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AN INEQUALITY FOR THE NUMBER OF ELEMENTS IN A SUM OF TWO SETS OF LATTICE POINTS

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For a fixed positive integer n, let Q be the set of all n-dimensional lattice points $(x_1, \cdots, x_n)$ with each x_i a nonnegative integer and at least one x_i positive. A finite nonempty subset R of Q is called a *fundamental set* if for every $(r_1, \cdots, r_n)$ in R, all vectors $(x_1, \cdots, x_n)$ of Q with $x_i \leqq r_i$, $i=1, \cdots, n$, are also in R. If A is any subset of Q and R is any fundamental set, let $A(R)$ denote the number of vectors in $A \cap R$. Finally, if A is any proper subset of Q, let the *density* of A be the quantity

$$\alpha = \text{glb}\, \frac{A(R)}{Q(R) + 1}\,,$$

taken over all fundamental sets R for which $A(R) < Q(R)$. Then the theorem proved in this paper can be stated as follows.

THEOREM. Let A and B be subsets of Q, let C be the set of all vectors of the form a, b, or $a + b$ where $a \in A$ and $b \in B$, let α be the density of A, and let R be any fundamental set such that (1) there exists at least one vector in R which is not in C, and (2) for each b in $B \cap R$ (if any) there exists g in R but not in C such that $g - b$ is in Q. Then

$$C(R) \geqq \alpha[Q(R) + 1] + B(R)\,.$$

It will be seen that for $n = 1$ this theorem implies a result of H. B. Mann [2].

Let A and B be sets of positive integers, and for any positive integer x denote by $A(x)$ the number of integers in A which are not greater than x. Let the *modified density* (or *Erdös density*) of A be the quantity

$$\alpha = \underset{x \geqq k}{\text{glb}}\, \frac{A(x)}{x + 1}$$

where k is the smallest positive integer not in A. If $C = A + B$ is the set of all integers of the form a, b, or $a + b$, where a is in A and b is in B, and if x is a positive integer not in C, then Mann has shown [2] that

$$C(x) \geq \alpha x + B(x)\,.$$

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(Actually, Mann's work is sufficient to establish $C(x) \geqq \alpha(x+1) + B(x)$.) We will show that this theorem, with somewhat weaker hypotheses, can be extended to certain sets of n-dimensional lattice points.

Let Q be the set of all lattice points $\boldsymbol{x} = (x_1, \cdots, x_n)$ for which each component is a nonnegative integer and at least one component is positive. Define the sum of subsets of Q in the same manner as was done for sets of positive integers, addition of lattice points being done componentwise, and for any subsets A and B of Q let $A - B$ denote the set of all elements of A which are not in B. If A and S are subsets of Q and S is finite let $A(S)$ be the number of elements in $A \cap S$. Let $\boldsymbol{\omega}_i$ be that element of Q for which the ith component is 1 and the others are 0.

DEFINITION 1. A finite nonempty subset R of Q will be called a *fundamental set* if whenever $\boldsymbol{r} = (r_1, \cdots, r_n)$ is in R then all vectors $\boldsymbol{x} = (x_1, \cdots, x_n)$ of Q such that $x_i \leqq r_i$, $i = 1, \cdots, n$, are also in R.

DEFINITION 2. Let A be any proper subset of Q. Then the *density* of A is the quantity

$$\alpha = \operatorname{glb} \frac{A(R)}{Q(R)+1},$$

taken over all fundamental sets R for which $A(R) < Q(R)$.

2. Extension of Mann's result.

The theorem to be proved can now be stated as follows.

THEOREM. *Let A and B be subsets of Q, let $C = A + B$, and let α be the density of A. Let R be any fundamental set such that for each $\boldsymbol{b}$ in $B \cap R$ there exists $\boldsymbol{g}$ in $R - C$ such that $\boldsymbol{g} - \boldsymbol{b}$ is in Q, and $Q(R - C) \geqq 1$. Then*

