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CONVOLUTION TRANSFORMS WHOSE INVERSION FUNCTIONS HAVE COMPLEX ROOTS

JOHN DAUNS AND DAVID V. WIDDER

The convolution transform is defined by the equation

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt = (G*\varphi)(x) . \tag{1.1}$$

If the kernel $G(t)$ has a bilateral Laplace transform which is the reciprocal of an entire function $E(s)$, then $E(s)$ is called the inversion function of the transform. This terminology is appropriate in view of the fact that the transform (1.1) is inverted, in some sense, by the operator $E(D)$, where D stands for differentiation with respect to x:

$$E(D)f(x) = \varphi(x) . \tag{1.2}$$

It is the purpose of the present paper to prove (1.2) when the roots of $E(s)$ are allowed to be genuinely remote from the real axis.

Formula (1.2) was first proved by Widder [7] in 1947 for a large class of entire functions $E(s)$ and by Hirschman and Widder [3] in 1949 for the whole Laguerre-Pólya class. The latter functions have real roots only, indeed are the uniform limits of polynomials with real roots only, see p. 42 of [5].

In 1951 Hirschman and Widder [4] extended this inversion theory, allowing the roots of $E(s)$ to be complex. However, the roots were asymptotically real in the sense that their arguments clustered to 0 or to π. At the same time A. O. Garder [2] allowed the approach to the real axis to be slower. We require only that they should occur in pairs symmetric in the origin and in a sector inside the sector $|\tan(\arg s)| < 1$. More precisely:

$$E(s) = \prod_1^{\infty}\left(1 - \frac{s^2}{a_k^2}\right), \sum_{k=1}^{\infty} a_k^{-2} < \infty$$

$$|\arg a_k| \leqq \frac{\pi}{4} - \eta , \qquad 0 < \eta < \frac{\pi}{4} .$$

We wish also to call attention to some new asymptotic relations.

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If

$$G_{2n}(t) = \prod_1^n \left(1 - \frac{D^2}{a_k^2}\right) G(t) ,$$

we show that

$$G_{2n}(t) \sim k(t, v_n) \qquad (n \to \infty) \tag{1.3}$$

umiformly for $-\infty < t < \infty$. Here $k(t, v)$ is the fundamental solution of the heat equation,

$$k(t, v) = (4\pi v)^{-1/2} \exp(-t^2/4v) ,$$

with $-\infty < t < \infty$, with $Re\, v > 0$, with the square root one-at-one, and where v_n is given by

$$v_n = \sum_{n+1}^{\infty} a_k^{-2} .$$

In order to establish (1.3) we are obliged to make an additional assumption on the distribution of the roots of $E(s)$, see Condition B in § 4.

As a consequence of (1.3) we prove that

$$\int_{-\infty}^{\infty} | G_{2n}(t) | \, dt \sim (\cos^2 \varphi_n - \sin^2 \varphi_n)^{-1/2} \qquad (n \to \infty) , \tag{1.4}$$

where $\varphi_n = (1/2) \arg v_n^{-1}$. This result tends to indicate that present methods cannot be employed for the inversion of (1.1) if the roots of $E(s)$ lie outside the 45° sector used above.

Finally we compute explicitly the functions $G_{2n}(t)$ corresponding to $E(s) = \cos \alpha s$ where $|\arg \alpha| < \pi/2$. Here all roots lie on a line through the origin. In this case the integral (1.4) tends to infinity with n when $|\arg \alpha| \geqq \pi/4$. This result indicates clearly that our arguments must fail if the roots of $E(s)$ are not restricted to lie inside the 45° sector.

2. A first inversion theorem.

Let us introduce the following conventions.

