PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

SAMPLE FUNCTIONS OF CERTAIN DIFFERENTIAL PROCESSES ON SYMMETRIC SPACES

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In a recent paper, we have proved a formula characterizing the abstract Fourier-Stieltjes transform of an isotropic infinitely divisible probability measures on a symmetric space. The formula is the full analogue of the classical Lévy-Khinchine formula for the Fourier-Stieltjes transform of infinitely divisible probability measures on the real line.

Now, just as in the case of the line, an isotropic, infinitely divisible probability measure on a symmetric space gives rise in a natural way to a continuous one parameter convolution semigroup of such measures; and thence to a stochastic process with stationary independent "increments". It is the purpose of this paper to construct the sample functions of such a process. We shall exhibit the sample functions of such a process as limits with probability one (*uniformly* on compact subsets of the parameter set) of sequences of continuous Brownian trajectories interlaced with finitely many isotropic Poissonian jumps.

Our construction brings out clearly the significance of the Lévy measure of the process as a measure of the expected number of jumps of the path having a given size and occurring in unit time. (See details below.) It also follows from our construction that the sample function of these processes can be assumed to have only discontinuities of the first kind. This fact, however, was known and indeed a more general result of this kind was proved in [13] by J. Woll. Thus the main new results of this paper must be considered to be the actual construction of the sample paths, and the geometric information that it gives about the process.

Our results are inspired by the work of Itô [8]. Itô considers such processes on the line. However, the noncommutativity of the groups that concern us and the nonlinear nature of our spaces force us to use techniques quite different from his. Our methods are of independent interest and indeed they can be utilized to construct a theory of "addition" of isotropic random variables taking values in symmetric spaces.

We consider in this paper only the case of a noncompact symmetric space. Surprisingly enough, the compact case is somewhat more messy in technique, due to the fact that in compact symmetric spaces the conjugate locus of a given point interferes with a routine

Received January 10, 1964.

reworking of our construction. Galmarino's result in [2] for S^2 is therefore not contained in ours. However, his method is special to S^2 and does not seem to be susceptible to generalization.

We should like to acknowledge a valuable conversation with J. M. G. Fell recently and another one with H. P. McKean Jr. over a year ago.

2. **Preliminaries.** We are forced to ask of the reader some familiarity with the theory of symmetric spaces. We shall sketch here the notation we use frequently in this paper. Any symbol to which a meaning is not given here is to have the same meaning as in [3, § 2] or [6].

G will stand for a noncompact connected semi-simple Lie group with a finite centre and K for a maximal compact subgroup of G. $\mathfrak{g}_0$, $\mathfrak{k}_0$ are the Lie algebras of G, K respectively. The Cartan-Killing form B on $\mathfrak{g}_0 \times \mathfrak{g}_0$, given by $B(X, Y) = \text{Trace } (adX \ adY)$ where $X \to adX$ is the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_0$, is nondegenerate on $\mathfrak{g}_0 \times \mathfrak{g}_0$ and if $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ w.r.t. B, we have $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, B is negative definite on $\mathfrak{k}_0 \times \mathfrak{k}_0$, positive definite on $\mathfrak{p}_0 \times \mathfrak{p}_0$. This is called a Cartan decomposition of $\mathfrak{g}_0$. $\mathfrak{p}_0$ can be identified in a natural way with the tangent space at $\pi(e)$ to the manifold G/K, where $\pi : G \to G/K$ is the natural projection and e the identity of G. The restriction of B to $\mathfrak{p}_0 \times \mathfrak{p}_0$ gives by translation, a Riemannian structure for G/K endowed with which G/K is a symmetric space; conversely every symmetric space of the noncompact type arises in this way [6]. We shall have occasion to use the Iwasawa decomposition G, described as follows: $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{h}_{\mathfrak{p}0} \oplus \mathfrak{n}_0$, where $\mathfrak{h}_{\mathfrak{p}0}$ is a maximal abelian subspace of $\mathfrak{p}_0$, $\mathfrak{n}_0$ a nilpotent subalgebra of $\mathfrak{g}_0$ and $\mathfrak{n}_0$ is an *ideal* in $\mathfrak{h}_{\mathfrak{p}0} \oplus \mathfrak{n}_0$. If $A_{\mathfrak{p}}$, N are analytic subgroups of G with Lie algebras $\mathfrak{h}_{\mathfrak{p}0}$, $\mathfrak{n}_0$ respectively then $A_{\mathfrak{p}}$, N are simply connected, $A_{\mathfrak{p}} N$ is solvable, N is normal in $A_{\mathfrak{p}}N$ and $G = KA_{\mathfrak{p}}N$ in the sense that the map $(k, a, n) \to kan$ of $K \times A_{\mathfrak{p}} \times N$ to G is an analytic diffeomorphism of $K \times A_{\mathfrak{p}} \times N$ onto G. Given $x \in G$, we denote by $k(x)$, $H(x)$, $n(x)$ the unique elements of $K, \mathfrak{h}_{\mathfrak{p}0}, N$ such that $x = k(x)(\exp H(x))n(x)$. Note that by the very definition, we have $H(xy) = H(xk(y)) + H(y)$, cf. [4]; we shall use this later.

The left and right translations $L(x)$, $R(x)$ given by $y \to xy$, $y \to yx$ respectively, induce operations on functions, measures, distributions etc. on G; e.g. $f^{L(x)}(y) = f(xy) = (f \cdot L(x))(y)$. We shall say that a function, measure etc. is *spherical* or *isotropic* on G if it is invariant under $L(k), R(k), k \in K$ i.e. $f^{L(k)} = f^{R(k)} = f$.

A G-valued random variable (r.v.) ξ is a map of a probability space $(\Omega, \mathscr{B}, P)$ to G, measurable relative to the σ-field of Borel subsets of G; ξ carries the measure on Ω to a probability measure F_ξ on G which we call the distribution of ξ. A r.v. is called spherical if its distribution is spherical. It is easily seen that a G-valued spherical r.v. can be thought of as a G/K valued r.v. whose distribution is invariant under the left action of K, and conversely. We find it more convenient to work on G as far as possible. *All r.v.s in this paper will be G-valued spherical r.v.s unless expressly stated otherwise.*

Given independent r.v.s ξ_1, ξ_2 their product $\xi_1 \cdot \xi_2$ is also a r.v., and $F_{\xi_1 \cdot \xi_2} = F_{\xi_1} \cdot F_{\xi_2}$, the product on the right being convolution. It will be important to us that if ξ_1, ξ_2 are independent and spherical then $F_{\xi_1 \cdot \xi_2} = F_{\xi_2 \cdot \xi_1}$, because convolution of spherical measures is commutative [6].

