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ON THE PHASE-SHIFT FORMULA FOR THE SCATTERING OPERATOR

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Let us denote by $S_{|k|}$ the scattering operator attached to a fixed value $|k|^2$ of the kinetic energy. We shall show by a method different from that of Buslaev [2] that $S_{|k|}$ is the identity plus a Hilbert-Schmidt integral operator under a weaker assumption on $q(x)$, and give a way of unique determination of the phase shifts in terms of which $S_{|k|}$ can be represented as the orthogonal direct sum of multiplication operators by a number with absolute value equal to unity.

The scattering operator as well as the wave operators concerned with the Schrödinger equation in 3-space has been an object of various investigations among which M. Sh. Birman and M. G. Krein have established in [1] that the scattering operator corresponding to any fixed value of the incident kinetic energy is equal to the identity plus a trace-class operator if the difference of the resolvents is in the trace class or if it is completely continuous and the difference of some integral power of the resolvents is in the trace class. They have also introduced what they call the spectral shift function that is very similar to the phase shift we shall define later. Their method is quite abstract so that it may be applicable in principle to a wide range of problems. The concrete case of the Schrödinger operator, however, may be of interest, too and admit of more concrete approaches. In this connection V. S. Buslaev [2] has studied the Schrödinger operator $-\Delta + q(x)$, where the potential $q(x)$ has been assumed to be real-valued and infinitely differentiable and to decrease near infinity faster than any power of $|x|^{-1}$, and presented an explicit way of obtaining the scattering operator in the form of the identity plus an integral operator when the kinetic energy is fixed. In the special case of a spherically symmetric potential T. A. Green and O. E. Lanford, III [3] derived with mathematical rigor the phase-shift formula for the scattering operator.

Under our assumption on $q(x)$ stated below, that $S_{|k|}$ equals $I+$ (Hilbert-Schmidt operator) is included in the more general theorem mentioned above of Birman and Krein [1], though the approaches are different. Since we shall base our argument principally on the eigenfunction expansion and an identity involving the wave and scattering operators,

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we want to collect here some results on the eigenfunction expansion from [4] together with our assumption on $q(x)$.

We assume:

(A) $q(x)$ is a real-valued function of $x \in E = R^3$ which is locally Hölder continuous except at a finite number of singularities and is in $L_2(E)$. Moreover, there exist positive constants h, C_0 and R_0 such that $|q(x)| \leqq C_0 |x|^{-3-h}$ for $|x| \geqq R_0$.[1]

Then the operator $A = -\Delta + q(x)$ with $D(A) = C_0^\infty(E)$[2] determines in $L_2(E)$ a unique self-adjoint extension H that is lower semibounded, while we denote by H_0 the correspnding operator for the case $q(x) \equiv 0$. It is known that $H = H_0 + V$ and $D(H) = D(H_0) \subset D(V)$, where V is the operator in $L_2(E)$ of multiplication by $q(x)$.

There exist the (improper) eigenfunctions $\varphi(x, k)$, $k \in M$,[3] associated with the (improper) eigenvalues $|k|^2 > 0$ of the Schrödinger equation $-\Delta\varphi + q(x)\varphi = |k|^2\varphi$, that have the asymptotic expansion:

$$(1.1) \quad \begin{cases} \varphi(x, k) = e^{ik\cdot x} - \dfrac{2\pi}{|x|} e^{i|k||x|} F(|k|, -\omega_k, -\omega_x) + o\left(\dfrac{1}{|x|}\right), \\ F(|k|, \omega, \omega') = \dfrac{1}{8\pi^2} \displaystyle\int_E \varphi(x, |k|, -\omega) q(x) e^{i|k|\omega'\cdot x} dx \text{ [4]} \end{cases}$$

and that have the property:

$$(1.2) \quad \begin{cases} \varphi(x, k) \text{ is bounded and uniformly continuous in } x \in E \text{ and} \\ k \in D,\ D \text{ being any compact domain of } M \text{ not containing} \\ \text{the origin.} \end{cases}$$

The eigenfunctions of H_0 are $e^{ik\cdot x}$ and the eigenfunction expansion in this case involves the ordinary Fourier transforms

$$(1.3) \quad \hat{f}_0(k) = (2\pi)^{-3/2} \operatorname{lim} \int_E e^{-ik\cdot x} f(x) dx \qquad (f(x) \in L_2(E))\text{ [5]}$$

and the whole $L_2(E)$ maps onto $L_2(M)$ in a one-to-one way. However

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[1] This assumption is stronger than (A) of [4], where we assumed that

$$q(x) = o(|x|^{-2-h}) . \quad |x| = \left(\sum_{j=1}^3 (x^j)^2\right)^{\frac{1}{2}}, \quad x = (x^1, x^2, x^3) \in E .$$

[2] $D(A)$ = domain of A.

