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ON ALGEBRAIC HOMOGENEOUS SPACES

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Let V be a homogeneous space with respect to a connected algebraic group G and (A, α) an Albanese variety of V. Then, for any points a and a' of $A, \alpha^{-1}(a)$ is a homogeneous space, of dimension $= \dim V - \dim A$, with respect to the maximal connected linear normal algebraic subgroup L of G and there exists an everywhere defined birational transformation of $\alpha^{-1}(a)$ onto $\alpha^{-1}(a')$. We have (a) $\dim A = 0 \Leftrightarrow V$ is considered as a homogeneous space with respect to a connected linear algebraic group $\Leftrightarrow$ The isotropy group of any point on V contains D (where D is the smallest normal algebraic subgroup of G giving rise to a linear factor group); (b) $\dim A = \dim V \Leftrightarrow V$ is considered as a homogeneous space with respect to an abelian variety $\Leftrightarrow$ The isotropy group of any point on V contains L. More generally, for a connected normal algebraic subgroup N of G, we can define a quotient variety W_N of V by N with a natural mapping φ_N and then, for any points Q and Q' of W_N, $\varphi_N^{-1}(Q)$ is a homogeneous space with respect to N and there exists an everywhere defined birational transformation of $\varphi_N^{-1}(Q)$ onto $\varphi_N^{-1}(Q')$. When $N = L$, there exists a bijective birational mapping of W_L onto A and (W_L, φ_L) is an Albanese variety of V. On the other hand, when $N = D$ and V is complete, W_D is a rational variety and $\varphi_D^{-1}(Q)$ is birationally equivalent to the direct product of an abelian variety and a rational variety. In the case where the definition field k of the homogeneous space V is finite, there exists a homogeneous space W with respect to L, defined over k, such that we have (the number of rational points on V over k) = (the number of rational points on A over k) $\times$ (the number of rational points on W over k). In particular, if V is complete then the conjecture of Lang and Weil on the zeros of the congruence zeta-function of V follows from the above result.

It is known as the structure theorem of Chevalley (cf. [7]) that a connected algebraic group G has a maximal connected linear normal algebraic subgroup L such that the factor group G/L is an abelian variety. One can prove this by considering the relation between G and an Albanese variety of G; i.e. L is characterized as the inverse image of a single point on the Albanese variety by a canonical mapping (cf. [3] and [7]).

In this paper, we shall consider a homogeneous space V with

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respect to a connected algebraic group G, i.e. an algebraic variety V on which G operates regularly and transitively (cf. [7]) and prove some analogous results. Let A be an Albanese variety of V and α a canonical mapping of V into A. We shall show that, for any points a and a' on A, the inverse image $\alpha^{-1}(a)$ is a homogeneous space with respect to the maximal connected linear normal algebraic subgroup L of G and there exists an everywhere defined, birational transformation of $\alpha^{-1}(a)$ onto $\alpha^{-1}(a')$. (Theorem 1) Then we shall give some necessary and sufficient conditions for (a) $\dim A = 0$, and for (b) $\dim A = \dim V$. (Theorem 2) Next we consider a connected normal algebraic subgroup N of G and a homogeneous space W_N with respect to G/N, which will be defined in the beginning of § 2. There is a natural mapping φ_N of V onto W_N. We shall show that, for any points Q and Q' on W_N, the inverse image $\varphi_N^{-1}(Q)$ is a homogeneous space with respect to N and there exists an everywhere defined, birational transformation of $\varphi_N^{-1}(Q)$ onto $\varphi_N^{-1}(Q')$. (Theorem 3) There exists a bijective birational mapping of W_N onto A when $N = L$. Finally we shall show, using these results, that the problem of counting the number of rational points of homogeneous spaces over finite fields is reduced to the case where the transformation groups are linear. (Theorem 4)