The theory of spherical functions developed by Harish-Chandra [5] and others is a powerful tool for analysis on symmetric spaces. For $\nu \in E_C$ the space of complex valued linear functionals on $\mathfrak{h}_{\mathfrak{p}0}$, let $\varphi_\nu(x)$ be the corresponding *elementary* spherical function (see [3, § 3] where a fuller description is found), defined by

$$\varphi_\nu(x) = \int_K \exp\left[i\,\nu(H(xk)) - \rho(H(xk))\right] dk \tag{2.1}$$

where ρ is the half-sum of the positive roots. Then $\varphi_\nu(e) = 1$, and $\int_K \varphi_\nu(xky)dk = \varphi_\nu(x) \cdot \varphi_\nu(y)$. φ_ν is analytic and is an eigenfunction of every differential operator on G which commutes with left translations by elements of G and right translations by elements of K. For a spherical measure μ on G, we defined in [3] its Fourier-Stieltjes transform by

$$\hat{\mu}(\lambda) = \int_G \varphi_\lambda(x) d\mu(x),\ \lambda \in E_R \tag{2.2}$$

E_R being the real valued linear functionals on $\mathfrak{h}_{\mathfrak{p}0}$. It was shown in [3] that $\hat{\mu}$ determines μ. Further, calling a spherical probability measure μ infinitely divisible if for each integer n, $\mu = \theta^n$ where θ is a spherical probability measure and the product is the convolution product, we proved in [3] the theorem:

THEOREM 2.1. *A spherical probability measure μ on G is infinitely divisible if and only if*

$$\hat{\mu}(\lambda) = \exp\left\{P_D(\lambda) - \int_{|x|>0} [1 - \varphi_\lambda(x)]\, dL(x)\right\} \tag{2.3}$$

where L is a spherical measure on G such that $\int [|x|^2/1 + |x|^2] dL(x) < \infty$, and $P_D(\lambda)$ is the eigenvalue, corresponding to the eigenfunction φ_λ, of an elliptic second order differential operator in $D(G/K)$ which *annihilates constants.*

Here, $|x|$ stands for the distance of xK from eK in the natural metric on G/K.

A few consequences of Theorem 2.1 may be noted here as they will be needed in § 5.

First of all, as observed in [3], given a spherical probability measure μ, $\hat{\mu}$ may be defined not only for $\lambda \in E_R$ by $\int \varphi_\lambda(x)d\mu(x)$, but also for all those $\nu \in E_C$ for which $\int \varphi_\nu(x)d\mu(x)$ makes sense. For $\lambda \in E_R$ consider the linear functional $\lambda - i\rho \in E_C$. It is easy to check that $\int \varphi_{\lambda - i\rho}(x)d\mu(x)$ makes sense. Indeed, it is equal to $\int \exp i\lambda(H(x))d\mu(x)$; so that if a random variable ξ has the distribution μ, then $\hat{\mu}(\lambda - i\rho) = \int \varphi_{\lambda - i\rho}(x)d\mu(x)$ is the ordinary Fourier-Stieltjes transform of the distribution of the $\mathfrak{h}_{\mathfrak{p}0}$-valued random variable $H(\xi)$.

In particular, if ξ is a r.v. whose distribution F_ξ is infinitely divisible, then one may conclude that, D, L being as above, the distribution of $H(\xi)$ has the ordinary Fourier-Stieltjes transform given by, $(\lambda \in E_R)$,

$$(2.4)\quad (F_\xi)(\lambda - i\rho) = \exp\left\{P_D(\lambda - i\rho) - \int_{|x|>0} [1 - \varphi_{\lambda - i\rho}(x)]\, dL(x)\right\}$$
$$= \exp\left\{P_D(\lambda - i\rho) - \int_{|x|>0} [1 - \exp i\lambda(H(x))] dL(x)\right\}$$

using (2.1).

It follows from this that the real and imaginary parts of

$$\int_{|x|>0} [1 - \exp i\lambda(H(x))] dL(x)$$

both exist as convergent integrals, from which,
since $\sin z \sim z$ and $1 - \cos z \sim z^2$ near $z = 0$, it is possible to conclude that the integrals

$$\int_{0<|x|<1} H(x)dL(x), \int_{0<|x|<1} \| H(x) \|^2 dL(x)$$

both exist. (Here, as elsewhere $\|\ \|$ stands for the norm on $\mathfrak{h}_{\mathfrak{p}0}$ given by the Cartan-Killing form.) This remark will be crucial in § 5.

Lastly, it is not hard to show, though we did not mention it in [3], that the only second order elliptic differential operators in $D(G/K)$ which annihilate constants are just positive multiples of the Laplace-Beltrami operator of G/K, if we assume G/K irreducible, which we may do with impunity. Indeed if $D = (c/2)\Delta$ where $c > 0$ and Δ is the Laplace-Beltrami operator, then $P_D(\lambda) = (-c/2)\{\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle\}$ where $\langle \lambda, \lambda \rangle$ is the inner product given by the Cartan-Killing form.

We may normalize $c = 1$ for future purposes.

As a final remark on notation, for a real linear functional $\lambda \in E_R$, we denote by H_λ the unique element $\in \mathfrak{h}_{\mathfrak{p}_0}$ such that $B(H_\lambda, H) = \lambda(H)$; $H \in \mathfrak{h}_{\mathfrak{p}_0}$. Thus $\langle \lambda, \lambda \rangle = B(H_\lambda, H_\lambda) = \lambda(H_\lambda)$.

3. Brownian paths interlaced with isotropic Poisson jumps.

DEFINITION 3.1. Let $X = \{x(t), t \in [0, \infty)\}$ be a separable stochastic process. We say that X is differential if given $0 \leqq t_1 < t_2 < t_3 < t_4 < \infty$, $x(s)^{-1}x(t)$ and $x(u)^{-1}x(v)$ are independent for all $s, t \in [t_1, t_2]$ and $u, v \in [t_3, t_4]$, $s \leqq t$, $u \leqq v$. It is said to have stationary increments if $x(0)^{-1}x(s)$ and $x(t)^{-1}x(t+s)$ have the same distribution for any $t, s \geqq 0$.

If X is differential with stationary increments and if F^t is the distribution of $x(0)^{-1}x(t)$, it follows without difficulty that F^t is infinitely divisible and by applying Theorem 2.1, we conclude that for $\lambda \in E_R$,

$$(3.1) \qquad (\hat{F}^t)(\lambda) = \exp t \left\{ P_D(\lambda) - \int_{|x|>0} [1 - \varphi_\lambda(x) dL(x)] \right\} .$$

as in § 2.

DEFINITION 3.2. We say that $X = \{x(t), t \in [0, \infty)\}$ is a Gauss process if $L \equiv 0$ above and that it is a Lévy process if $D = 0$.

A Gauss process $b(t)$ is none other than a Brownian motion on G/K, essentially, i.e. $\pi(b(t))$ is a Brownian motion on G/K as defined in [7] for example. The properties of such a process are well known. It can be seen e.g. by appealing to [7], [12] that if $B = \{b(t), t \in [0, \infty)\}$ is a Brownian motion, then it has continuous sample functions almost surely. By its very definition,[1]

$$E(\varphi_\lambda(b(0)^{-1}b(t)) = \exp tP_D(\lambda) = \exp - \frac{c}{2} t\{\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle\}$$

cf. § 2. Conversely given a second order elliptic differential operator $D \in D(G/K)$ which annihilates constants, $D = c/2D$ and one may construct as in [7] or [11] the process $B = \{b(t), t \in [0, \infty)\}$ which stands in the above relation to D. We assume $c = 1$ without loss of generality.

Given (D, L) as described above, we first of all construct a Brownian motion $B = \{b(t), t \in [0, \infty)\}$ given by D as above. Let $\{\Omega', P'\}$ be the sample space for this process. We shall now explain what we mean be interlacing a Brownian path with an isotropic jump of *size x at time t_1*. Let $x \in G$, and $t \to b(t)$ be a Brownian path. The path $b(t)$ is to be unchanged for times $t < t_1$. At time t_1 it jumps

[1] E stands for the expectation operator.

to $b(t_1)kxk'$ with probability $dk\,dk'$. After time t_1 it continues as before so that at time $t \geqq t_1$ it is at $b(t_1)kxk'b(t_1)^{-1}b(t)$, with probability $dk\,dk'$. It is clear that to describe this more rigorously we shall have to enlarge the sample space a little. This will be done in detail later. For a jump of size x we shall refer to $|x|$ as the *length* of the jump.