[3] $M = R^3$, but it will be convenient to distinguish the "momentum" or "Fourier" space M = set of all wave vectors k from the "configuration" space E.

[4] $\omega_x = x/|x|$, $\omega_k = k/|k|$. All ω's are unit vectors. $\varphi(x, |k|, \omega) = \varphi(x, k)$ if $k = (|k|, \omega)$. $k\cdot x$ is the scalar product of k and x.

The first equation of (1.1) follows immediately from Lemma 3.2 (p. 11) of [4] and the integral equation (see [4], p. 17)

$$\varphi(x, k) = e^{ik\cdot x} - \frac{1}{4\pi} \int \frac{e^{i|k||x-y|}}{|x-y|} q(y)\varphi(y, k) dy .$$

[5] $\operatorname{lim} \int_E \cdots$ = limit in the mean of $\int_{|x| \leqq N} \cdots$ for $N \to \infty$.

$\varphi(x, k)$ can map only a part of $L_2(E)$ onto $L_2(M)$. Nemely let $H = \int_{-\infty}^{\infty} \lambda dE_\lambda$ and let $\hat{f}(k) = (2\pi)^{-3/2} \lim \int_E \overline{\varphi(x, k)} f(x) dx$ for $f(x) \in L_2(E)$. Then $\hat{f}(k)$ is in $L_2(M)$ and does not depend on the projection of $f(x)$ on $E_0 L_2(M)$, so that the mapping $f(x) \rightarrow \hat{f}(k)$ transforms $(I - E_0)L_2(E)$ in a one-to-one manner onto $L_2(M)$. Of course the Parseval relation and the inverse transform formula hold:

$$f\,\|^2 = \|\hat{f}\|^2_{L_2(M)}\,, \quad (f, g) = (\hat{f}, \hat{g})_{L_2(M)}\,,$$
$$f(x) = (2\pi)^{-3/2} \lim \int_M \varphi(x, k)\hat{f}(k)dk\,,$$[6]

where $f(x) \in (I - E_0)L_2(E)$. Moreover, we have the diagonal representation of H that reads as follows:

$$(1.4) \qquad (Hf)(x) = (2\pi)^{-3/2} \lim \int_M |\,k\,|^2 \varphi(x, k)\hat{f}(k)dk \\ \times\ (f(x) \in (I - E_0)L_2(E) \cap D(H))\,.$$

Next let us take a look at the wave and scattering operators. Under our assumption (A) the wave operators

$$(1.5) \qquad W_\pm = \operatorname*{strong\ limit}_{t \to \pm\infty} e^{itH} e^{-itH_0}$$

are known to exist and be isometries with domain $L_2(E)$ and range $(I - E_0)L_2(E)$, which enables us to define the scattering operator

$$(1.6) \qquad S = W_+^* W_-$$

which is unitary. Some fundamental properties of these operators are:

$$(1.7) \qquad \begin{cases} (I - E_0)W_\pm = W_\pm\,, \quad HW_\pm = W_\pm H_0\,, \\ W_\pm^* H = H_0 W_\pm^*\,, \quad SH_0 \subset H_0 S\,, \end{cases}$$

and the following representation for W_+ will also be used later:

$$(1.8) \qquad (W_+ f)(x) = (2\pi)^{-3/2} \lim \int_M \overline{\varphi(x, -k)} \hat{f}_0(k) dk\,.$$

2. Representation of the scattering operator in the Fourier space.

LEMMA 1. *For any $u \in D(H) = D(H_0)$ and $v \in L_2(E)$*

$$(2.1) \qquad (Su, v) = (u, v) - i \int_{-\infty}^{\infty} (e^{itH_0} W_+^* V e^{-itH_0} u, v) dt\,.$$

[6] $\|\ \ \|$ and $(\ ,\ )$ are the usual norm and inner product in $L_2(E)$. The subscript $L_2(M)$ shows that the norm and inner product are taken in $L_2(M)$. $\lim \int_M \cdots =$ limt in the mean of $\int_{N^{-1} \leqq |k| \leqq N} \cdots$ for $N \rightarrow \infty$.

Proof. We have

$$\frac{d}{dt}(e^{itH}e^{-itH_0}u, v) = i(e^{itH}Ve^{-itH_0}u, v) .$$

Integrating the above relation from $-\infty$ to $+\infty$ we obtain in view of (1.5)

$$([W_+ - W_-]u, v) = i\int_{-\infty}^{\infty}(e^{itH}Ve^{-itH_0}u, v)dt .$$

If we replace v by W_+v, we have on the one hand, according to the definition (1.6) of S,

$$([W_+ - W_-]u, W_+v) = (u, v) - (Su, v)$$

and on the other

$$i\int_{-\infty}^{\infty}(e^{itH}Ve^{-itH_0}u, W_+v)dt = i\int_{-\infty}^{\infty}(W_+^*e^{itH}Ve^{-itH_0}u, v)dt$$
$$= i\int_{-\infty}^{\infty}(e^{itH_0}W_+^*Ve^{-itH_0}u, v)dt ,$$

where we have used (1.7). Combining these two we arrive at (2.1), which was to be proved.