V being a homogeneous space with respect to a connected algebraic group G, we shall denote, for any point (g, P) on $G \times V$, the point of V obtained by operating g to P with gP. Also, for an algebraic subgroup K of G, we denote by π_K the canonical separable mapping of G onto the quotient variety G/K. If K is normal in G, then π_K is a rational homomorphism. Let k be a field of definition for V, G and the operation of G on V; moreover, we assume that there exists a rational point P_0 on V over k. We denote by H the set of all the elements g of G such that $gP_0 = P_0$ and call it the isotropy group of P_0 (in G). Since H is the inverse image of P_0 by the rational mapping $g \rightarrow gP_0$ of G onto V, H is a k-closed algebraic subgroup of G. Then, from the definition of quotient varieties, there exists a bijetive rational mapping of G/H onto V. When the characteristic of k is positive, this rational mapping is not necessarily birational, i.e. g being a generic point of G over k, $k(\pi_H(g))$ is a finite, purely inseparable extension of $k(gP_0)$. Let L be the maximal connected linear normal algebraic subgroup of G and D the smallest normal algebraic subgroup of G giving rise to a linear factor group, which are assumed to be defined over k (cf. [7]). D is contained in the center of G. We assume that an Albanese variety A and a canonical mapping α of V into A are defined over k. By a suitable translation if necessary, we may assume that $\alpha(P_0) = 0$ (the identity element of the group A). Since V is nonsingular, α is everywhere defined.

1. Albanese varieties of homogeneous spaces. Let g be a generic point of G over k. Then we have $k(g, gP_0) = k(g, P_0) = k(g)$ and so $k(g)$ contains $k(gP_0)$ and $k(\alpha(gP_0))$. Hence we can define a rational mapping β of G into A, defined over k, by the relation $\beta(g) = \alpha(gP_0)$. Then we have $\beta(g') = \alpha(g'P_0)$ for any point g' on G.

We start with several lemmas.

LEMMA 1. *β is a set-theoretically surjective rational homomorphism of G onto A and α is a set-theoretically surjective rational mapping of V onto A.*

Proof. Denoting by e the identity element of G, we have $\beta(e) = \alpha(eP_0) = \alpha(P_0) = 0$ and so β is a rational homomorphism. For any point a on A, there exist some points $P_1, \cdots, P_t$ on V such that $a = \alpha(P_1) + \cdots + \alpha(P_t)$. We have $P_i = g_iP_0$ with some g_i in G and so $\alpha(P_i) = \alpha(g_iP_0) = \beta(g_i)$. Then, as β is a homomorphism, we have $a = \beta(g_1) + \cdots + \beta(g_t) = \beta(g_1 \cdots g_t) = \alpha((g_1 \cdots g_t)P_0)$.

COROLLARY. *We have the inequality* $\dim A \leqq \dim V$.[1]

LEMMA 2. *Let π be the canonical rational mapping of G/H onto G/LH such that $\pi_{LH} = \pi \circ \pi_H$. Then $(G/LH, \pi)$ is an Albanese variety of G/H.*[2]

Proof. Since L contains the commutator subgroup of G, LH is a normal algebraic subgroup of G. Moreover, as there exists a rational homomorphism of G/L onto G/LH, G/LH is an abelian variety which is generated by G/H and π. Let f be a rational mapping of G/H into an abelian variety B. We may assume that $f \circ \pi_H$ is a rational homomorphism of G into B; then $f \circ \pi_H$ induces the 0-homomorphism on LH. So there exists a rational homomorphism λ of G/LH into B such that $f \circ \pi_H = \lambda \circ \pi_{LH} = \lambda \circ \pi \circ \pi_H$, i.e. we have $f = \lambda \circ \pi$.

LEMMA 3. *The subgroup LH is the kernel of β.*

Proof. Let M be the kernel of β. Since we have $\beta(h) = \alpha(hP_0) = \alpha(P_0) = 0$ for any point h on H, we have $LH \subset M$. Let g be a generic point of G over k. Then, with a suitable power q of the characteristic of k, we have $\bar{k}(\pi_H(g)) \supset \bar{k}(gP_0) \supset \bar{k}((\pi_H(g))^{(q)})$.[3] Moreover

[1] Therefore, if an algebraic curve is a homogeneous space with respect to a connected algebraic group, then the genus is equal to 0 or 1.

[2] Of course, this lemma holds for any algebraic subgroup H of G.

[3] If $q=0$, then (q) is the identity transformation of the ambient spaces. If $q>0$, then (q) denotes the rational transformation of the ambient spaces induced by the endomorphism of the universal domain: $\xi \to \xi^q$.