Given the Lévy measure L on $G-K$, denote by $dt \times L$ the product measure on $[0, \infty) \times G - K$, where dt is Lebesgue measure on $[0, \infty)$. In what follows, we denote by B^* a Borel subset of $[0, \infty) \times G - K$ which has the following two properties

(i) B^* has finite $dt \times L$ measure

(ii) there exists a $\delta > 0$ such that if $(t, x) \in B^*$, then $|x| \geqq \delta$.

We now describe what we mean by stationary random selection of points of $G-K$ according to the measure L; namely, we select randomly points $(\tau, x) \in [0, \infty) \times G - K$ so that

(3.2) Given $B^* \subset [0, \infty) \times G - K$, the number of selected points which lie in B^* is a random variable with a Poisson distribution whose mean is $(dt \times L)(B^*)$.

(3.3) If $B_i^*, i = 1, \cdots l$ are disjoint subsets of $[0, \infty) \times G - K$ then the numbers $\#_i$ of selected points which lie in B_i^* are all mutually independent random variables.

See in this connection [9]. Since $dt \times L$ is σ-finite, it is almost sure that we shall select only countably many points. The above description can be formalized somewhat awkwardly as follows. We let our sample space Ω'' be the space of all sequences $\omega'' = \{(\tau_j, x_j)\}_{j=1}^{\infty}$ (with $(\tau_j, x_j) \in [0, \infty) \times G - K$) such that

$$|x_{j+1}| \leqq |x_j| \text{ and } \tau_j < \tau_{j+1}\, j = 1, 2, \cdots .$$

(This ordering is for convenience only.) We can build on this space a Probability measure P'' by requiring

$$(3.4) \qquad \begin{aligned} &P''(\omega'' \mid \# B^*(\omega'') = l) \\ &= \exp - (dt \times L)(B^*) \quad ((dt \times L)(B^*)^l / l\,!) \end{aligned}$$

$$(3.5) \qquad \begin{aligned} &P''(\bigcap_{i=1}^{r} \{\omega'' \mid \# B_i^*(\omega'') = l_i)\}) \\ &= \prod_{i=1}^{r} P''(\omega'' \mid \# B_i^*(\omega'') = l_i)\,, \end{aligned}$$

$$\text{if} \quad B_i^* \bigcup B_j^* = \Phi \text{ when } i \neq j$$

where $\# B^*(\omega'')$ stands for the number of terms (τ_j, x_j) in the sequence $\omega'' = \{(\tau_j, x_j)\}_{j=1}^{\infty}$ which belong to B^*, and $l, l_i, i = 1, \cdots r$ are non-negative integers. P'' can now be extended to an appropriate Borel field of subsets of Ω'' in the usual way. We propose to omit specific mention of the underlying σ-field without risk of confusion.

Given the Brownian motion $\{b(t), t \in [0, \infty)\}$ our idea is to make a stationary random selection of points (τ_j, x_j) according to the Lévy measure L, independently of the Brownian motion and then to interlace the Brownian path by isotropic jumps of sizes x_j at times τ_j $j = 1, \cdots$. As seen above, this involves choices of k_j, k'_j at each interlacing. Precisely, let $\Omega = \Omega' \times \Omega'' \times \prod_{i=1}^{\infty} K \times \prod_{i=1}^{\infty} K$ and endow it with the product measure $P = P' \times P'' \times \prod dk \times \prod dk$ where dk is the normalized Haar measure of K. The space (Ω, P) will serve for our construction. A generic point $\omega \in \Omega$ will now furnish us with a Brownian path $t \to b(t, \omega)$; a sequence $(\tau_j(\omega), x_j(\omega))_{i=1}^{\infty}$ of points of $[0, \infty) \times G - K$, (selected stationarily and randomly according to the measure L); and sequences $k_j(\omega), k'_j(\omega), j = 1, \cdots$ of points of K, these last being distributed uniformly over K; all these random objects being independent mutually. For typographical convenience we shall omit the underlying point ω.

Now let δ_n be a sequence such that $1 \geqq \delta_n \geqq \delta_{n+1} > 0$ and $\delta_n \to 0$ as $n \to \infty$. For any real number $\alpha > 0$, let

$$B^*(\alpha, t) = \{(s, x) \mid (s, x) \in [0, \infty) \times G - K, s \leqq t, |x| > \alpha\} .$$

Given $\omega \in \Omega$, let $b(t)$, $\{(\tau_j, x_j)\}_{j=1}^{\infty}, k_j, k'_j$ be the items that it furnishes. Let $j(\alpha, t)$ be the largest integer j such that $(\tau_l, x_l) \in B^*(\alpha, t)$ for all $l \leqq j$. Now define for $\alpha \geqq 1$, $n = 1,2, \cdots$.

$$\begin{aligned} y_n^{\alpha}(t) = b(0)^{-1}b(\tau_{j(\alpha,t)+1})\cdot k_{j(\alpha,t)+1}\cdot x_{j(\alpha,t)+1}\cdot k'_{j(\alpha,t)+1} \\ \cdot b(\tau_{j(\alpha,t)+1})^{-1}\cdot b(\tau_{j(\alpha,t)+2})\cdot k_{j(\alpha,t)+2}\cdot x_{j(\alpha,t)+2} \\ \cdot k'_{j(\alpha,t)+2}\cdots k_{j(\delta_n,t)}\cdot x_{j(\delta_n,t)}\cdot k'_{j(\delta_n,t)}\cdot b(\tau_{j(\delta_n,t)})^{-1}b(t) \end{aligned} \tag{3.6}$$

$$\begin{aligned} y_n(t) = b(0)^{-1}b(\tau_1)k_1x_1k'_1b(\tau_1)^{-1}b(\tau_2)k_2x_2k'_2\cdots \\ \cdots k_{j(\delta_n,t)}\cdot x_{j(\delta_n,t)}\cdot k'_{j(\delta_n,t)}b(\tau_{j(\delta_n,t)})^{-1}b(t) . \end{aligned} \tag{3.7}$$

In spite of its unprepossessing appearance, $t \to y_n(t)$ is just the Brownian path interlaced with independent isotropic jumps of length $> \delta_n$ occurring before time t, selected stationarily randomly according to the Lévy measure L; while $y_n^{\alpha}(t)$ is just the Brownian path into which those jumps of lengths between δ_n and α have been interlaced. It is clear that $j(\alpha, t)$ is finite with probability one because of (2.3), so that the path $t \to y_n(t)$ has with probability one finitely many jumps in $[0, t]$ at times $\tau_1, \tau_2, \cdots \tau_{j(\delta_n,t)}$.

We shall show below that the sequence $\pi(y_n(t))$ will converge as $n \to \infty$ with probability one uniformly on compact subsets of $[0, \infty)$ to a process whose law is given by (3.1).