THEOREM 1. *The scattering operator S is given the following representation in the Fourier space:*

$$(Su)\hat{}_0(k) = \hat{u}_0(k) - i\int_\Omega |k|\, F(|k|, \omega, \omega')\hat{u}_0(|k|, \omega')d\omega' \qquad (2.2)$$[7]

a.e., where $F(|k|, \omega, \omega')$ is given by (1.1), $\omega = \omega_k = k/|k|$ *and Ω is the totality of all unit vectors ω'.*

Proof. We see from Lemma 1 that for $u \in D(H)$ and $v \in L_2(E)$

$$(Su, v) - (u, v) = -i\int_{-\infty}^{\infty}(Ve^{-itH_0}u, W_+e^{-itH_0}v)dt . \qquad (2.3)$$

Using (1.8) and the diagonal representation of H_0 (which is obtained

[7] For the definition of $\hat{u}_0(k)$ see (1.3).

In (2.2) $F(|k|, \omega, \omega')$ can be replaced by $F(|k|, -\omega', -\omega)$, which results in the integration with respect to the incident directions according to (1.1). Indeed we have in general

$$\int\varphi(x, |k|, \omega)q(x)e^{i|k|\omega'\cdot x}dx$$
$$= \int\varphi(x, |k|, \omega)q(x)\Big[\varphi(x, |k|, \omega') + \frac{1}{4\pi}\int\frac{e^{i|k|\,|x-y|}}{|x-y|}q(y)\varphi(y, |k|, \omega')dy\Big]dx$$

and the right hand side is symmetric in ω and ω'.

from (1.4) with $\varphi(x, k)$ replaced by $e^{ik\cdot x}$), we have

$$(2.4)\qquad (W_+e^{-itH_0}v)(x) = (2\pi)^{-3/2}\lim\int_M \overline{\varphi(x, -k_1)}e^{-it|k_1|^2}\hat{v}_0(k_1)dk_1 ,$$

$$(2.5)\qquad (Ve^{-itH_0}u)(x) = (2\pi)^{-3/2}q(x)\lim\int_M e^{ik_2\cdot x}e^{-it|k_2|^2}\hat{u}_0(k_2)dk_2 .$$

On the other hand

$$(2.6)\qquad J \equiv (Su, v) - (u, v) = \int_M [(Su)\hat{}_0(k)\overline{\hat{v}_0(k)} - \hat{u}_0(k)\overline{\hat{v}_0(k)}]dk ,$$

as is seen from the Parseval relation for the ordinary Fourier transforms. Now let us suppose that $\hat{u}_0$ and $\hat{v}_0$ are in $C_0^\infty(M)$,[8] which in view of the condition that $q(x) \in L_1$ allows us to freely interchange the integration order except for the integration with respect to t. Putting together (2.3), (2.4), (2.5) and (2.6) we have

$$\begin{aligned}(2.7)\qquad J &= -i(2\pi)^{-3}\int_{-\infty}^{\infty} dt\int_E q(x)dx\int_M e^{ik_2\cdot x}e^{-it|k_2|^2} \\ &\quad\times \hat{u}_0(k_2)dk_2\int_M \varphi(x, -k_1)e^{it|k_1|^2}\overline{\hat{v}_0(k_1)}dk_1 \\ &= -i(2\pi)^{-3}\lim_{t\to\infty}\int_{-t}^{t} dt\int_M\int_M dk_2dk_1e^{it(|k_1|^2-|k_2|^2)} \\ &\quad\times \hat{u}_0(k_2)\overline{\hat{v}_0(k_1)}\int_E e^{ik_2\cdot x}q(x)\varphi(x, -k_1)dx \\ &= -i(2\pi)^{-3}\lim_{t\to\infty}\int_M\int_M dk_2dk_1\frac{2\sin t(|k_1|^2 - |k_2|^2)}{|k_1|^2 - |k_2|^2} \\ &\quad\times \hat{u}_0(k_2)\overline{\hat{v}_0(k_1)}\int_E e^{ik_2\cdot x}q(x)\varphi(x, -k_1)dx ,\end{aligned}$$

where in the last expression the t-integration has been interchanged with the k-integrations, which is possible because of the t-integration being taken over a finite interval. We can now proceed formally as follows: making use of a symbolic relation $\lim_{t\to\infty}(1/\pi)(\sin t\lambda/\lambda) = \delta(\lambda)$, $\delta(\lambda)$ being Dirac's delta function,

$$\begin{aligned}(2.8)\qquad J &= -\frac{i}{4\pi^2}\int_M\int_M \delta(|k_1|^2 - |k_2|^2)\hat{u}_0(k_2)\overline{\hat{v}_0(k_1)} \\ &\quad\times\left[\int_E e^{ik_2\cdot x}q(x)\varphi(x, -k_1)dx\right]dk_2dk_1 \\ &= -\frac{i}{8\pi^2}\int_M dk\,|k|\,\overline{\hat{v}_0(k)}\int_\Omega \hat{u}_0(|k|, \omega_1)\end{aligned}$$