$(G/LH)^{(q)}$ is an abelian variety with $(\pi_{LH}(g))^{(q)}$ as a generic point over $\bar{k}$ and $(\pi_{LH}(g))^{(q)}$ is rational over $\bar{k}((\pi_H(g))^{(q)})$. Hence, by Lemmas 1 and 2 and by the universal mapping property of Albanese varieties, we have $\bar{k}(\pi_{LH}(g)) \supset \bar{k}(\alpha(gP_0)) \supset \bar{k}((\pi_{LH}(g))^{(q)})$. On the other hand, since β induces a bijective rational homomorphism of G/M onto A, we have $\bar{k}(\pi_{LH}(g)) \supset \bar{k}(\pi_M(g)) \supset \bar{k}(\alpha(gP_0)) = \bar{k}(\beta(g))$. Hence $\bar{k}(\pi_{LH}(g))$ is purely inseparable and separable over $\bar{k}(\pi_M(g))$ and so we have $\bar{k}(\pi_{LH}(g)) = \bar{k}(\pi_M(g))$, which implies that $LH = M$.

Now we define the operation of G on A by $g \circ a = \beta(g) + a$ for any point (g, a) on $G \times A$. Then, by Lemma 1, A is a homogeneous space with respect to G and the mapping α commutes with the operations of G on V and A, i.e. we have $\alpha(gP) = g \circ \alpha(P)$ for any point (g, P) on $G \times V$. In fact, denoting $P = g'P_0$ with some g' in G, we have $\alpha(gP) = \beta(gg') = \beta(g) + \beta(g') = \beta(g) + \alpha(g'P_0)$. Therefore we can apply Proposition 1 of Rosenlicht [7] to the homogeneous spaces V and A and we have the following results. Let Γ_α be the graph of α in $V \times A$. Then, for any subvariety X of A, $Y = \mathrm{pr}_V((V \times X) \cdot \Gamma_\alpha)$ is defined and has the dimension $= \dim V - \dim A + \dim X$. Moreover the point set of Y coincides with the set-theoretical inverse image $\{P \in V;\ \alpha(P) \in X\}$ of X by α.

For a point a on A, we denote by $W(a)$ the set $\{P \in V;\ \alpha(P) = a\}$, i.e. the set-theoretical inverse image of a by α. Then, by the above results, $W(a) = \mathrm{pr}_V((V \times a) \cdot \Gamma_\alpha)$ is a $k(a)$-closed algebraic set of dimension $= \dim V - \dim A$. We take an element g_a of G such that $\beta(g_a) = a$ (see Lemma 1). For a point $P = gP_0$ in $W(a)$, we have $\beta(g) = \alpha(P) = a$ and so, by Lemma 3, g is in the coset $g_a LH$, i.e. we have $g = g_a lh$ with some l in L and h in H; so we have $P = g_a lP_0$. Conversely, for a point $P = g_a lP_0$ with some l in L, we have $\alpha(P) = \beta(g_a l) = \beta(g_a) = a$, i.e. P is in $W(a)$. Hence, as L is normal in G, we see that $W(a)$ coincides also with the point set $\{lg_aP_0;\ l \in L\}$, i.e. the L-orbit of g_aP_0 in V. Since L is connected, $W(a)$ is a $k(a)$-closed irreducible subvariety of V and is a homogeneous space with respect to L. Since $W(a)$ is the image of a linear algebraic group L by a rational mapping, it has a trivial Albanese variety (i.e. the irregularity of $W(a) = 0$). For any points a and a' on A, it is easily verified that there exists an everywhere defined, birational transformation which maps $W(a)$ onto $W(a')$; in fact, $g_a lP_0 \to (g_{a'} g_a^{-1}) g_a lP_0 = g_{a'} lP_0$ is such a transformation. Finally, if U is an irreducible subvariety of V, which is a homogeneous space with respect to a connected linear algebraic group, then, as U has a trivial Albanese variety, $\alpha(U)$ consists of a single point and so U is contained in $W(\alpha(U))$. Therefore we have the following

THEOREM 1. *For a point a on A, we denote by $W(a)$ the point* *set* $\{P \in V;\ \alpha(P) = a\}$ *in* V, i.e. *the set-theoretical inverse image of* a *by* α. *Then* $W(a)$ *is a* $k(a)$*-closed irreducible subvariety, of dimension* $= \dim V - \dim A$, *and has a trivial Albanese variety. Moreover* $W(a)$ *is a homogeneous space with respect to* L *and is maximal among all the subvarieties of* V, *which are homogeneous spaces with respect to connected linear algebraic groups* (*in the sense that such subvarieties are contained in some* $W(a)$). *For any points* a *and* a' *on* A, *there exists an everywhere defined, birational transformation of* $W(a)$ *onto* $W(a')$.