Condition A. The sequence $a_1, a_2, \cdots$ of complex constants satisfies Condition A if

$$\sum_1^{\infty} | a_k |^{-2} < \infty \quad \text{and} \quad | \arg a_k | \leqq \frac{\pi}{4} - \eta$$

for some η in $0 < \eta < \pi/4$. It is assumed that the a_k are arranged in an order of nondecreasing real parts with $Re\, a_1 > 0$, i.e.

$$0 < Re\, a_1 \leqq Re\, a_k \leqq Re\, a_{k+1} \qquad (k = 1, 2, \cdots) .$$

DEFINITION. The class of entire functions A consists of all entire functions $E(s)$ of the form

$$E(s) = \prod_1^\infty \left(1 - \frac{s^2}{a_k^2}\right)$$

where the roots a_k satisfy condition A.

For example, $\cos(2+i)s$ belongs to the class A.

We now state the main theorem of the present section, a result that will be improved in § 3 by more complicated methods.

THEOREM 2.1. *If for* $-\infty < t < \infty$

1.
$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{E(s)}\, ds \qquad (E(s) \in A)\ .$$

2. *$\varphi(t)$ is bounded on compact sets and*

$$\varphi(t) = O(e^{\sigma|t|}) \qquad (|t| \to \infty, 0 < \sigma < Re\, a_1)\ .$$

3.
$$f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)\, dt\ ,$$

then

$$\lim_{n\to\infty} \prod_1^n \left(1 - \frac{D^2}{a_k^2}\right) f(x) = \varphi(x)$$

at any point $t = x$ of continuity of $\varphi(t)$.

We shall establish this result by the series of Lemmas 2.2, 2.3, and 2.4.

Consider a fixed function $E(s)$ in the class A. Then let $E_{2n}(s)$ be defined by

$$E_{2n}(s) = \prod_{n+1}^\infty \left(1 - \frac{s^2}{a_k^2}\right) \qquad (n = 0, 1, 2, \cdots)\ . \tag{2.1}$$

Define S_n by

$$S_n = \sum_{n+1}^\infty |a_k|^{-2} \qquad (n = 0, 1, 2, \cdots)\ . \tag{2.2}$$

Let $G_{2n}(t)$ and $G(t)$ be defined by

$$G_{2n}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{E_{2n}(s)}\, ds\ , \quad G(t) = G_0(t) \qquad (-\infty < t < \infty;\ n = 0, 1, 2, \cdots)\ . \tag{2.3}$$

If $P_{2n}(D)$ is defined as

$$P_{2n}(D) = \prod_{1}^{n} \left(1 - \frac{D^2}{a_k^2}\right) \qquad (n = 0, 1, 2, \cdots), \tag{2.4}$$

then the next lemma will show that the integral (2.3) converges, that

$$P_{2n}(D)G(t) = G_{2n}(t),$$

and furthermore it will give lower bounds of the function $E_{2n}(s)$ in terms of both s and n. It will become clear later that exactly these lower bounds are the ones needed to obtain the required information about the kernels $G_{2n}(t)$.

LEMMA 2.2. *Let the roots* $a_k = r_k e^{i\beta_k}$, η, $E_{2n}(s)$, *and* S_n *be as in Condition A and equations* (2.1) *and* (2.2).

A. *Let* $re^{i\theta}$ *with* $r>0$ *be any point in the angular sector defined by*

$$|\tan\theta| \geqq \tan\left(\frac{\pi}{2} - \frac{\eta}{2}\right).$$

Then

$$|E_{2n}(re^{i\theta})| \geqq 1 + r^2 S_n \sin\eta$$

and also

$$|E_{2n}(re^{i\theta})| \geqq 1 + r^4 \sin^2\eta \sum_{n<i<j<\infty} r_i^{-2} r_j^{-2}.$$

B. *Define* K *to be the constant*

$$K = \frac{1}{2}\sin\frac{\eta}{2}.$$

Let n *be arbitrary,* $n = 0, 1, 2, \cdots$, *but fixed. Let* $re^{i\theta}$ *with* $r > 0$ *be any point in the triangular region defined by the inequalities*

$$|\tan\theta| \leqq \tan\left(\frac{\pi}{2} - \frac{\eta}{2}\right), \qquad |r\cos\theta| \leqq K S_n^{-1/2}.$$