[8] By $C_0^\infty(M)$ is meant the totality of infinitely differentiable functions of $k \in M$ whose support is compact and does not contain the origin.

$$\times \left[\int_E e^{i|k|\omega_1 \cdot x} q(x) \varphi(x, |k|, -\omega) dx \right] d\omega_1$$

$$= -i \int_M \left[\int_\Omega |k| F(|k|, \omega, \omega') \hat{u}_0(|k|, \omega') d\omega' \right] \overline{\hat{v}_0(k)} dk$$

(by (1.1)).

A justification of the above procedure is the following. Viewing the expression

$$\hat{u}_0(k_2) \overline{\hat{v}_0(k_1)} \left[\int_E e^{ik_2 \cdot x} q(x) \varphi(x, -k_1) dx \right]$$

as a function of $|k_2|$ we can show that this is Hölder continuous in $|k_2|$. For locally Hölder continuous functions it is known that Fourier's single integral formula holds.[9] (For more details see the Appendix.)

Since $\hat{v}_0(k)$ is arbitrary in $C_0^\infty(M)$, (2.2) follows from (2.6), (2.7) and (2.8) for $\hat{u}_0(k) \in C_0^\infty(M)$. However, the integral operator on the right side of (2.2) always makes sense for $\hat{u}_0(k) \in L_2(M)$, since $F(|k|, \omega, \omega')$ is a continuous function of ω and ω' on Ω for any fixed $|k| > 0$, as is easily seen from (1.1) and (1.2),[10] and hence defines a completely continuous operator of the Hilbert-Schmidt type on $L_2(\Omega)$, and since any $\hat{u}_0(k) \in L_2(M)$ can be regarded as an element of $L_2(\Omega)$ for almost every fixed value of $|k|$. Thus (2.2) holds for any $u \in L_2(E)(\hat{u}_0 \in L_2(M))$.

Now let us consider operators $S_r (r > 0)$ on $L_2(\Omega)$ defined by

$$(S_r u)(\omega) = u(\omega) - i \int_\Omega r F(r, \omega, \omega') u(\omega') d\omega' (u(\omega) \in L_2(\Omega)) .$$

As has been pointed out in the above proof of Theorem 1, S_r is equal to the identity plus a completely continuous operator of the Hilbert-Schmidt type and hence is invertible if it has no nontrivial null vector. Moreover, we can assert that S_r is unitary. Indeed fix any $r > 0$, and let $\delta_{r,\varepsilon}(s)$ be a real-valued smooth function of $s > 0$ with its support contained in the interval $(r - \varepsilon, r + \varepsilon)$ and with the property

$$\int_0^\infty \delta_{r,\varepsilon}(s)^2 ds = 1 \qquad (0 < \varepsilon < r) .$$

$u(\omega) \cdot \delta_{r,\varepsilon}(|k|)$ lies in $L_2(M)$ for any $u \in L_2(\Omega)$ and the unitary character of S implies

$$\int_0^\infty s^2 \delta_{r,\varepsilon}(s)^2 ds \int_\Omega |S_s u(\omega)|^2 d\omega = \| S(u \cdot \delta_{r,\varepsilon}) \|^2 = \| u \cdot \delta_{r,\varepsilon} \|^2$$

[9] See Zygmund [5], Chapter II, §6 and Chapter XVI, §1. A complete justification of this fact will be given in the Appendix.

[10] The absolute convergence of the integral defining $F(|k|, \omega, \omega')$ is seen from the fact that $q(x) \in L_1(E)$ (note that we have assumed $q(x) = o(|x|^{-3-h})$).

$$= \int_0^\infty s^2 \delta_{r,\varepsilon}(s)^2 ds \int_\Omega | u(\omega) |^2 d\omega \,.$$

Thus if we let $\varepsilon \to 0$, we obtain from the left and right sides

$$\int_\Omega | S_r u(\omega) |^2 d\omega = \int_\Omega | u(\omega) |^2 d\omega \,.$$

This shows that S_r is an isometry and hence unitary, because this also shows the nonexistence of nontrivial null-vectors. Thus we have the following

THEOREM 2. *The scattering operator S is a continuous sum of $S_{|k|}$ with the weight function $| k |^2$:*

$$|| Su ||^2 = \int_0^\infty || S_{|k|} u(| k |, \cdot) ||^2_{L_2(\Omega)} \, | k |^2 d \, | k | \,,$$

where each $S_{|k|}$ is a unitary operator on $L_2(\Omega)$ of the form $S_{|k|} = I - i \, | k | \, F_{|k|}$, $F_{|k|}$ being a completely continuous integral operator on $L_2(\Omega)$ of the Hilbert-Schmidt type.