COROLLARY. *If* V *is complete, then* $W(a)$ *is a rational variety.*

Proof. $W(a)$ is a complete homogeneous space with respect to a connected linear algebraic group L and so is rational (cf. [4] and [8]).

For a connected algebraic group G, we have easily the following results (cf. [7]):

$$\dim(\text{Albanese variety of } G) = 0 \Leftrightarrow G \text{ is linear} \Leftrightarrow D = \{e\}\ .$$

$$\dim(\text{Albanese variety of } G) = \dim G \Leftrightarrow G \text{ is an abelian variety} \Leftrightarrow L = \{e\}\ .$$

Now we prove analogous results for homogeneous spaces.

THEOREM 2. (a) *The following three conditions are equivalent*:

(i) $\dim A = 0$;

(ii) *V is considered as a homogeneous space with respect to a connected linear algebraic group*;

(iii) *The isotropy group of any point on V contains D.*

(b) *The following three conditions are equivalent*:

(i) $\dim A = \dim V$;

(ii) *V is considered as a homogeneous space with respect to an abelian variety* (i.e. *V is the image of an abelian variety by a bijective birational mapping* (cf. [10]));

(iii) *The isotropy group of any point on V contains L.*

Proof. For the assertion (iii) of (a) and (b), as L and D are normal in G and all the isotropy groups are conjugate to H, we have only to consider the isotropy group H of P_0.

(a) If $\dim A = 0$, then we have $W(0) = V$ and V is a homogeneous space with respect to L by Theorem 1. If V is a homogeneous space with respect to a connected linear algebraic group, then clearly V has a trivial Albanese variety, i.e. $\dim A = 0$. By Lemma 3, we have $\dim A = 0$ if and only if we have $G = LH$, i.e. if and only if H contains D (cf. [7]).

(b) If $\dim A = \dim V$, then we have $\dim LH = \dim H$ and H contains the connected component of the identity element of LH and so contains L. Hence H is normal in G and V is considered as a homogeneous space with respect to G/H, which is an abelian variety as a homomorphic image of G/L by a rational mapping. If V is a homogeneous space with respect to an abelian variety, clearly we have $\dim V = \dim A$.

COROLLARY. *In the case where k is algebraically closed and V is complete, V has a trivial Albanese variety if and only if V is a rational variety over k.*

Proof. It follows from the fact that complete homogeneous spaces with respect to connected linear algebraic groups are rational (cf. [4] and [8]) and from Theorem 2.

2. Quotients with respect to normal algebraic subgroups. Let N be a normal algebraic subgroup of G, defined over k. Then, by [7], N operates regularly on V and the algebraic factor group G/N operates on the variety W' of N-orbits on V. It is easily verified that W' is a pre-homogeneous space with respect to G/N. Then, by a theorem of Weil, there exist a homogeneous space $W_N = W$ with respect to G/N, which is birationally equivalent to W' over k, and a generically surjective separable rational mapping $\varphi_N = \varphi$ of V to W, both defined over k, which have the following properties: g and P being independent generic points of G and V over k, we have $\varphi(gP) = \pi_N(g)\varphi(P)$; if we have $\varphi(P_1) = \varphi(P_2)$ for generic points P_1 and P_2 of V over k, then we have $P_2 = gP_1$ with some g in N.[4]

The homogeneous space W with respect to G/N is also considered as a homogeneous space with respect to G by the operation $gQ = \pi_N(g)Q$ for any point (g, Q) on $G \times W$. Then, as we have $\varphi(gP) = g\varphi(P)$ for independent generic points g and P of G and V over k, we can also apply Proposition 1 of [7] to the homogeneous spaces V and W and we have the following results on W and φ. φ is an everywhere defined, surjective separable rational mapping, defined over k, and, for any point (g, P) on $G \times V$, we have $\pi_N(g)\varphi(P) = g\varphi(P) = \varphi(gP)$. Let Γ_φ be the graph of φ in $V \times W$. Then, for any subvariety W_1 of W, $V_1 = \mathrm{pr}_V((V \times W_1)\cdot\Gamma_\varphi)$ is defined and has the dimension $= \dim V - \dim W + \dim W_1$. Moreover the point set of V_1 coincides with the set-theoretical inverse image $\{P \in V; \varphi(P) \in W_1\}$ of W_1 by φ.