Then

$$|E_{2n}(re^{i\theta})| \geqq \frac{2}{3}.$$

Proof. A typical term of the infinite product $E_{2n}(re^{i\theta})$ satisfies

$$[1 - r^2 r_k^{-2} e^{2i(\theta-\beta_k)}][1 - r^2 r_k^{-2} e^{-2i(\theta-\beta_k)}] = 1 - 2r^2 r_k^{-2}\cos 2(\theta - \beta_k) + r^4 r_k^{-4}.$$

Since in case A, the argument θ satisfies either $\pi/2 - \eta/2 \leqq \theta \leqq \pi/2 + \eta/2$ or $-\pi/2 - \eta/2 \leqq \theta \leqq -\pi/2 + \eta/2$, and since the argument β_k of any root satisfies $-\pi/4 + \eta \leqq -\beta_k \leqq \pi/4 - \eta$, it follows that in case A we have $-\cos 2(\theta - \beta_k) \geqq \sin \eta$. Consequently, by multiplying out the infinite product, we obtain

$$|E_{2n}(re^{i\theta})| \geqq \prod_{n+1}^{\infty} 1 + r^2 r_k^{-2} \sin \eta > 1 + r^2 S_n \sin \eta .$$

Similarly, we also obtain the second inequality in A.

For the proof of B, take $k > n$ and restrict $re^{i\theta} = \sigma + iy$ to the angular sector $|y| \leqq |\sigma| \cot \eta/2$. By using the latter inequality, we see that a typical term of the infinite product $E_{2n}(\sigma + iy)$ has the lower bound

$$\left|1 - \frac{(\sigma + iy)^2}{r_k^2 e^{2i\beta_k}}\right| \geqq 1 - \frac{\sigma^2 + y^2}{r_k^2} \geqq 1 - \frac{\sigma^2}{r_k^2}\left(1 + \cot^2 \frac{\eta}{2}\right) .$$

This latter lower bound is positive. The inequalities $r_k^2 S_n > 1$ and $|\sigma| \leqq K S_n^{-1/2}$ imply that

$$\frac{\sigma^2}{r_k^2}\left(1 + \cot^2 \frac{\eta}{2}\right) = \frac{\sigma^2}{4^2 K r_k^2} < \frac{1}{4} .$$

By use of the latter and by multiplying out the infinite product we obtain

$$|E_{2n}(\sigma + iy)| > 1 - \sum_{p=1}^{\infty} 4^{-p} S_n^{-p} \sum_{n < k(1) < \cdots < k(p) < \infty} r_{k(1)}^{-2} \cdots r_{k(p)}^{-2} ,$$

where the indices $k(1), \cdots, k(p)$ range over the integers.

Use of the inequality

$$\sum_{n < k(1) < \cdots < k(p) < \infty} r_{k(1)}^{-2} \cdots r_{k(p)}^{-2} < S_n^p$$

leads to

$$|E_{2n}(re^{i\theta})| \geqq \frac{2}{3} .$$

Thus conclusion B has been established.

The next lemma gives some facts about the kernels $G_{2n}(t)$. Once the lower bound given by part A of last lemma is available, the next lemma can be proved exactly as in the case of real roots a_k, see [6; p. 265] and [5; p. 108]; we omit the proof.

LEMMA 2.3. *Let $E_{2n}(s)$, $G_{2n}(t)$, and $P_{2n}(D)$ be defined by* (2.1), (2.3) *and* (2.4). *In particular, the roots a_k defining $E_{2n}(s)$ satisfy condition A, and consequently*

$$0 < Re\, a_k \leqq Re\, a_{k+1} \qquad (k = n + 1, n + 2, \cdots) .$$

Let $n = 0, 1, 2, \cdots$ *be arbitrary.*

A. *For any* σ *in* $|\sigma| < Re\, a_{n+1}$,

$$G_{2n}(t) = P_{2n}(D)G(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{st}}{E_{2n}(s)}\,ds\ .$$