3. Phase shifts. Phase-shift formula. Let us consider our whole problem now starting from the potential $q_\varepsilon(x) = \varepsilon q(x)$ with a real parameter ε instead of $q(x)$ and agree to add a subscript ε to everything concerned; e.g. $\varphi_\varepsilon(x, k)$, H_ε, S_ε, $S_{\varepsilon,|k|}$ etc. We have considered in [4] a Banach space B of all continuous functions $u(x)$ tending uniformly to 0 as $| x | \to \infty$, with the norm $|| u ||_B = \max_{x \in E} | u(x) |$, and operators $T_\kappa (Im\kappa \geq 0)$ on B:

$$(T_\kappa f)(x) = -\frac{1}{4\pi} \int_E \frac{e^{i\kappa|x-y|}}{| x - y |} q(y) f(y) dy \,.$$

Clearly $T_{\kappa,\varepsilon} = \varepsilon T_\kappa$ has the same properties as T_κ.

LEMMA 2. *In addition to the properties* (1.2) $\varphi_\varepsilon(x, k)$ *has the property that it depends continuously on* $\varepsilon (0 \leqq \varepsilon \leqq 1)$ *uniformly for* $x \in E$ *and* k *varying over a compact domain of* M *not including* 0.

Proof. $\varphi_\varepsilon(x, k)$ is expressible in the following way:[11]

$$\varphi_\varepsilon(x, y) = e^{ik \cdot k} + v_\varepsilon(x, k) \,,$$
$$v_\varepsilon(\cdot, k) = (I - T_{|k|,\varepsilon})^{-1} p_\varepsilon(\cdot, k) \,,$$

$$p_\varepsilon(x, k) = \varepsilon p(x, k) = -\frac{\varepsilon}{4\pi} \int_E \frac{e^{i|k|\,|x-y|}}{| x - y |} q(y) e^{ik \cdot y} dy \,.$$

[11] See [4], pp. 17-18.

It is, therefore, enough to prove the assertion for $v_\varepsilon(x, k)$. We have

$$\begin{aligned} & |v_{\varepsilon_1}(x, k) - v_{\varepsilon_2}(x, k)| \\ & \leqq \| (I - T_{|k|,\varepsilon_1})^{-1} p_{\varepsilon_1}(\cdot, k) - (I - T_{|k|,\varepsilon_2})^{-1} p_{\varepsilon_2}(\cdot, k) \|_B \\ & \leqq \| (I - T_{|k|,\varepsilon_1})^{-1} - (I - T_{|k|,\varepsilon_2})^{-1} \|_B \| p_{\varepsilon_1}(\cdot, k) \|_B \\ & \quad + \| (I - T_{|k|,\varepsilon_2})^{-1} \|_B \| p_{\varepsilon_1}(\cdot, k) - p_{\varepsilon_2}(\cdot, k) \|_B . \end{aligned}$$

As is easily seen, $\| p_{\varepsilon_1}(\cdot, k) \|_B$ is bounded and $\| p_{\varepsilon_1}(\cdot, k) - p_{\varepsilon_2}(\cdot, k) \|_B \leqq$ const. $|\varepsilon_1 - \varepsilon_2|$ in the domain of the variables specified in the lemma. On the other hand, one can see from the continuity in ε of $T_{|k|,\varepsilon}$ and from the existence of $(I - T_{|k|,\varepsilon})^{-1}$ that $(I - T_{|k|,\varepsilon})^{-1}$ is continuous in ε in the operator norm and necessarily bounded in the same domain.[12] The above inequality together with these remarks shows the asserted continuity.

LEMMA 3. *$F_\varepsilon(|k|, \omega, \omega')$ is uniformly continuous in the totality of ε, $|k|$, ω and ω' for $0 \leqq \varepsilon \leqq 1$, $0 < \alpha \leqq |k| \leqq \beta < \infty$ and $\omega, \omega' \in \Omega$.*

Proof. The continuity in $|k|$, ω and ω' follows from Lemma A in the Appendix, where we should note that the modulus of continuity is a bounded function of $\varepsilon (0 \leqq \varepsilon \leqq 1)$, which is seen from the boundedness in ε of $\varphi_\varepsilon(x, k)$ which has been incidentally shown in the statement of Lemma 2. What remains is to prove the continuity in ε. But this is a direct consequence of Lemma 2 and the absolute integrability of the defining integral for $F_\varepsilon(|k|, \omega, \omega')$, which completes the proof of the lemma.