$$C(R) \geqq \alpha[Q(R)+1] + B(R).$$

Proof. Let the elements of Q be ordered so that $(x_1, \cdots, x_n) > (y_1, \cdots, y_n)$ if $x_1 > y_1$, or if $x_1 = y_1, \cdots, x_k = y_k, x_{k+1} > y_{k+1}$. Consider a nonempty set $S = R' - R''$, where R' and R'' are fundamental sets, and let $\boldsymbol{\delta}_1 = (\delta_{11}, \cdots, \delta_{1n}), \cdots, \boldsymbol{\delta}_u = (\delta_{u1}, \cdots, \delta_{un})$ be all the vectors of S such that for each $i = 1, \cdots, n$ and for each $j = 1, \cdots, u$ we have either (1) $\boldsymbol{\delta}_j - \boldsymbol{\omega}_i$ is in R'', or (2) $\boldsymbol{\delta}_j - \boldsymbol{\omega}_i = \boldsymbol{0} = (0, \cdots, 0)$, or (3) $\delta_{ji} = 0$. There must be at least one such vector in S, for S is a nonempty finite set, and hence has a least element (in our ordering). This least element will satisfy the given conditions. Also, it is easily seen that if $(s_1, \cdots, s_n)$ is any vector in S then for at least one of the $\boldsymbol{\delta}_j$ we have $\delta_{ji} \leqq s_i$, $i = 1, \cdots, n$.

From this it follows that if for each $j = 1, \cdots, u$ we let

$$S_j = \{s = (s_1, \cdots, s_n) \mid s \in S, s_i \geqq \delta_{ji}, i = 1, \cdots, n\},$$

then $S = S_1 \cup \cdots \cup S_u$. Also, let $S'_j = \{s - \boldsymbol{\delta}_j \mid s \in S_j, s \neq \boldsymbol{\delta}_j\}$ and let $S' = S'_1 \cup \cdots \cup S'_u$. Each S'_j, and therefore also S', is either a fundamental set or is empty.

LEMMA 1. $Q(S') + 1 \leqq Q(S)$.

Proof of Lemma 1. The lemma is obvious if $n = 1$, since then $u = 1$ also. Hence assume $n \geqq 2$. Let λ_1 be a mapping defined so that

$$S_j\lambda_1 = \{s - \delta_{j1}\omega_1 \mid s \in S_j\},\ j = 1, \cdots, u,$$
$$S\lambda_1 = S_1\lambda_1 \cup \cdots \cup S_u\lambda_1 .$$

Partition S into sets $T_{c_2 \cdots c_n}$ such that

$$T_{c_2 \cdots c_n} = \{s = (x_1, c_2, \cdots, c_n) \mid s \in S\},$$

and let

$$T_{c_2 \cdots c_n}\lambda_1 = \{s = (x_1, c_2, \cdots, c_n) \mid s \in S\lambda_1\},$$
$$k_{c_2 \cdots c_n} = \max_{1 \leqq j \leqq u} (\max \{x_1 - \delta_{j1} \mid (x_1, c_2, \cdots, c_n) \in S_j\}).$$

Then $Q(T_{c_2 \cdots c_n}\lambda_1) = k_{c_2 \cdots c_n} + 1$ or $Q(T_{c_2 \cdots c_n}\lambda_1) + 1 = k_{c_2 \cdots c_n} + 1$, according as $\mathbf{0} \notin T_{c_2 \cdots c_n}\lambda_1$ or $\mathbf{0} \in T_{c_2 \cdots c_n}\lambda_1$, and $k_{c_2 \cdots c_n} + 1 \leqq Q(T_{c_2 \cdots c_n})$.

Hence $Q(S\lambda_1) \leqq Q(S)$, and $Q(S\lambda_1) + 1 \leqq Q(S)$ if $\mathbf{0} \in S\lambda_1$.

Now define mappings $\lambda_2, \cdots, \lambda_n$ such that

$$S_j\lambda_1 \cdots \lambda_{i-1}\lambda_i = \{s - \delta_{j/i}\omega_i \mid s \in S_j\lambda_1 \cdots \lambda_{i-1}\},$$

$i = 2, \cdots, n$, and obtain as above

$$Q(S\lambda_1 \cdots \lambda_i) + \theta_i \leqq Q(S\lambda_1 \cdots \lambda_{i-1}) + \theta_{i-1} \leqq Q(S),$$

where $\theta_1 = 0$ or 1 according as $\mathbf{0} \notin S\lambda_1 \cdots \lambda_i$ or $\mathbf{0} \in S\lambda_1 \cdots \lambda_i$. This establishes the lemma.

DEFINITION 3. A set S will be said to be of type I if

(1) S is a fundamental set,

(2) $Q(S - C) \geqq 1$, and

(3) for all $\boldsymbol{b}$ in $B \cap S$ (if any) and all $\boldsymbol{g}$ in $S - C$ we have $\boldsymbol{g} - \boldsymbol{b}$ contained in Q.