We show that the isotropy group of $\varphi(P_0)$ in G is the subgroup NH. Let g be an element of the isotropy group, i.e. we have $g\varphi(P_0) = \varphi(gP_0) = \varphi(P_0)$. Taking a generic point g' of G over $k(g)$, we have

[4] In a recent paper [q], Rosenlicht has shown the existence of a "quotient space V/N" defined over k which is a homogeneous space with respect to G/N. We may take this V/N as W_N.

$\varphi(g'gP_0) = \varphi(g'P_0)$ and so $ng'gP_0 = g'P_0$ with some n in N. As N is normal in G, we see that g is in NH. The converse assertion is trivial by the definition of the operation of G on W. Hence there exists a bijective rational mapping of G/NH onto W and W has the dimension $= \dim G - \dim NH$.

For a point Q on W, we denote by $X(Q)$ the set $\{P \in V; \varphi(P) = Q\}$, i.e. the set-theoretical inverse image of Q by φ. Then, by the above results, $X(Q) = \mathrm{pr}_V((V \times Q) \cdot \Gamma_\varphi)$ is a $k(Q)$-closed algebraic set of dimension $= \dim V - \dim W = \dim NH - \dim H$. If $P = gP_0$ and $P' = g'P_0$ are any two points on $X(Q)$, then, as we have $\varphi(gP_0) = \varphi(g'P_0)$, $g^{-1}g'$ is in NH and so we have $P' = n'P$ with some n' in N. Conversely, for any point P in $X(Q)$ and any point n' in N, we have $\varphi(n'P) = \pi_N(n')\varphi(P) = Q$ and so $n'P$ is in $X(Q)$. Hence $X(Q)$ is the N-orbit of any point in it.

Now we assume that N is connected. Then, clearly, $X(Q)$ is a $k(Q)$-closed irreducible subvariety of V and is a homogeneous space with respect to N. Moreover, for any points Q and Q' on W, there exists an everywhere defined, birational transformation of $X(Q)$ onto $X(Q')$. Therefore we have the following

THEOREM 3. *Let N be a connected normal algebraic subgroup of G, defined over k. Then there exist a homogeneous space W_N with respect to the algebraic factor group G/N, defined over k and of dimension* $= \dim G - \dim NH$, *and an everywhere defined, surjective separable rational mapping φ_N of V onto W_N, defined over k, which have the following properties.*[5] *For a point Q on W_N, we denote by $X_N(Q)$ the point set $\{P \in V; \varphi_N(P) = Q\}$ in V,* i.e. *the set-theoretical inverse image of Q by φ_N. Then $X_N(Q)$ is a $k(Q)$-closed irreducible subvariety of V, of dimension* $= \dim V - \dim W_N = \dim NH - \dim H$, *and is a homogeneous space with respect to N. For any points Q and Q' on W_N, there exists an everywhere defined, birational transformation of $X_N(Q)$ onto $X_N(Q')$.*

We consider the case where $N = L$. Then the homogeneous space W_L with respect to G/L is an abelian variety (cf. [10]) and, as α is L-invariant, we can prove that there exists a bijective birational mapping φ of W_L onto A such that $\alpha = \varphi \circ \varphi_L$. Hence Theorem 1 also follows from Theorem 3. Next we consider the case where $N = D$. Then W_D is a homogeneous space with respect to the connected linear algebraic group G/D and so W_D has a trivial Albanese variety. On the other hand, for any point Q on W_D, $X_D(Q)$ is a homogeneous space with

[5] W_N is also a homogeneous space with respect to G and φ_N commutes with the operations of G on V and W_N.