Now we proceed to define the phase shifts appropriately. $S_{\varepsilon,|k|}$, which is defined on $L_2(\Omega)$ and corresponds to the potential $q_\varepsilon(x)$, can be expressed as $S_{\varepsilon,|k|} = I - i|k| F_{\varepsilon,|k|}$, where $F_{\varepsilon,|k|}$ is the integral operator with the kernel $F_\varepsilon(|k|, \omega, \omega')$. We shall show that $S_{\varepsilon,|k|}$ is continuous in ε and $|k|$ in the sense of the operator norm for $0 \leqq \varepsilon \leqq 1$ and $|k| > 0$. Indeed we have for $u(\omega) \in L_2(\Omega)$

$$\begin{aligned} & \| (S_{\varepsilon_1,|k_1|} - S_{\varepsilon_2,|k_2|}) u \|^2_{L_2(\Omega)} = \| (|k_1| F_{\varepsilon_1,|k_1|} - |k_2| F_{\varepsilon_2,|k_2|}) u \|^2_{L_2(\Omega)} \\ & \leqq 2(|k_1| - |k_2|)^2 \| F_{\varepsilon_1,|k_1|} u \|^2_{L_2(\Omega)} + 2|k_2|^2 \| (F_{\varepsilon_1,|k_1|} - F_{\varepsilon_2,|k_2|}) u \|^2_{L_2(\Omega)} \\ & \leqq 2(|k_1| - |k_2|)^2 \| u \|^2_{L_2(\Omega)} \int_\Omega \int_\Omega |F_{\varepsilon_1}(|k_1|, \omega, \omega')|^2 d\omega d\omega' \\ & \quad + 2|k_2|^2 \| u \|^2_{L_2(\Omega)} \int_\Omega \int_\Omega |F_{\varepsilon_1}(|k_1|, \omega, \omega') - F_{\varepsilon_2}(|k_2|, \omega, \omega')|^2 d\omega d\omega' . \end{aligned}$$

With the aid of Lemma 3 the right-hand side can be made arbitrarily small by choosing $|\varepsilon_1 - \varepsilon_2|$ and $||k_1| - |k_2||$ small enough.

Let $\delta_{\varepsilon,|k|,n}$, $n = 1, 2, \cdots$ $(|\delta_{\varepsilon,|k|,1}| \geqq |\delta_{\varepsilon,|k|,2}| \geqq \cdots \geqq 0)$ be the eigen-

[12] See the first few lines of p. 16 of [4].

values of $F_{\varepsilon,|k|}$, enumerated according to their (finite) multiplicity. In view of the continuity of $S_{\varepsilon,|k|}$, and accordingly of $F_{\varepsilon,|k|}$, the $\delta_{\varepsilon,|k|,n}$ are seen to be continuous functions of ε and $|k|$. We have $\delta_{\varepsilon,|k|,n} \to 0$ ($n \to \infty$) because of the complete continuity of $F_{\varepsilon,|k|}$. Also we can choose the associated eigenvectors $\Phi_{\varepsilon,|k|,n}(\omega) \in L_2(\Omega)$ so that they be continuous in ε and $|k|$. From the fact that $S_{\varepsilon,|k|}$ is unitary (Theorem 2) it follows that $1 - i\,|k|\,\delta_{\varepsilon,|k|,n}$ are the eigenvalues of $S_{\varepsilon,|k|}$ with absolute value equal to 1, and hence $1 - i\,|k|\,\delta_{\varepsilon,|k|,n} = e^{2i\eta_{\varepsilon,|k|,n}}$, where $\eta_{\varepsilon,|k|,n}$ are real. In the case $\varepsilon = 0$ all the eigenvalues of $S_{0,|k|}$ are 1. We can, therefore, determine $\eta_{\varepsilon,|k|,n}$ uniquely in view of the continuity in ε and $|k|$, by the condition $\lim_{\varepsilon\to 0} \eta_{\varepsilon,|k|,n} = 0$. These considerations enable us to give the following

THEOREM 3. *The following phase-shift formula for the scattering operator holds:*

$$(Su)\hat{}_0(k) = S_{|k|}u(|k|, \omega) = \sum_{n=1}^{\infty} e^{2i\eta_{|k|,n}}(u, \Phi_{|k|,n})_{L_2(\Omega)}\Phi_{|k|,n}(\omega) ,$$

where $\Phi_{|k|,n} = \Phi_{1,|k|,n}(\varepsilon = 1)$ *and where* $\eta_{|k|,n} = \eta_{1,|k|,n}(\varepsilon = 1)$ *are quantities called the phase-shifts which are uniquely determinable by the condition* $\lim_{\varepsilon\to 0} \eta_{\varepsilon,|k|,n} = 0$ *and are continuous in* $|k|$.