DEFINITION 4. A set S will be said to be of type II if

(1) there exist fundamental sets R', R'' such that $S = R' - R''$,

(2) $B(S) \geqq 1$ and $Q(S - C) \geqq 1$, and

(3) for all $\boldsymbol{b}$ in $B \cap S$ and $\boldsymbol{g}$ in $S - C$ we have $\boldsymbol{g} - \boldsymbol{b}$ contained in Q.

LEMMA 2. *If S is any set of type II then*

$$C(S) \geqq \alpha Q(S) + B(S) .$$

Proof of Lemma 2. Define the sets S'_j and S' as above. Let $\boldsymbol{b} = (b_1, \cdots, b_n)$ be the largest vector such that

(1) $\boldsymbol{b}$ is in $B \cap S$, and

(2) $b_1 + \cdots + b_n = \max\{x_1 + \cdots + x_n \mid (x_1, \cdots, x_n) \in B \cap S\}$. Likewise, let $\boldsymbol{g} = (g_1, \cdots, g_n)$ be the largest vector such that

(1) $\boldsymbol{g}$ is in $S - C$, and

(2) $g_1 + \cdots + g_n = \max\{y_1 + \cdots + y_n \mid (y_1, \cdots, y_n) \in S - C\}$.

Let $B(S) = \rho \geqq 1$, $Q(S - C) = \sigma \geqq 1$, $Q(S' - A) = \tau$. The set $\{\boldsymbol{g} - \boldsymbol{x} \mid \boldsymbol{x} \in B \cap S\}$ contains ρ elements of Q (Definition 4, part 3), none of which is in A. We show that these are in S': If $\boldsymbol{x} = (x_1, \cdots, x_n)$ is in $B \cap S$ then $\boldsymbol{x}$ is in S_j for some j such that $1 \leqq j \leqq u$. Hence $\delta_{ji} \leqq x_i \leqq g_i$ for all $i = 1, \cdots, n$, and $\boldsymbol{g}$ is in S_j. $\boldsymbol{0} \neq \boldsymbol{g} - \boldsymbol{x} = (\boldsymbol{g} - \boldsymbol{\delta}_j) - (\boldsymbol{x} - \boldsymbol{\delta}_j)$. But $\boldsymbol{g} - \boldsymbol{\delta}_j$ is in S'_j and S'_j is a fundamental set. Hence $\boldsymbol{g} - \boldsymbol{x}$ is in S'_j, therefore in S'.

Likewise, the (possibly empty) set $\{\boldsymbol{y} - \boldsymbol{b} \mid \boldsymbol{y} \in S - C,\ \boldsymbol{y} \neq \boldsymbol{g}\}$ contains $\sigma - 1$ elements, all of which are in $S' - A$. We must show that the two sets are disjoint. Hence suppose that for some $\boldsymbol{y} \neq \boldsymbol{g}$ and, therefore, $\boldsymbol{x} \neq \boldsymbol{b}$, we have

$$\boldsymbol{g} - \boldsymbol{x} = \boldsymbol{y} - \boldsymbol{b} .$$

Equating the ith components and transposing gives the n equations

$$\text{(V)} \qquad \begin{aligned} g_1 + b_1 &= y_1 + x_1 \\ g_2 + b_2 &= y_2 + x_2 \\ &\vdots \\ g_n + b_n &= y_n + x_n \end{aligned}$$

and

$$g_1 + \cdots + g_n + b_1 + \cdots + b_n = y_1 + \cdots + y_n + x_1 + \cdots + x_n .$$

Because of the way in which $\boldsymbol{g}$ and $\boldsymbol{b}$ were chosen, this implies

$$g_1 + \cdots + g_n = y_1 + \cdots + y_n \quad \text{and} \quad b_1 + \cdots + b_n = x_1 + \cdots + x_n .$$

Therefore $\boldsymbol{g} > \boldsymbol{y}$ and $\boldsymbol{b} > \boldsymbol{x}$, and at least one of the n equations of (N) must fail to hold. We now have

$$\begin{aligned} \tau &\geqq \sigma - 1 + \rho , \\ Q(S) - \sigma &\geqq Q(S) - \tau - 1 + \rho , \\ Q(S) - \sigma &\geqq Q(S') - \tau + Q(S) - Q(S') - 1 + \rho . \end{aligned}$$