4. Convergence in distribution.

THEOREM 4.1

$$E(\varphi_\lambda(y_n(t))) = \exp t\left\{P_D(\lambda) - \int_{|x|>\delta_n}[1-\varphi_\lambda(x)]dL(x)\right\}. \tag{4.1}$$

Proof. Since $\{b(t), t \in [0, \infty)\}$ is differential and since (τ_j, x_j), k_j, k_j', $\{b(t), t \in [0, \infty)\}$ are mutually independent, it follows that the r.v.s

$$b(0)^{-1}b(\tau_1)\,;\ k_1x_1k'_1\,;\ b(\tau_1)^{-1}b(\tau_2)\,;\ k_2x_2k'_2\cdots$$

are mutually independent and spherical. Hence their distributions commute under convolution; cf. our remarks in § 2. Therefore the distribution of $y_n(t)$ is the same as that of

$$b(0)^{-1}b(t)k_1x_1k'_1\cdot k_2x_2k'_2\cdots k_{j(\delta_n,t)}x_{j(\delta_n,t)}k'_{j(\delta_n,t)}\,.$$

Therefore

$$\begin{aligned} E(\varphi_\lambda(y_n(t))) &= E(\varphi_\lambda(b(0)^{-1}b(t)k_1x_1k'_1\, k_2x_2k'_2\cdots k_{j(\delta_n,t)}x_{j(\delta_n,t)}k'_{j(\delta_n,t)})) \\ &= E(\varphi_\lambda(b(0)^{-1}b(t)))\ E\left(\prod_{j=1}^{j(\delta_n,t)}\varphi_\lambda(k_jx_jk'_j)\right) \\ &= E(\varphi_\lambda(b(0)^{-1}b(t)))\ E\left(\prod_{j=1}^{j(\delta_n,t)}\varphi_\lambda(x_j)\right) \end{aligned} \tag{4.2}$$

because φ_λ is spherical. Since

$$E(\varphi_\lambda(b(0)^{-1}b(t)) = \exp\{tP_D(\lambda)\}\,, \tag{4.3}$$

it remains to compute $E\left(\prod_{j=1}^{j(\delta_n,t)}\varphi_\lambda(x_j)\right)$.

Denote by $C^*(n,\alpha)$ the set $\{x \mid \delta_n < |x| \leqq \alpha\}$, α being a real number $\geqq 1$ say. Then it is clear that

$$[0,t]\times C^*(n,\alpha) = B^*(\delta_n,t) - B^*(\alpha,t), \text{ and so, } E\left(\prod_{j=1}^{j(\delta_n,t)}\varphi_\lambda(x_j)\right)$$

$$= \lim_{\alpha\to\infty} E\left(\prod_{j=j(\alpha\ t)}^{j(\delta_n\ t)}\varphi_\lambda(x_j)\right) \text{ recall } j(\infty,t)=1)\,.$$

Let now, $\Delta_1, \Delta_2, \cdots, \Delta_{p_\alpha}$ be a partition of the closure of $C^*(n,\alpha)$ by subsets of diameter $\leqq \varepsilon$, ε being >0, and let $z_1, \cdots z_{p_\alpha} \in G$ be points in $\Delta_1, \cdots, \Delta_{p_\alpha}$. It is known that φ_λ is a uniformly continuous function on G, therefore we can conclude that $|\varphi_\lambda(z_p) - \varphi_\lambda(z'_p)| = o(\varepsilon)$ as $\varepsilon \to o$, uniformly for $z'_p \in \Delta_p$ $1 \leqq p \leqq p_\alpha$, $\alpha = 1, 2, \cdots$.

Suppose that exactly l_p of the points

$$(\tau_j, x_j),\ j = j(\alpha,t)+1, \cdots, j(\delta_n,t)$$

fall in $[0, t] \times \Delta_p$, $p = 1, \cdots, p_\alpha$. The probability of this event is

$$\prod_{p=1}^{p_\alpha} \exp - tL(\Delta_p)\cdot(\{tL(\Delta_p)\}^{l_p}/l_p\,!) \tag{4.4}$$

On the other hand, if exactly l_p of these points are in Δ_p, the product

$$\prod_{j=j\,(\alpha,t)}^{j(\delta_n\,t)} \varphi_\lambda(x_j) = \prod_{p=1}^{p_\alpha} \{\varphi_\lambda(z_p) + o(\varepsilon)\}^{l_p}$$

by our remark above. We therefore have

$$\begin{aligned} &E\left(\prod_{j=j(\alpha,t)}^{j(\delta_n,t)} \varphi_\lambda(x_j)\right) \\ &= \lim_{\varepsilon\to 0} \sum_{1,\cdots,p_\alpha=0}^{\infty} \prod_{p=1}^{p_\alpha} \{\varphi_\lambda(z_p) + o(\varepsilon)\}^{l_p} \exp - \{tL(\Delta_p)\}\,\{tL(\Delta_p)\}^{l_p}/l_p\,! \\ &= \lim_{\varepsilon\to 0} \prod_{p=1}^{p_\alpha} \exp\,[- tL(\Delta_p) + tL(\Delta_p)\{\varphi_\lambda(z_p) + o(\varepsilon\}] \\ &= \lim_{\varepsilon\to 0} \exp - \ t\left\{ \sum_{p=1}^{p_\alpha} [1 - \varphi_\lambda(z_p)]L(\Delta_p) + o(\varepsilon)L(C^*(n, \alpha) \right\} \\ &= \exp\left\{\lim_{\varepsilon\to 0} - t\left\{ \sum_{p=1}^{p_\alpha} (1 - \varphi_\lambda(z_p))L(\Delta_p) + o(\varepsilon)L(C^*(n, \alpha))\right\}\right\} \\ &= \exp - t\int_{\delta_n<|x|\leq\alpha} [1 - \varphi_\lambda(x)]dL(x)\,. \end{aligned} \tag{4.5}$$

Letting $\alpha \to \infty$, and remembering (4.2), (4.3) we have the assertion of the lemma.

COROLLARY 4.1

$$\lim_{n\to\infty} E(\varphi_\lambda(y_n(t))) = \exp t\left\{ P_D(\lambda) - \int_{|x|>0} [1 - \varphi_\lambda(x)]dL(x)\right\}.$$

5. Convergence with probability one. We begin with a lemma which has independent interest.

LEMMA 5.1. *Suppose we have a sequence y_n, $n = 1, 2, \cdots$ contained in a compact subset A of G such that for each $z \in G$ and each positive definite elementary spherical function Φ, the sequence $\Phi(zy_n)$ converges. Then the sequence $\pi(y_n)$ converges in G/K where π is the natural projection of G onto G/K.*

Proof. Suppose f is a continuous function with compact support in G such that $f(xk) = f(x)$ for all $k \in K$. We first of all claim that such an f can be approximated uniformly on its support B by a finite linear combination of left translates of a spherical function. To see this let f_n be an approximate identity in $L_1(G)$. Then as is well known, $f \cdot f_n \to f$ uniformly on B, where the dot stands for the convolution. Let now $F_n(x) = \int_K \int_K f_n(kxk')dkdk'$. Then F_n is spherical. Further, using the fact that $f(xk) = f(x)$ for $k \in K$, it can be easily shown that $f \cdot F_n \to f$ uniformly on B as $n \to \infty$. Since

$$(f \cdot F_n)(x) = \int_G f(y)F_n(y^{-1}x)dy \ ,$$

and since B is compact, we may approximate this last integral by suitable Riemann sums to get the following conclusion. Given $\varepsilon > 0$, there exist complex numbers $a_1 \cdots a_r$, elements $g_1 \cdots g_r \in G$ and a function $F \in C(G)$ with $F(kxk') = F(x)$ for $k, k' \in K$ such that

$$|f(x) - \sum_{i=1}^{r} a_i F(g_i x)| < \varepsilon \ \ x \in B \ . \tag{5.1}$$

We next claim that if F is any function $\varepsilon C(G)$ such that $F(kxk') =$ F(x) $k, k' \in K$, then given $\varepsilon > 0$ and a compact subset $C \subset G$, there exist complex numbers $b_1 \cdots b_l$ and elementary positive definite spherical functions $\Phi_1, \cdots, \Phi_l$ such that

$$|F(x) - \sum_{i=1} b_i \Phi_i(x)| < \varepsilon \quad x \in C \ . \tag{5.2}$$

To see this let $D = \{k_1 x k_2 \mid k_1, k_2 \in K, x \in C\}$; then D is compact. By Godement's theorem [10, p. 403] there exist complex numbers $b_1, \cdots, b_m$ and elementary positive definite functions $\Phi_1 \cdots \Phi_m$ on G such that

$$|F(g) - \sum_{i=1}^{m} b_i \Phi_i(g)| < \varepsilon \quad g \in D \ . \tag{5.3}$$