In the spherically symmetric potential case we can first expand the wave function into the sum of spherical harmonics components, and discuss each component radial function separately. As a result we see[13] that in each component space the scattering operator is a multiplication by a function of the form $e^{2i\eta_k}$ (k is real positive) if we pass to the (one-dimensional) Fourier transform of the radial function. In this case we can take as $\Phi_{|k|,n}(\omega)$ the spherical harmonics. Thus $\Phi_{|k|,n}(\omega)$ will play a rôle similar to that of the spherical harmonics, though the latter are independent of $|k|$.

We can also deduce a formula for the scattering cross-section $Q(|k|)$ corresponding to the situation where plane waves of the kinetic energy $|k|^2$ are incident upon the scatterer with potential $q(x)$. $Q(|k|)$ is defined as a quantity proportional to the integral over all directions ω_x of the square of the coefficient of the $|x|^{-1}$ term in the asymptotic expansion

$$\varphi(x, k) = e^{ik\cdot x} - |x|^{-1}2\pi e^{i|k||x|}F(|k|, -\omega_k, -\omega_x) + o\left(\frac{1}{|x|}\right).$$

It is not difficult to show by means of the phase-shift formula (Theorem 3) that $Q(|k|)$ is equal to $(1/|k|^2)\sum_{n=1}^{\infty} \sin^2 \eta_{|k|,n}$ up to a constant. This is

[13] Cf. Green-Lanford, III [**3**].

well known when the potential is spherically symmetric.

APPENDIX. Rigorous derivation of (2.8).

We shall give a justification of the use of the delta function in the proof of Theorem 1.

LEMMA A. *$F(r_1, \omega_1, r_2, \omega_2) \equiv \int_E e^{ir_1\omega_1\cdot x} q(x)\varphi(x, r_2, -\omega_2)dx$ is uniformly Hölder continuous in r_1 for any fixed r_2 and in r_2 for any fixed r_1, where $0 < \alpha \leqq r_1, r_2 \leqq \beta < \infty$ and $\omega_1, \omega_2 \in \Omega$.*

Proof. We first show that $v(x, r, \omega) = \varphi(x, r, \omega) - e^{ir\omega\cdot x}$ is uniformly Hölder continuous in $r(\alpha \leqq r \leqq \beta, \omega \in \Omega, x \in E)$. (For the notation see the beginning part of the proof of Lemma 2. Note that we consider now the case $\varepsilon = 1$.) We have

$$\text{(A.1)} \quad v(\cdot, r_1, \omega) - v(\cdot, r_2, \omega) = [(I - T_{r_1})^{-1} - (I - T_{r_2})^{-1}]p(\cdot, r_1, \omega) + (I - T_{r_2})^{-1}[p(\cdot, r_1, \omega) - p(\cdot, r_2, \omega)] .$$

We can see as in the proof of Lemma 2 that

$$\| (I - T_{r_1})^{-1} - (I - T_{r_2})^{-1} \|_B \leqq C_1 | r_1 - r_2 | ,$$

where C_1 is a constant independent of r_1 and r_2. On the other hand $\| p(\cdot, r_1, \omega) \|_B \leqq C_2$, C_2 being independent of r_1 and ω. Thus we have $| [(I - T_{r_1})^{-1} - (I - T_{r_2})^{-1}]p(x, r_1, \omega) | \leqq C_3 | r_1 - r_2 |$ with C_3 independent of any variable. As to the second term of (A.1) we have

$$\begin{aligned} | p(x, r_1, \omega) - p(x, r_2, \omega) | &\leqq \frac{1}{4\pi} \int_E \frac{| e^{ir_1|x-y|} - e^{ir_2|x-y|} |}{| x - y |} | q(y) | dy \\ &\quad + \frac{1}{4\pi} \int_E | e^{ir_1\omega\cdot y} - e^{ir_2\omega\cdot y} | \frac{| q(y) |}{| x - y |} dy \\ &\leqq C_4 | r_1 - r_2 | \int_E | q(y) | dy \\ &\quad + C_5 | r_1 - r_2 | \int_E \frac{| y |}{| x - y |} | q(y) | dy . \end{aligned}$$

The last integral can be seen to be bounded by a constant independent of $x \in E$ (see [4], Lemma 3.1, p. 11). $\| (I - T_{r_2})^{-1} \|_B$ is bounded independently of r_2 (see the proof of Lemma 2). Thus we have

$$| v(x, r_1, \omega) - v(x, r_2, \omega) | \leqq C_6 | r_1 - r_2 | ,$$

where C_6 does not depend on x and ω, which shows the asserted uniform Hölder continuity (with exponent equal to 1).