We recall that $Q(S) - Q(S') - 1 \geqq 0$, and that S' is a fundamental set. Hence

$$\begin{aligned} C(S) &\geqq A(S') + Q(S) - Q(S') - 1 + B(S) \\ &\geqq \alpha[Q(S') + 1] + \alpha[Q(S) - Q(S') - 1] + B(S) \\ &= \alpha Q(S) + B(S) . \end{aligned}$$

LEMMA 3. *If S is any set of type I then*

$$C(S) \geqq \alpha[Q(S) + 1] + B(S) .$$

Proof of Lemma 3. (i) Suppose $B(S) = 0$. Then

$$C(S) = A(S) \geqq \alpha[Q(S) + 1] + B(S) .$$

(ii) Suppose $B(S) \geqq 1$. Define $\boldsymbol{b}$ and $\boldsymbol{g}$ as in the proof of Lemma 2. Let $B(S) = \rho$, $Q(S - C) = \sigma$, $Q(S - A) = \tau$. Again the two sets $\{\boldsymbol{g} - \boldsymbol{x} \mid \boldsymbol{x} \in B \cap S\}$ and $\{\boldsymbol{y} - \boldsymbol{b} \mid \boldsymbol{y} \in S - C,\ \boldsymbol{y} \neq \boldsymbol{g}\}$ give $\sigma - 1 + \rho$ elements not in A, which now will be in S. Also $\boldsymbol{g}$ is in $S - C$, hence is in $S - A$, but is in neither of the two sets above. This implies that

$$\begin{aligned} \tau &\geqq \sigma + \rho , \\ Q(S) - \sigma &\geqq Q(S) - \tau + \rho , \\ C(S) &\geqq A(S) + B(S) \geqq \alpha[Q(S) + 1] + B(S) . \end{aligned}$$

We can now return to the proof of the theorem. Let R be any fundamental set satisfying the hypotheses of the theorem. We will use induction on the number of elements in $R - C$.

(i) Let $Q(R - C) = 1$. Then R is a set of type I, and we may apply Lemma 3.

(ii) Assume the the theorem holds for any fundamental set R' satisfying the hypotheses of the theorem and such that $Q(R' - C) < k$, $k \geqq 2$, and let $Q(R - C) = k$. If $B(R) = 0$ then R is of type I, so assume $B(R) \geqq 1$.

Let $\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_k$ be the k vectors in $R - C$, $T_j = \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{g}_j$ or $\boldsymbol{g}_j - \boldsymbol{x} \in Q\}$, $j = 1, \cdots, k$. If $\boldsymbol{b} \in T_j$ for all $j = 1, \cdots, k$ and all $\boldsymbol{b}$ in $B \cap R$ then again R is of type I, so assume (by re-numbering, if necessary) that $B(R - T_1) > 0$. Let J be the maximum j such that $B(R - (T_1 \cup \cdots \cup T_j)) > 0$. Then $\boldsymbol{b} \in B$ and $\boldsymbol{b} \in R - (T_1 \cup \cdots \cup T_J)$ implies $\boldsymbol{b} \in T_{J+1}$. We observe that $J < k$, since $\boldsymbol{b} \in R - (T_1 \cup \cdots \cup T_k)$ would imply that there does not exist $\boldsymbol{g}$ in $R - C$ such that $\boldsymbol{g} - \boldsymbol{b}$ is in Q, contrary to hypothesis. Also, $\boldsymbol{g}_{J+1} \notin T_1 \cup \cdots \cup T_J$.

Let $W_0 = T_1 \cup \cdots \cup T_J$. If $R - W_0$ is not of type II, there exists $\boldsymbol{b} \in B \cap T_{J+1}$ and a subscript i_1 such that $i_1 > J + 1$, $\boldsymbol{b} \notin T_{i_1}$. Let $W_1 = W_0 \cup T_{i_1}$. If $R - W_1$ is not of type II, we may repeat the above to form $W_2 = W_1 \cup T_{i_2}$, and so on. Eventually we must obtain a set W_m such that $R - W_m$ is of type II, $m \geqq 0$.

But W_m is a fundamental set satisfying the hypotheses of the theorem, and $Q(W_m - C) < k$ since $g_{J+1} \notin W_m$. Hence

$$C(W_m) \geqq \alpha[Q(W_m) + 1] + B(W_m) .$$

Also,

$$C(R - W_m) \geqq \alpha Q(R - W_m) + B(R - W_m) .$$

Adding, we obtain

$$C(R) \geqq \alpha[Q(R) + 1] + B(R) .$$

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