Then if $x \in C$, we have

$$\left| \int_K \int_K F(kxk')dk\,dk' - \sum_{i=1}^{m} b_i \int_K \int_K \Phi_i(kxk')dkdk' \right| < \varepsilon \ . \tag{5.4}$$

If we suppose that of the functions $\Phi_1 \cdots \Phi_m$ the first l (say) are of class 1 (see [6, p. 414] for the definition) it is easy to show that $\Phi_i(kxk') = \Phi_i(x)$ $1 \leqq i \leqq l$, $k, k' \in K$ and $\int_K \int_K \Phi_i(kxk')dkdk' = 0$ for $l+1 \leqq i \leqq m$. Since $F(kxk') = F(x)$ $k, k' \in K$, it follows from (5.4) that

$$|F(x) - \sum_{i=1}^{l} b_i \Phi_i(x)| < \varepsilon \quad x \in C \tag{5.5}$$

which is what we claimed.

Putting together (5.1) and (5.5) we see that given $\varepsilon > 0$, we can find complex numbers $c_1, \cdots, c_m$, elements $z_1, \cdots, z_m \in G$ and elementary positive definite spherical functions $\Phi_1 \cdots \Phi_m$ such that

$$(5.6) \qquad |f(x) - \sum_{i=1}^{m} c_i\Phi_i(z_i x)| < \varepsilon \quad x \in B .$$

Now we turn to the proof of the Lemma. $\pi(A)$ is compact in G/K. Suppose that the sequence $\pi(y_n)$ has two cluster points say y' y'' in $\pi(A)$, $y' \neq y''$. We can find a nonzero function $\tilde{f}$ continuous with compact support in G/K such that y', y'' ε the interior of supp $\tilde{f}$ and $\tilde{f}(y') \neq \tilde{f}(y'')$. Let $f = \tilde{f}\circ\pi$. Then f is continuous on G, we may take it to have compact support in G, and $f(xk) = f(x)$; $k \in K$. By (5.6), we have, given $\varepsilon < 0$

$$(5.7) \qquad |f(y) - \sum c_i\Phi_i(z_i y)| < \varepsilon \quad y \in \text{supp f} .$$

If $y_{n'}, y_{n''}$ are subsequences of y_n such that $\pi(y_{n'}) \to y'$, $\pi(y_{n''}) \to y''$, then since we know by hypothesis that $\Phi_i(z_i\, y_n)$ converges for each i as $n \to \infty$, (5.7) implies easily that $\tilde{f}(y') = \tilde{f}(y'')$, contradicting the choice of f. Thus $\pi(y_n)$ cannot have two distinct cluster points and the lemma is proved.

LEMMA 5.2. *Suppose that ξ_1, ξ_2 are independent r.v.'s; let $z \in G$. Then the $\mathfrak{h}_{\mathfrak{p}_0}$-valued random variable $H(z\xi_1\,\xi_2)$ has the same distribution as $H(z\xi_1) + H(\xi_2)$.*

Proof. For $\lambda \in E_R$

$$\begin{aligned}(5.8) \qquad & E(\exp i\lambda(H(z\xi_1\,\xi_2))) \\ &= \iint \exp i\lambda(H(zxy))dF_{\xi_1}(x)dF_{\xi_2}(y) \\ &= \iint \exp i\lambda(H(zxk(y)) + H(y))dF_{\xi_1}(x)\,dF_{\xi_2}(y) \\ &= \int \exp i\lambda(H(y))\cdot\Big[\int \exp i\lambda(H(zxk(y)))dF_{\xi_1}(x)\Big]\,dF_{\xi_2}(y) .\end{aligned}$$

The integral in the square brackets is seen to be independent of y on making the substitution $x \to xk(y)^{-1}$ and remembering $F_{\xi_1}^{R(k)} = F_{\xi_1}$, $k \in K$. Hence

$$\begin{aligned} & E(\exp i\lambda(H(z\xi_1\xi_2)) \\ (5.9) \qquad &= \int \exp i\lambda(H(y))dF_{\xi_2}(y)\int \exp i\lambda(H(zx))dF_{\xi_1}(x) \\ &= E(\exp i\lambda(H(\xi_2)))\cdot E(\exp i\lambda(H(z\xi_1))) ,\end{aligned}$$

which finishes the proof.

REMARK. Only the fact that $F_{\xi_1}^{R(k)} = F_{\xi_1}$ is used in the above proof, and not the *sphericity* of F_{ξ_1}. This remark becomes important when general nonspherical random variables taking values in G/K are considered.

COROLLARY 5.3. *If $\xi_i, i = 1, \cdots, n$ are mutually independent r.v.'s and $z \in G$, then $H(z\xi_1\xi_2 \cdots \xi_n)$ has the same distribution as $H(z\xi_1) + H(\xi_2) \cdots + H(\xi_n)$.*

LEMMA 5.4. *Let ξ_i, η_i $i = 1, \cdots, n$ be mutually independent r.v.'s such that $E(H(\xi_i)), E(H(\eta_i))$ exist. Let $z \in G, k \in K$ be fixed. Then,*

$$\begin{aligned}(5.10) \qquad & E(H(z\xi_1\eta_1\xi_2\eta_2) \cdots \xi_n\eta_n k) \mid \xi_1, \xi_2, \cdots, \xi_n) \\ &= H(z\xi_1\xi_2 \cdots \xi_n k) + E(H(\eta_1\eta_2 \cdots \eta_n)) .\end{aligned}$$

Proof. It will be enough to show that if $f_1, f_2, \cdots, f_n$ are bounded complex valued Borel functions on G then

$$\begin{aligned}(5.11) \qquad & E(f_1(\xi_1)f_2(\xi_2) \cdots f_n(\xi_n)H(z\xi_1\eta_1\xi_2\eta_2 \cdots \xi_n\eta_n k)) \\ &= E(f_1(\xi_1) \cdots f_n(\xi_n)H(z\xi_1\xi_2 \cdots \xi_n k)) \\ &\quad + E(f_1(\xi_1) \cdots f_n(\xi_n))E(H(\eta_1\eta_2 \cdots \eta_n)) .\end{aligned}$$

The left side

$$\begin{aligned}(5.12) \qquad = \iint \cdots \int f_1(x_1) \cdots f_n(x_n)H(zx_1y_1 \cdots x_ny_nk)dF_{\xi_1}(x_1) \cdots \\ dF_{\xi_n}(x_n)dF_{\eta_1}(y_1) \cdots dF_{\eta_n}(y_n) .\end{aligned}$$