B. *Let* a_n *as a zero of* $E_{2n}(s)$ *be of multiplicity* $\mu + 1$. *Then there is a polynomial* $p(t)$ *of degree* μ *such that for any* k *in* $-Re\, a_{n+1} < k < Re\, a_{n+1}$ *and any integer* $\nu = 0, 1, 2, \cdots$ *the following holds*

$$\left(\frac{d}{dt}\right)^{\nu} G(t) = \left(\frac{d}{dt}\right)^{\nu}[p(t)e^{-|t|a_{n+1}}] + O(e^{-k|t|})\ , \qquad (|t| \to \infty)\ .$$

C. *For all* $s = \sigma + i\tau$ *with* $|\sigma| < Re\, a_{n+1}$ *and* $-\infty < \tau < \infty$

$$\frac{1}{E_{2n}(s)} = \int_{-\infty}^{\infty} e^{-st}G_{2n}(t)\,dt\ , \qquad \int_{-\infty}^{\infty} G_{2n}(t)\,dt = 1\ .$$

In the next lemma a sufficiently good upper bound of the kernel $G_{2n}(t)$ in terms of both t and n is proved in order to have an inversion formula as an immediate consequence.

LEMMA 2.4. *Let* $G_{2n}(t)$ *and* S_n *be as defined by equations* (2.3) *and* (2.2). *Then there exist constants* M *and* K *independent of both* n *and* t *such that*

$$|G_{2n}(t)| \leqq MS_n^{-1/2}\exp(-KS_n^{-1/2}|t|) \qquad (-\infty < t < \infty, n = 0, 1, 2, \cdots)\ .$$

Proof. Use of the fact that $G_{2n}(t)$ is an even function of t and use of Lemma 2.3 shows that

$$G_{2n}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-(\sigma+iy)t}}{E_{2n}(\sigma+iy)}\,dy$$

provided σ satisfies $0 < \sigma < r_{n+1}\cos\beta_{n+1}$ (where $r_{n+1}e^{i\beta_{n+1}}$ is that root of $E_{2n}(s)$ with smallest positive real part). Let K be as in Lemma 2.2 the constant $K = (1/2)\sin(\eta/2)$. Assume for the rest of the proof that σ is restricted to $0 < \sigma \leqq KS_n^{-1/2}$. Then since $\cos\beta_{n+1} > 1/\sqrt{2}$, it follows that

$$0 < \sigma \leqq (1/2)\sin(\eta/2)r_{n+1} < r_{n+1}\cos\beta_{n+1}\ .$$

By setting $A = \tan(\pi/2 - \eta/2)$ and using the lower bounds of Lemma 2.2 we obtain

$$|G_{2n}(t)| \leqq \frac{3A}{2\pi}\sigma e^{-\sigma t} + \frac{e^{-\sigma t}}{\pi}\int_{\sigma A}^{\infty} \frac{1}{1 + y^2 S_n \sin\eta}\,dy\ .$$

Replace the lower limit σA in the last integral by 0, set $\sigma = KS_n^{-1/2}$

and let M be the constant $M = 3AK/2\pi + (1/2)(\sin\eta)^{-1/2}$. Since $G_{2n}(t)$ is an even function, the last inequality shows that for all n and t, the function $G_{2n}(t)$ satisfies the conclusion of the theorem

$$|G_{2n}(t)| \leqq MS_n^{-1/2}\exp(-KS_n^{-1/2}|t|) .$$

REMARK. In the previous lemma the constants M and K are functions of η only. As η tends to 0, M tends to ∞ and K tends to 0, thus making the upper bound of the theorem meaningless as η tends to 0. These are phenomena which are typical of the theory and which we will encounter again.

Now we are in a position to prove Theorem 2.1.

