part of the proof.

Proof of sufficiency. We shall show now that $K_n(x)$ is not only bounded below when $1 - g(t)$ is positive definite but is, in fact, positive for all n and x.

$$K_n(x) = \operatorname{Im} \frac{e^{ix/2}}{2 \sin x/2} \int_0^1 [1 - (1 - e^{ix})t]^n dg(t) .$$

Let $z = 1 - e^{ix}$. Then

$$K_n(x) = \operatorname{Im} \frac{iz}{4 \sin^2 x/2} \int_0^1 (1 - zt)^n dg(t)$$

$$= Re \frac{z}{4 \sin^2 x/2} \int_0^1 (1 - zt)^n dg(t) .$$

Let $f_n(t) = (1 - zt)^{n+1}$ in $0 \leq t \leq 1$ and $(1 + \bar{z}t)^{n+1}$ in $-1 \leqq t < 0$. Therefore

$$K_n(x) = \frac{-1}{8(n+1) \sin^2 x/2} \int_{-1}^1 f'_n(t) dg(t) .$$

It suffices to show that

$$\int_{-1}^1 f'_n(t) dg(t) \leq 0 .$$

Let $G(t) = f_0(t)e^{-ixt}$. Since $G(-t) = G(1 - t)$ for $0 \leqq t \leqq 1$, $f_0(t)$ may be defined for t, $|t| > 1$, so that $G(t)$ will be periodic of period 1.

Now

$$\int_{-1}^1 G(t) e^{-2\pi ikt} dt = 2 \int_0^1 [1 - (1 - e^{ix})t] e^{-ixt} e^{-2\pi ikt} dt$$

$$= \frac{4(1 - \cos x)}{(x + 2\pi k)^2} .$$

Consequently

$$G(t) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi ikt} \text{ where each } C_k \geqq 0 .$$

$G(t)$, therefore, is positive definite, and since the product of two positive definite functions is positive definite, it follows that $f_0(t)$ is positive definite. Also, each $f_n(t)$ is positive definite if it is defined for t, $|t| > 1$, by

$$f_n(t) = e^{i(n+1)xt}[G(t)]^{n+1} .$$

Therefore

$$f_n(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos \lambda_k t + ib_k \sin \lambda_k t)$$

where $a_k \geqq 0$ and $\lambda_k > 0$, $k = 1, 2, \cdots$, and

$$f'_n(t) = \sum_{k=1}^{\infty} (-a_k \lambda_k \sin \lambda_k t + ib_k \lambda_k \cos \lambda_k t) ,$$

so that

$$\int_{-1}^{1} f'_n(t)dg(t) = \sum_{k=1}^{\infty}\int_{-1}^{1}(-a_k\lambda_k \sin \lambda_k t + ib_k\lambda_k \cos \lambda_k t)dg(t) .$$

Since $f'_n(t)$ is of bounded variation, its Fourier series is boundedly convergent [9, P. 408] and the order of summation and integration may be interchanged [1, P. 74].

$$\int_{-1}^{1} \cos At\, dg(t) = 0 \text{ since } g(t) \text{ is even, and}$$

$$\int_{-1}^{1} \sin At\, dg(t) = A\int_{-1}^{1} \cos At(1 - g(t))dt$$

$$= A\int_{-\infty}^{\infty} e^{iAt}(1 - g(t))dt$$

which is positive for positive A [2, P. 26] since $1 - g(t)$ is positive definite and belongs to $L^1(-\infty, \infty)$. Finally

$$\int_{-1}^{1} f'_n(t)dg(t) \leq 0$$

and the theorem is proved.

This result, about positive kernels, may be compared with Kuttner's result in [5].

It is worth noting that we have proved

$$Re\, z\int_{0}^{1}(1 - zt)^n dg(t) \geqq 0 ,$$

where $z = 1 - e^{ix}$, whenever $1 - g(t)$ is positive definite. This provides some new inequalities which must be satisfied by a class of positive definite functions which is encountered quite often. For example, when $n = 1$ and $x = \pi$, we obtain

$$\int_{0}^{1}(1 - 2t)dg(t) \geq 0 .$$

References

1. J. C. Burkill, *The Lebesgue integral*, Camb. Tracts in Math. and Math. Phys. no. 40, Cambridge, 1958.

2. R. R. Goldberg, *Fourier transforms*, Cambridge, 1961.

3. G. H. Hardy and W. W. Rogosinski, *Fourier series*, 2nd ed., Cambridge Tracts no. 38, Cambridge, 1962.

4. B. Kuttner, *Note on the Gibbs phenomenon*, J. London Math. Soc. **20** (1945), 136-139.

5. ———, *A further note on the Gibbs phenomenon*, J. London Math. Soc. **22** (1947), 295-298.

6. J. Mann, *Hausdorff means and the Gibbs phenomenon*, Doctoral Dissertation, Yeshiva University, 1964.

7. D. J. Newman, *The Gibbs phenomenon for Hausdorff means*, Pacific J. Math. **12**

(1962), 367-370.
8. O. Szász, *Gibbs phenomenon for Hausdorff means*, Trans. Amer. Math. Soc. **69** (1950), 440-456.
9. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford, 1939.
10. D. V. Widder, *The Laplace transform*, Princeton, 1941.

CITY COLLEGE AND YESHIVA UNIVERSITY