Because of the sphericity of $F_{\xi i}, F_{\eta_i}$, this integral is invariant under the substitution $x_i \to x_ik_i, k_i \in K$. Hence we can and do assume to begin with that $f_i(x) = f_i(xk)$ $x \in G, k \in K$, (if necessary by replacing $f_i(x)$ by $\int_K f_i(xk)dk$). Now in the above integral let us subject the y_i to the substitutions $y_i \to k_iy_ik'_i$. Remembering that F_{ξ_i} is spherical we get that the left side of (5.11) (call it I)

$$\begin{aligned} &= \iint \cdots \int f_1(x_1) \cdots f_n(x_n) \cdot H(zx_1k_1y_1k'_1x_2k_2y_2k'_2 \cdots x_nk_ny_nk'_nk) \\ &\quad dF_{\xi_1}(x_1) \cdots dF_{\xi_n}(x_n) \cdot dF_{\eta_1}(y_1) \cdots dF_{\eta_n}(y_n) . \\ (5.13) \qquad &= \iint \cdots \int f_1(x_1) \cdots f_n(x_n) \cdot \iint \cdots \int \\ &\quad H(zx_1k_1y_1k'_1\, x_2k_2y_2k'_2 \cdots x_nk_ny_nk'_nk)dk_1 \cdots dk_ndk'_1 \cdots dk'_n \\ &\quad dF_{\xi_1}(x_1) \cdots dF_{\xi n}(x_n) \cdot dF_{\eta_1}(y_1) \cdots dF_{\eta_n}(y_n) .\end{aligned}$$

Now, using $H(xy) = H(xk(y)) + H(y)$, it is quite easy to check that $\iint H(xkyk')\,dk\,dk' = \int H(xk)dk + \int H(yk')dk'$, so

$$\begin{aligned}
&\iint\cdots\int H(zx_1k_1y_1k_1'\cdots y_nk_n'k)dk_1\cdots dk_n dk_1'\cdots dk_n' \\
&= \int H(zx_1k_1)\,dk_1 + \int H(x_2k_2)dk_2 \cdots + \int H(x_nk_n)dk_n \\
&\quad + \int H(y_1k_1')dk_1' \cdots + \int H(y_nk_n')dk_n' \qquad (5.14)\\
&= \iint\cdots\int H(zx_1k_1x_2k_2\cdots x_nk_n)dk_1dk_2\cdots dk_n \\
&\quad + \iint\cdots\int H(y_1k_1'y_2k_2'\cdots y_nk_n')dk_1'\cdots dk_n'
\end{aligned}$$

$$\begin{aligned}
I = &\iint\cdots\int f_1(x_1)\cdots f_n(x_n)\Big[\iint\cdots\int H(z_1x_1k_1x_2k_2\cdots x_nk_n) \\
&dk_1\cdots dk_n\Big]dF_{\xi_1}(x_1)\cdots dF_{\xi_n}(x_n)dF_{\eta_1}(y_1)\cdots dF_{\eta_n}(y_n) \qquad (5.15)\\
&+ \iint\cdots\int f_1(x_1)f_2(x_2)\cdots f_n(x_n)\Big[\iint\cdots\int H(y_1k_1'\cdots y_nk_n') \\
&dk'_1\cdots dk'_n\Big]dF_{\xi_1}(x_1)\cdots dF_{\xi_n}(x_n)dF_{\eta_1}(y_1)\cdots dF_{\eta_n}(y_n)\,.
\end{aligned}$$

Using the fact that F_{η_i} are probability measures and that $f_i(x) = f_i(xk)$ for all $k \in K$, we have

$$\begin{aligned}
I = &\iint\cdots\int f_1(x_1)\cdots f_n(x_n)H(zx_1x_2\cdots x_nk)dF_{\xi_1}(x_1)\cdots dF_{\xi_n}(x_n) \\
&+ \Big[\iint\cdots\int f_1(x_1)\cdots f_n(x_n)dF_{\xi_1}(x_1)\cdots dF_{\xi_n}(x_n)\Big] \\
&\times\Big[\iint\cdots\int H(y_1y_2\cdots y_n)dF_{\eta_1}(y_1)\cdots dF_{\eta_n}(y_n)\Big] \qquad (5.16)\\
&= E(f_1(\xi_1)f_2(\xi_2)\cdots f_n(\xi_n)H(z\xi_1\xi_2\cdots\xi_nk)) \\
&+ E(f_1(\xi_1)\cdots f_n(\xi_n))E(H(\eta_1\eta_2\cdots\eta_n))\,,
\end{aligned}$$

which is (5.11). (Note that in the first step of (5.16) we introduced k with impunity because in (5.15) the measure dk_n is right invariant.)

COROLLARY 5.5. *If $\xi_i, i = 1, \cdots, n$ are independent r.v.'s such that $E(H(\xi_i))$ exists, then if $z \in G, k \in K$ the sequence*

$$Z_n = H(z\xi_1\cdots\xi_nk) - E(H(z\xi_1\cdots\xi_nk))$$

is a martingale sequence.

The proof follows directly from the lemma.

LEMMA 5.6. *Let $\{b(t), t \in [0, \infty)\}$ be the Brownian motion. Then*

$$E(H(b(0)^{-1}b(t)) = tH_\rho\,. \tag{5.17}$$

Proof.

$$\begin{aligned} &E(\exp i\lambda(H(b(0)^{-1}b(t)))) \\ &= \int_G \exp i\lambda(H(x))\cdot dF_{b(0)^{-1}b(t)}(x) \\ &= \int_G\int_K \exp i\lambda(H(xk))dk\cdot dF_{b(0)^{-1}b(t)}(x) \\ &= \int_G \varphi_{\lambda-i\rho}(x)\cdot dF_{b(0)^{-1}b(t)}(x) \\ &= \exp -\frac{t}{2}\{\langle\lambda - i\rho, \lambda - i\rho\rangle + \langle\rho, \rho\rangle\} \\ &= \exp -\frac{t}{2}\langle\lambda, \lambda\rangle + it\lambda(H_\rho)\,. \end{aligned} \tag{5.18}$$

This is clearly the Fourier-Stieltjes transform of a Gaussian distribution on $\mathfrak{h}_{\mathfrak{p}_0}$, with mean at tH_ρ, concluding the proof.

COROLLARY 5.7. *If $z \in G$, then*

$$E(H(zb(0)^{-1}b(t))) = \int_K H(zk')dk' + tH_\rho\,,$$

and

$$E(\| H(zb(0)^{-1}b(t)) - E(H(zb(0)^{-1}b(t))) \|^2) = l\cdot t + C_z$$

where $l = \dim \mathfrak{h}_{\mathfrak{p}_0}$, and C_z is a constant depending only on z.

Proof. Both assertions follow by observing that

$$\begin{aligned} &E(\exp i\lambda(H(zb(0)^{-1}b(t)))) \\ &= \int_G \varphi_{\lambda-i\rho}(zx)dF_{b(0)^{-1}b(t)}(x) \\ &= \int_G\int_K \varphi_{\lambda-i\rho}(zkx)dk\,dF_{b(0)^{-1}b(t)}(x),\ \textit{since } F \textit{ is spherical} \\ &= \varphi_{\lambda-i\rho}(z)\cdot\int_G \phi_{\lambda-i\rho}(x)dF_{b(0)^{-1}b(t)}(x) \\ &= \varphi_{\lambda-i\rho}(z)\cdot\exp -\frac{t}{2}\langle\lambda, \lambda\rangle + it\lambda(H_\rho) \\ &= \left[\int_K \exp i\lambda(H(zk))dk\right]\left[\exp -\frac{t}{2}\langle\lambda, \lambda\rangle + it\lambda(H_\rho)\right]. \end{aligned} \tag{5.19}$$

Thus we know that this is the Fourier-Stieltjes transform of the distribution of $H(zb(0)^{-1}b(t))$. The two quantities of the corollary are merely the mean and the trace of the covariance matrix of this distribution and the corollary follows after a painless computation.