Proof. By letting M_0 be the constant guaranteed by hypothesis 2 of Theorem 2.1, i.e. for any fixed x and all t,

$$|\varphi(x-t)-\varphi(x)| \leqq M_0 e^{\sigma|t|} ,$$

and by using Lemma 2.3 we find that for any $\delta > 0$

$$|P_{2n}(D)(G*\varphi)(x)-\varphi(x)| \leqq \sup_{|t|<\delta}|\varphi(x-t)-\varphi(x)|\int_{-\infty}^{\infty}|G_{2n}(t)|\,dt$$
$$+ M_0\int_{\delta<|t|<\infty}|G_{2n}(t)|\,e^{\sigma|t|}\,dt .$$

Replacement of $|G_{2n}(t)|$ by its upper bound given by Lemma 2.4,

$$|G_{2n}(t)| \leqq MS_n^{-1/2}\exp(-KS_n^{-1/2}|t|) ,$$

and use of the continuity of $\varphi(t)$ at $t = x$ immediately give the theorem.

3. A second inversion theorem. We now remove the boundedness condition on $\varphi(t)$, assumed in Theorem 2.1, assuming here instead only local integrability. The inversion formula will be valid not only at points of continuity of $\varphi(t)$ but at all points of the Lebesgue set for that function.

THEOREM 3.1. *If $G(t)$ and $f(x)$ are defined as in Theorem 2.1 with $\varphi(t) \in L^1$ in every finite interval and if*

$$\int_0^t \varphi(u)du = O(e^{\sigma|t|}) \qquad (|t|\to\infty,\ 0<\sigma<Re\,a_1) ,$$

then

$$\lim_{n\to\infty}\prod_1^n\left(1-\frac{D^2}{a_k^2}\right)f(x)=\varphi(x)$$

for all x in the Lebesgue set for $\varphi(t)$.

We first prove a result about the derivative of $G_{2n}(t)$.

LEMMA 3.2. *Let the roots $a_k = r_k e^{i\beta_k}$, η, $G_{2n}(t)$, and S_n be defined by Condition A and equations (2.3) and (2.2). Then there exist constants M_1, K_1, M_2, and K_2 independent of both n and t such that for all $n = 0, 1, 2, \cdots$ the following holds:*

A. *If n satisfies $S_n \geqq 4r_{n+1}^{-2}$, then*

$$|G'_{2n}(t)| \leqq M_1 S_n^{-1} \exp(-K_1 S_n^{-1/2} |t|) \qquad (-\infty < t < \infty) .$$

B. *If n satisfies $S_n < 4r_{n+1}^{-2}$, then*

$$|G'_{2n}(t)| \leqq M_2 r_{n+1}^2 \exp(-K_2 r_{n+1} |t|) \qquad (-\infty < t < \infty) .$$

Proof. First conclusion A will be proved. Let K be the constant

$$K = \frac{1}{2} \sin \frac{\eta}{2} .$$

Restrict σ to $0 < \sigma \leqq KS_n^{-1/2}$. The latter guarantees that $0 < \sigma < r_{n+1} \cos \beta_{n+1}$ and hence $G'_{2n}(t)$ is given by

$$G'_{2n}(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-(\sigma+iy)t}(\sigma + iy)}{E_{2n}(\sigma + iy)} dy .$$

With $A = \tan(\pi/2 - \eta/2)$, the above becomes

$$(1) \qquad |G'_{2n}(t)| \leqq \frac{e^{-\sigma t}}{2\pi} \int_{-\sigma A}^{\sigma A} \frac{\sigma + |y|}{|E_{2n}(\sigma + iy)|} dy + \frac{e^{-\sigma t}}{2\pi} \int_{\sigma A < |y| < \infty} \frac{\sigma + |y|}{|E_{2n}(\sigma + iy)|} dy .$$

The assumption that $S_n \geqq 4r_{n+1}^{-2}$ guarantees that $S_n - r_k^{-2} \geqq 1/2\, S_n$ for all $k > n$. Hence the second lower bound given by part A of Lemma 2.2 becomes

$$|E_{2n}(\sigma + iy)| \geqq 1 + \frac{1}{2} y^4 \sin^2 \eta \sum_{k=n+1}^{\infty} \frac{1}{r_k^2}\left(S_n - \frac{1}{r_k^2}\right) \geqq 1 + \left(\frac{1}{2} y^2 S_n \sin \eta\right)^2 .$$