Now let us return to the function $F(r_1, \omega_1, r_2, \omega_2)$ which can be written as the sum $G(r_2) + H(r_2)$, where

$$G(r_2) = \int_E e^{ir_1\omega_1\cdot x + ir_2\omega_2\cdot x} q(x) dx \ ,$$

$$H(r_2) = \int_E e^{ir_1\omega_1\cdot x} q(x) v(x, r_2, \omega_2) dx \ .$$

First we estimate $|G(r_2') - G(r_2'')|$. We have

$$\begin{aligned} |G(r_2') - G(r_2'')| &\leqq \int_E |{}^{ir_2'\omega_2\cdot x} - e^{ir_2''\omega_2\cdot x}|\,|q(x)|\,dx \\ &\leqq 2^{1-\theta} |r_2' - r_2''|^\theta \int_E |x|^\theta\,|q(x)|\,dx \ , \end{aligned}$$

where we have made use of the inequality $|e^{ia} - e^{ib}| \leqq 2^{1-\theta}|a - b|^\theta$, and where $0 < \theta < h$ (see the assumption (A)) so that the integral

$$\int_E |x|^\theta\,|q(x)|\,dx$$

is finite, which in turn gives us the estimate

$$|G(r_2') - G(r_2'')| \leqq C_7 |r_2' - r_2''|^\theta$$

with C_7 independent of the other variables. For $H(r_2)$ we can easily obtain from the uniform Hölder continuity in r or $v(x, r, \omega)$ shown before, the estimate

$$|H(r_2') - H(r_2'')| \leqq C_6 |r_2' - r_2''| \int_E |q(x)|\,dx \ .$$

These two estimates together prove the lemma (the Hölder continuity in r_1 can be treated similarly).

The following lemma is concerned with Fourier's single integral formula, which is stated in the new eddition of A. Zygmund's book [5], but not in the old one. For completeness' sake we shall give a proof of it.

LEMMA B. *Let $f(x)$ be integrable over $(-\infty, \infty)$ and uniformly Hölder continuous over $[\alpha, \beta]$. Then*

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\lambda(x-y)}{x-y} f(y) dy$$

uniformly for $x \in [\alpha', \beta']$, $\alpha < \alpha' < \beta' < \beta$.

Proof. If we note the well known formula

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\lambda(x-y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} du = 1 \ ,$$

what we have to show turns out to be the following:

$$\text{(A.2)} \qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\lambda(x-y)}{x-y}[f(x)-f(y)]dy \to 0 \,, \qquad \text{as } \lambda\to\infty$$

uniformly for $x\in[\alpha',\beta']$.

We split the integral into three parts:

$$\text{(A.3)} \qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\lambda(x-y)}{x-y}[f(x)-f(y)]dy = \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} \,,$$

where we have chosen $\delta>0$ such that $x\pm\delta$ lie in $[\alpha,\beta](x\in[\alpha',\beta'])$. For any given $\varepsilon>0$ we have with θ as the exponent of Hölder continuity

$$\left|\int_{x-\delta}^{x+\delta}\right| \leqq \text{const.}\int_{x-\delta}^{x+\delta}\frac{1}{|x-y|}|x-y|^{\theta}dy = \text{const.}\frac{2\delta^{\theta}}{\theta} < \varepsilon$$

by choosing δ sufficiently small. The last term of (A.3) is the sum of two integrals:

$$\int_{x+\delta}^{\infty} = \frac{1}{\pi}f(x)\int_{x+\delta}^{\infty}\frac{\sin\lambda(x-y)}{x-y}dy - \frac{1}{\pi}\int_{\delta}^{\infty}\sin\lambda y\,\frac{f(x-y)}{y}dy \,.$$

The first integral tends uniformly to 0 as $\lambda\to\infty$, since it equals

$$\frac{1}{\pi}f(x)\int_{\lambda\delta}^{\infty}\frac{\sin u}{u}du \,.$$

As to the second integral, since

$$\int_{\delta}^{\infty}\left|\frac{f(x-y)}{y}\right|dy$$

is continuous in $x\in[\alpha',\beta']$, the Riemann-Lebesgue lemma holds uniformly with respect to x. A similar argument is applicable to the first integral on the right side of (A.3). Thus we have obtained the result (A.2).

Now let us derive the first line of (2.8). In (2.7) we have the function $F(|k_2|,\omega_2,|k_1|,\omega_1)\hat{u}_0(|k_2|,\omega_2)\overline{\hat{v}_0(|k_1|,\omega_1)}$. We first integrate with respect to ω_1 and ω_2. Then this will turn out to be a function of the form $f(|k_1|,|k_2|)$ which is uniformly Hölder continuous, for instance, in $|k_1|$ for $|k_1|,|k_2|$ in some finite interval exclusive of 0, as is seen from Lemma A and from $\hat{u}_0,\hat{v}_0$ being assumed to be in $C_0^{\infty}(M)$. Finally the application of Lemma B and the theorem on uniform convergence yields the required result.

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