LEMMA 5.8. *With the notation of* § 3, *we have for* $z \in G, k \in K$,

$$E(H(zy_n^1(t)k)) = E(H(zb(0)^{-1}b(t))) + t\int_{\delta_n<|x|<1} H(x)dL(x) . \tag{5.20}$$

Proof. By sphericity, $E(H((zy^1{}_n(t)) = E(H(zy_n'(t)))$. Now recall the definition of $y_n^1(t)$ viz. (3.6). Because $\{b(t), t \in [0, \infty)\}$ is differential and because $k_jx_jk_j'$ $j = j(1, t) + 1, \cdots, j(\delta_n, t)$ are mutually independent, it follows from Corollary 5.3 that

$$E(H(zy_n^1(t))) = E(H(zb(0)^{-1}b(t))) + E\Big(\sum_{j=j(1,t)}^{j(\delta_n\ t)} H(k_jx_jk_j')\Big) . \tag{5.21}$$

The second term can be computed by exactly the same method as the one followed in theorem 4.1. We omit the computation. The result is

$$t\int_K\int_K\int_{\delta_n<|x|\leq 1} H(kxk')dL(x)dkdk'.$$

Remembering that L is spherical, this is $t\int_{\delta_n<|x|\leq 1} H(x)dL(x)$ finishing the proof.

LEMMA 5.9. *Let* $z \in G, k \in K$ *and write*

$$Z_n(t) = H(zy_n^1(t)\cdot k) - E(H(zy_n^1(t))) .$$

Then for a fixed n, $Z_n(t)$ *is a martingale in the parameter* t; *for a fixed* t, *it is a martingale in the parameter* n.

Proof. Let $s < t$, then $y_n^1(t) = y_n^1(s)\cdot\zeta_n(s, t)$ where

$$\zeta_n(s,t)=b(s)^{-1}b(\tau_{j(\delta_n,s)+1})k_{j(\delta_n,s)+1}x_{j(\delta_n,s)+1}+1k'_{j(\delta_n,s)+1}\cdots b(\tau_{j(\delta_n,t)})^{-1}b(t). \tag{5.22}$$

Note that because $\{b(t), t \in [0, \infty)\}$ is differential, $\zeta_n(s, t)$ is independent of $y_n^1(u)$ $u < s$. Moreover, we have

$$\begin{aligned} &E(H(\zeta_n(s, t)))\\ &\quad = E(H(b(s)^{-1}b(t))) + E\Big(\sum_{j=j(\delta_n,s)+1}^{j(\delta_n,t)} H(k_jx_jk_j')\Big)\\ &\quad = E(H(b(0)^{-1}b(t-s))) + E\Big(\sum_{j=j(1,t)}^{j(\delta_n,t)} H(k_jx_jk_j')\Big)\\ &\qquad - E\Big(\sum_{j=j(1,s)}^{j(\delta_n,s)} H(k_jx_jk_j')\Big)\\ &\quad = (t-s)H_\rho + (t-s)\int_{\delta_n<|x|\leq 1} H(x)dL(x) \end{aligned} \tag{5.23}$$

by Lemmas 5.2, 5.6 and 5.8.

Using Lemma 5.4, we get

$$\begin{aligned} &E(H(zy_n^1(t)k) \mid y_n^1(u)\ u \leqq s) \\ &= E(H(zy_n^1(s)\zeta_n(s,t)k) \mid y_n^1(u)\ u \leqq s) \\ &= H(zy_n^1(s)k) + E(\zeta_n(s,t)) \\ &= H(zy_n^1(s)k) + (t-s)H_\rho + (t-s)\int_{\delta_n<|x|\leqq 1} H(x)dL(x)\,. \end{aligned} \tag{5.24}$$

On the other hand, since

$$E(H(zy_n^1(t)k)) = \int_K H(zk')dk' + tH_\rho + t\int_{\delta_n<|x|<1} H(x)dL(x)$$

by virtue of Corollary 5.7 and Lemma 5.8, (5.24) now implies the first assertion of the present lemma.

Turning to the second assertion, fix t. The only difference between $y_n^1(t)$ and $y_{n+1}^1(t)$ is that the latter has interlaced jumps of lengths in $[\delta_{n+1}, \delta_n]$ while the former has no such jumps. Indeed by (3.6) we see that

$$y_{n+1}^1(t) = y_n^1(\tau_{j(\delta_n,t)})\theta_n(t) \tag{5.25}$$

where

$$\begin{aligned} \theta_n(t) = b(\tau_{j(\delta_n,t)})^{-1}b(\tau_{j(\delta_n,t)+1})k_{j(\delta_n,t)+1}x_{j(\delta_n,t)+1}\cdots \\ \cdots k'_{j(\delta_{n+1},t)})b(\tau_{j(\delta_{n+1},t)})^{-1}b(t)\,. \end{aligned} \tag{5.26}$$

We now use Lemma 5.4 with the random variables

$$b(\tau_j)^{-1}b(\tau_{j+1})\ j=1,\cdots,j(\delta_{n+1},t),\ \ k_jx_jk'_j,\ j=1,\cdots,j(\delta_n,t)$$

playing the roles of the ξ's in that lemma and the r.v.'s

$$k_jx_jk'_j,\ j=j(\delta_n,t)+1,\cdots,j(\delta_{n+1},t)$$

playing the role of the η's of that lemma. Recalling also Lemma 5.2, we have

$$\begin{aligned} &E\left(H(zy_{n+1}^1(t)k)\ \middle|\ \begin{matrix} b(s) & s\leqq t \\ k_j,\tau_j,k'_j,x_j, & j=1,\cdots,j(\delta_n,t)\end{matrix}\right) \\ &= H(zy_n^1(t)k) + E\Big(\sum_{j=j(\delta_n,t)+1}^{j(\delta_{n+1},t)} H(k_jx_jk'_j)\Big) \\ &= H(zy_n^1(t)k) + E\Big(\sum_{j=j(1,t)}^{j(\delta_{n+1},t)} H(k_jx_jk'_j)\Big) \\ &\qquad - E\Big(\sum_{j=j(1,t)}^{j(\delta_{n+1},t)} H(k_jx_jk'_j)\Big) \\ &= H(zy_n^1(t)k) + t\int_{\delta_{n+1}<|x|\leqq 1} H(x)dL(x) \end{aligned} \tag{5.27}$$

$$- t\int_{\delta_n<|x|<1} H(x)dL(x)$$

in view of Lemma 5.8, this last easily leads to the second assertion of the present lemma.

LEMMA 5.10

$$E(\| Z_n(t) \|^2) = lt + t\int_{\sigma_n<x\leq 1} \| H(x) \|^2 dL(x) + C_z \tag{5.28}$$

where $l = \dim \mathfrak{h}_{\mathfrak{p}_0}$, *and* C_z *is the constant of Corollary* 5.7.