Use of the last inequality and the estimate of part B of Lemma 2.2 in equation (1) gives

$$(2) \qquad |G'_{2n}(t)| \leqq \frac{3}{2\pi} A(1 + A)\sigma^2 e^{-\sigma t} + \frac{e^{-\sigma t}}{\pi} \int_{\sigma A}^{\infty} \frac{\sigma + y}{1 + \left(\frac{1}{2} y^2 S_n \sin \eta\right)^2} dy .$$

Replace the limit σA by 0 in the last integral; define the constants c_1 and c_2 as

$$c_1 = \int_0^\infty \frac{1}{1+u^4}\,du\,, \qquad c_2 = \int_0^\infty \frac{u}{1+u^4}\,du\,;$$

let K_1 and M_1 be the constants

$$K_1 = K\,, \quad M_1 = \frac{3}{2\pi}A(1+A)K^2 + \frac{c_1\sqrt{2}}{\pi}(\sin\eta)^{-1/2}K + \frac{2c_2}{\pi\sin\eta}\,,$$

and set $\sigma = KS_n^{-1/2}$. Then equation (2) gives

$$|\,G'_{2n}(t)\,| \leqq M_1 S_n^{-1}\exp\left(-K_1 S_n^{-1/2}\,|\,t\,|\right)$$

for all t and all n satisfying $S_n \geqq 4r_{n+1}^{-2}$.

For the proof of part B, $G_{2n}(t)$ has to be expressed in the form

$$G_{2n}(t) = (g * G_{2n+2})\,(t) \tag{3}$$

where $g(t)$ is the function

$$g(t) = \frac{1}{2}a_{n+1}e^{-a_{n+1}|t|} \qquad (-\infty < t < \infty)\,.$$

Differentiation of (3) under the integral sign and an integration by parts gives

$$G'_{2n}(t) = -\frac{a_{n+1}^2}{2}\int_{-\infty}^{\infty}\frac{u}{|\,u\,|}e^{-a_{n+1}|u|}G_{2n+2}(t-u)\,du\,\cdot \tag{4}$$

By use of the estimate

$$|\,G_{2n+2}(t)\,| \leqq MS_{n+1}^{-1/2}\exp\left(-KS_{n+1}^{-1/2}\,|\,t\,|\right)$$

of Lemma 2.4, equation (4) becomes

$$\begin{aligned} &|\,G'_{2n}(t)\,| \\ &\quad \leqq \frac{1}{2}Mr_{n+1}^2\int_{-\infty}^{\infty}\exp\left(-\frac{1}{\sqrt{2}}r_{n+1}\,|\,u\,|\right)S_{n+1}^{-1/2}\exp\left(-KS_{n+1}^{-1/2}\,|\,u-t\,|\right)du\,. \end{aligned} \tag{5}$$

By integrating equation (5) by parts we obtain

$$\begin{aligned} |\,G'_{2n}(t)\,| &\leqq MK^{-1}r_{n+1}^2\exp\left(-\frac{1}{\sqrt{2}}r_{n+1}\,|\,t\,|\right) \\ &\quad \frac{M}{2\sqrt{2}\,K}r_{n+1}^3\int_{-\infty}^{\infty}\exp\left(-\frac{1}{\sqrt{2}}r_{n+1}\,|\,u\,| - KS_{n+1}^{-1/2}\,|\,u-t\,|\right)du\,. \end{aligned} \tag{6}$$

If n satisfies $S_n < 4r_{n+1}^{-2}$ as in conclusion B, then

$$S_{n+1} - r_{n+1}^{-2} = S_n - 2r_{n+1}^{-2} < 2r_{n+1}^{-2} \quad \text{and} \quad S_{n+1}^{-1/2} > r_{n+1}/\sqrt{3}\,.$$