respect to a connected commutative algebraic group D and so there exists a bijective birational mapping of a connected commutative algebraic group onto $X_D(Q)$ (cf. [10]). Consequently there exists a birational mapping of the direct product of an abelian variety and a rational variety to $X_D(Q)$ (cf. [7]). Hence we have the following

PROPOSITION 1. *W_L is an Albanese variety of V and φ_L is a canonical mapping. If V is complete, then W_D is a rational variety and $X_D(Q)$ is birationally equivalent to the direct product of an abelian variety and a rational variety.*

3. Certain conditions on transformation groups. It is easily verified that there exists a connected algebraic group G', which is a homomorphic image of G by a rational mapping, such that V is a homogeneous space with respect to G' and G' operates on V effectively. Such a group G' has the several properties which follow from the corresponding properties of G: for example, the connectedness of the isotropy group of any point on V, the solvability of the maximal connected linear algebraic subgroup and the property that G is generated by an abelian variety and a linear algebraic group.

PROPOSITION 2. *If G operates on V effectively, then we have*
(1) $H \cap G = \{e\}$, *where C is the center of G.*
(2) *H is linear.*
(3) *The Albanese variety of G is isogenous to A.*[6]

Proof. The assertion (1) is trivial. As we have $H_0 \cap D \subset H \cap D \subset H \cap C$ (H_0 is the connected component of the identity element of H), we have $H_0 \cap D = \{e\}$. Hence H_0 is isogenous to the algebraic subgroup H_0D/D of a linear algebraic group G/D; so H_0 and H are linear. Then, as L contains H_0, G/L (= the Albanese variety of G) is isogenous to G/LH, which is isogenous to A by Lemma 3.

PROPOSITION 3. *If V is complete, then V is considered as a homogeneous space with respect to a connected algebraic group G^*, which is the homomorphic image of G by a rational mapping and is generated by an abelian variety and a connected linear algebraic group. If k is a finite field, then we have the same assertion.*

Proof. Since D is contained in C the connected component $(L \cap D)_0$ of the identity element of $L \cap D$ is solvable and so, if V is complete,

[6] Of course, the assertions (1) and (2) are also true for the isotropy group of any point on V.

any element of $(L \cap D)_0$ operates trivially on V (cf. [2]). Hence V is considered as a homogeneous space with respect to the algebraic factor group $G/(L \cap D)_0$, which is generated by an abelian variety and a linear group (cf. [7]). If k is finite, then G itself is generated by an abelian variety and a linear group (cf. [1]).

PROPOSITION 4. We assume that V is considered as a homogeneous space with respect to a connected algebraic group G^*, which is generated by an abelian variety A^* and a connected linear algebraic group L^*, both defined over k. (*This is the case if V is complete or k is finite* (*Proposition* 3).) Then there exist a homogeneous space W^* with respect to L^*, defined over k and of dimension $= \dim V - \dim A$, and an everywhere defined, surjective separable rational mapping φ^* of V onto W^*, defined over k, which have the following properties. For a point Q^* on W^*, we denote by $X^*(Q^*)$ the point set $\{P \in V;\ \varphi^*(P) = Q^*\}$ in V, i.e. the set-theoretical inverse image of Q^* by φ^*. Then $X^*(Q^*)$ is a $k(Q^*)$-closed irreducible subvariety of V and there exists a bijective birational mapping of an abelian variety isogenous to A onto $X^*(Q^*)$. For any points Q^* and $Q^{*\prime}$ on W^*, there exists an everywhere defined, birational transformation of $X^*(Q^*)$ onto $X^*(Q^{*\prime})$.

Proof. By the remark stated in the beginning of § 3, we may replace G^* by a homomorphic image G' of G^* by a rational mapping, which operates on V transitively and effectively and is generated by an abelian variety A' and a connected linear algebraic group L', both defined over k. Then, as A' is contained in the center of G', we have $A' \cap H' = \{e'\}$ (H' is the isotropy group of P_0 in G' and e' is the identity element of G') and the Albanese variety A' of G' is isogenous to A (see Proposition 2). We apply Theorem 3 to the normal algebraic subgroup A' of G' and put $W^* = W_{A'}$. Then, as G'/A' is the homomorphic image of L' and so of L^* by rational mappings, the homogeneous space W^* with respect to G'/A' is also a homogeneous space with respect to L^*. The dimension of W^* is equal to $\dim G' - \dim A'H' = \dim G' - \dim A' - \dim H' = \dim V - \dim A$. Moreover, for a point Q^* on W^*, $X^*(Q^*) = X_{A'}(Q^*)$ is a homogeneous space with respect to A' and, as the intersection of A' and the isotropy group of any point on V in G' consists of a single element e', there exists a bijective birational mapping of an abelian variety isogenous to A' onto $X^*(Q^*)$ (cf. [10]).