Proof. $E(\| Z_n(t) \|^2)$ is just the trace of the covariance matrix of the distribution of $H(zy_n^1(t)k)$, call it the variance (cf § 2). The distribution of $H(zy_n^1(t)k)$ is by Corollary 5.3, the same as that of

$$H(zb(0)^{-1}b(t)) + \sum_{j=j(1,t)}^{j(\delta_n,t)} H(k_jx_jk_j') .$$

Hence the variance of $H(zy_n^1(t)k)$ is the sum of the variances of $H(zb(0)^{-1}b(t))$ and $\sum_{j=j(1,t)}^{j(\delta_n,t)} H(k_jx_jk_j')$. The variance of $H(zb(0)^{-1}b(t))$ has already been computed in Corollary 5.7 to be $lt + C_z$. As for $\sum_{j=j(1,t)}^{j(\delta_n,t)} H(k_jx_jk_j')$, Fourier-Stieltjes transform of its distribution can be computed to be

$$\begin{aligned} & E \exp i\lambda\Big(\sum_{j=j(,t)}^{j=j_n(\delta_n,t)} H(k_jx_jk_j')\Big) \\ & = E\Big(\prod_{j=j(1,t)}^{j(\delta_n,t)} \exp i\lambda(H(k_jx_jk_j'))\Big) \\ & = E\Big(\prod_{j=j(1,t)}^{j(\delta_n,t)} \phi_{\lambda-i\rho}(x_j)\Big) \\ & = \exp - t \int_{\delta_n<|x|\leq 1} [1 - \phi_{\lambda-i\rho}(x)]dL(x) \\ & = \exp - t \int_{\delta_n<|x|\leq 1} [1 - \exp i\lambda(H(x))dL(x) \end{aligned} \tag{5.29}$$

where we have omitted computations sufficiently similar to ones gone before.

From a knowledge of its Fourier-Stieltjes transform the variance of $\sum H(x_jx_jk_j')$ can now be easily computed to be $t \int_{\delta_n<|x|\leq 1} \| H(x) \|^2 dL(x)$. This finishes the proof.

THEOREM 5.11. *Let* $z \in G$, $k \in K$. *There exists a sequence* $\delta_n \downarrow 0$ *such that* $H(zy_n^1(s)k)$ *converges with probability* 1 *as* $n \to \infty$, *uniformly for* $0 \leq s \leq t$ *and* $k \in K$.

Proof. We have seen above that if

$$Z_n(s) = H(zy_n^1(s)k) - E(H(zy_n^1(s))) ,$$

then for fixed n, $Z_n(s)$ is a martingale in s. It follows that $Z_{n+1}(s) - Z_n(s)$ is also a martingale in the parameter s. By the Martingale inequality of Doob [1, p. 314] we have

$$\begin{aligned} &P(\sup_{0 \leqq s \leqq t} \| Z_{n+1}(s) - Z_n(s) \| \geqq \varepsilon) \\ &\leqq \frac{1}{\varepsilon^2} E(\| Z_{n+1}(t) - Z_n(t) \|^2) . \end{aligned} \tag{5.30}$$

On the other hand, for fixed t, $Z_n(t)$ is a martingale in the parameter n. It follows that

$$E(\| Z_{n+1}(t) - Z_n(t) \|^2) = E(\| Z_{n+1}(t) \|^2) - E(\| Z_n(t) \|^2) . \tag{5.31}$$

(Indeed for a martingale Z_m we have always $E(\| Z_m - Z_n \|^2) = E(\| Z_m \|^2) - E(\| Z_n \|^2)$.) In view of Lemma 5.10, we get

$$E(\| Z_{n+1}(t) - Z_n(t) \|^2) = t \int_{\delta_{n+1} < |x| \leqq \delta_n} \| H(x) \|^2 dL(x) , \tag{5.32}$$

since $\int_{0<|x|\leqq 1} \| H(x) \|^2 dL(x) < \infty$ (cf. § 2) we may choose the sequence δ_n so that $t \int_{\delta_{n+1}<|x|\leqq\delta_n} \| H(x) \|^2 dL(x) < 2^{-n}$ for large n. Then (5.29) with $\varepsilon = 2^{-n/3}$ yields

$$P(\sup_{0 \leqq s \leqq t} \| Z_{n+1}(s) - Z_n(s) \| \geqq 2^{-n/3}) \leqq 2^{-n/3} . \tag{5.33}$$

It follows by the Borel-Cantelli Lemma that $Z_n(s)$ must converge as $n \to \infty$ uniformly in $0 \leqq s \leqq t$ with probability 1. Now,

$$\begin{aligned} Z_n(s) &= H(zy_n^1(s)k) - E(H(zy_n^1(s))) \\ &= H(zy_n^1(s)k) - \int_K H(zk')dk' - sH_\rho \\ &\quad - s \int_{\delta_n < |x| \leqq 1} H(x)dL(x) \end{aligned} \tag{5.34}$$

and since $\int_{0<|x|\leqq 1} H(x)dL(x)$ exists (cf. § 2), we conclude that $H(zy_n^1(s)k)$ must converge as $n \to \infty$ uniformly in $0 \leqq s \leqq t$ with probability 1. The estimates (5.32), (5.33) being uniform in $k \in K$, the theorem now follows.

COROLLARY 5.12. *$\{y_n^1(s)\}_{n=1}^\infty$ is contained in a compact subset of G for $0 \leqq s \leqq t$ with probability one.*

Proof. We know by the theorem that $H(y_n^1(s))$ converges as $n \to \infty$

uniformly for $0 \leqq s \leqq t$ almost surely. If some subsequence $y^1_{n_j}(s)$ were to $\rightarrow \infty$ on G with positive probability, then by considering the map $x \rightarrow H(x)$, it could be shown easily that $H(y^1_{n_j}(s)) \rightarrow \infty$ on $\mathfrak{h}_{\mathfrak{p}_0}$ with positive probability, contradicting the theorem. Therefore the corollary follows.

THEOREM 5.13. *The sequence $\pi(y_n(s))$ converges on G/K as $n \rightarrow \infty$ with probability one uniformly for $0 \leqq s \leqq t$. If $y_\infty(s)$ is the limit, then $y_\infty(s)$, $s \in [0, \infty)$ is differential process on G/K. If F^t is the distribution of $y_\infty(t)$ on G/K, then F^t is given by* (3.1).

Proof. If φ is an elementary positive definite spherical function on G, by Harish-Chandra's formula (2.1) we have

$$\varphi_\nu(x) = \int_K \exp\,[i\nu(H(xk)) - \rho(H(xk))]dk \tag{5.35}$$

for some complex valued linear functional ν on $\mathfrak{h}_{\mathfrak{p}_0}$. We have seen above that for $z \in G$ $k \in K$, $H(zy^1_n(s)k)$ converges as $n \rightarrow \infty$ uniformly for $0 \leqq s \leqq t$ and $k \in K$ almost surely. It follows from (5.35) that $\varphi(zy^1_n(s))$ must therefore converge as $n \rightarrow \infty$ uniformly for $0 \leqq s \leqq t$ almost surely. By Lemma 5.1, we conclude that $\pi(y^1_n(s))$ must do the same. But $y_n(s)$ and $y^1_n(s)$ differ only in that $y_n(s)$ has in it jumps of lengths bigger than 1. Since there are almost surely only finitely many of these for $s \leqq t$, it follows that $\pi(y_n(s))$ must converge as $n \rightarrow \infty$ almost surely, uniformly for $0 \leqq s \leqq t$. That $y_\infty(s)$ is differential is obvious though messy to check. The last assertion follows from Corollary 4.2.

In particular, the process given by (3.1) may always be assumed to have sample functions whose discontinuities are only jumps. The Lévy measure L has the interpretation that $tL(\Delta)$ is the expected number of jumps of sizes x which the path experiences till time t, for which $x \in \Delta$.

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