COROLLARY. *In the case where V is complete, W^* is a rational variety over $\bar{k}$ and $X^*(Q^*)$ is an abelian variety over $\overline{k(Q^*)}$ isogenous to A.*

Proof. It follows from Corollary of Theorem 2.

We consider the case where L is solvable over k. Then there exists a cross section σ of W_L to V with respect to φ_L (cf. [7]) and so we can prove that, for a generic point Q of W_L over k, $X_L(Q)$ coincides with the orbit $L\sigma(Q)$ and contains a generic point $l\sigma(Q)$ of V over k and we have $k(l\sigma(Q)) = k(l\sigma(Q), Q)$. Moreover, as a homogeneous space with respect to a connected solvable group L, $L\sigma(Q)$ is a rational variety over $k(Q)$ (cf. [8]). Hence we have the following

PROPOSITION 5. If L is solvable, then V is birationally equivalent to the direct product of the Albanese variety and a rational variety.

On the other hand, we have the following

PROPOSITION 6. If V is complete and L is solvable, then there exists a bijective birational mapping of an abelian variety onto V.

Proof. L has a fixed point $P = gP_0$ on V (cf. [2]) and so $L = g^{-1}Lg$ is contained in H. Then the assertion follows from Theorem 2 (b).

4. Rational points over finite fields. Finally we consider the case where k is a finite field with q elements. In this case, if V, G and the operation of G on V are all defined over k, then there exist a rational point P_0 on V over k (cf. [5]) and an Albanese variety (A, α) of V defined over k. Moreover G is generated by an abelian variety and a connected linear algebraic group, both defined over k. For a variety U defined over k, we denote by $N_k(U)$ the number of rational points on U over k. Let W^* be the variety defined in Proposition 4, which is a homogeneous space with respect to a connected linear algebraic group defined over k. Then we have the following

THEOREM 4. *We have* $N_k(V) = N_k(A)\cdot N_k(W^*)$.

Proof. We use the same notations as in the proof of Proposition 4, i.e. $G' = A'\cdot L'$ is a connected algebraic group, defined over k, which operates on V transitively and effectively. Here A' is an abelian variety isogenous to A over k and L' is a connected linear algebraic group defined over k. For a rational point P on V over k, we have $(\varphi^*(P))^{(q)} = \varphi^*(P^{(q)}) = \varphi^*(P)$, i.e. $\varphi^*(P)$ is a rational point on W^* over k.[7] Hence, denoting by $Q_1^*, \cdots, Q_t^*$ ($t = N_k(W^*)$) all the rational points on W^* over k, we see that each rational points on V over k is in one and only one $X^*(Q_i^*)$ ($i = 1, \cdots, t$). Since $X^*(Q_i^*)$ is a homogeneous space with respect to A' defined over a finite field $k = k(Q_i^*)$, there exists a rational point P_i on $X^*(Q_i^*)$ over k (cf. [5]). Then, as the intersection of A' and the isotropy group of P_i in G' consists of a single point e'

(see Proposition 2), the mapping $a' \to a'P_i$ of A' onto $X^*(Q_i^*)$ is bijective and rational over k. So we have $N_k(X^*(Q_i^*)) = N_k(A')$, which is equal to $N_k(A)$, as A' is isogenous to A over k. Hence we have $N_k(V) = N_k(A) \cdot N_k(W^*)$.

Moreover, if we assume that V is complete and W^* is rational over k (cf. Corollary of Theorem 2), then we see easily that the conjecture of Lang and Weil on the zeros of the congruence zeta-function of V follows from Theorem 4 (cf. [6]).

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[7] (q) is the rational mapping defined in the footnote (3).