Substitution of the latter together with the inequality $|t| - |u| \leqq |u - t|$ in equation (6) gives

$$(7) \qquad |G'_{2n}(t)| \leqq MK^{-1}r^2_{n+1} \exp\left(-\frac{1}{\sqrt{2}} r_{n+1}|t|\right)$$

$$+ \frac{M}{2\sqrt{2}K} r^3_{n+1} \exp\left(-\frac{1}{\sqrt{3}} K r_{n+1}|t|\right) \int_{-\infty}^{\infty} \exp\left(-\frac{\sqrt{3} - K\sqrt{2}}{\sqrt{6}} r_{n+1}|u|\right) du .$$

Let M_2 and K_2 be the constants

$$M_2 = MK^{-1} + M\sqrt{3}K^{-1}(\sqrt{3} - K\sqrt{2})^{-1}, \qquad K_2 = \frac{1}{\sqrt{3}} K .$$

Then equation (7) shows that

$$|G'_{2n}(t)| \leqq M_2 r^2_{n+1} \exp(-K_2 r_{n+1}|t|)$$

holds for all t and all n satisfying $S_n < 4r^{-2}_{n+1}$. Hence conclusion B has been ectablished.

Now we prove Theorem 3.1.

Proof. If $\psi(t)$ is given by

$$\psi(t) = \int_0^t [\varphi(x-u) - \varphi(x)]\, du \qquad (-\infty < t < \infty),$$

then by the hypotheses of Theorem 3.1 there is a constant M_0 for which

$$|\psi(t)| < M_0 e^{\sigma|t|} \qquad (-\infty < t < \infty).$$

If $t = x$ is in the Lebesgue set of $\varphi(t)$ then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\psi(t)| \leqq \varepsilon|t|$ for any t in $|t| \leqq \delta$. An integration by parts, easily justified by Lemma 2.3, yields

$$|P_{2n}(D)(G * \varphi)(x) - \varphi(x)| \leqq \varepsilon \int_{-\infty}^{\infty} |tG'_{2n}(t)|\, dt + M_0 \int_{\delta<|t|<\infty} |G'_{2n}(t)|\, e^{\sigma|t|} dt .$$

Replacing $|G'_{2n}(t)|$ by either one of the two upper bounds given by the last Lemma 3.2, we easily obtain the conclusion of the theorem.

4. Asymptotic estimates. For the estimates of the present section we need to place further restrictions on the roots of the inversion function.

Condition B. The sequence of complex constants $a_1, a_2, \cdots$ satisfies Condition A and in addition

$$\lim_{n\to\infty} |a_n|^{4/3} \sum_{n}^{\infty} |a_k|^{-2} = \infty .$$

For example the sequence $a_n = n$ satisfies Condition B. The sequence $a_n = 2^n$ satisfies Condition A but not Condition B. In the latter case the above limit becomes

$$\lim_{n\to\infty} 2^{4n/3}\left(\frac{4}{3\cdot 2^{2n}}\right) = 0 .$$

DEFINITION. The entire function $E(s)$ belongs to the class of functions B if

$$E(s) = \prod_1^\infty \left(1 - \frac{s^2}{a_k^2}\right)$$

where the roots of $E(s)$ satisfy Condition B.

We can now state the principal result of this section. To do so we adopt the notation of § 1 for the function $k(t, v)$. Set

$$S_n = \sum_{n+1}^\infty |\, a_k\,|^{-2} \tag{4.1}$$

and

$$v_n = \sum_{n+1}^\infty a_k^{-2} . \tag{4.2}$$

THEOREM 4.1. *If*

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{E(s)} ds \qquad (E(s) \in B) ,$$

$$G_{2n}(t) = \prod_1^n \left(1 - \frac{D^2}{a_k^2}\right) G(t)$$

then

$$G_{2n}(t) = k(t, v_n) + O(|\, a_{n+1}\,|^{-2} S_n^{-3/2}) \qquad (n \to \infty) \tag{4.3}$$

uniformly on $-\infty < t < \infty$.

Observe that the remainder term in (4.3) tends to zero with v_n under the assumption $E(s) \in B$.

LEMMA 4.2. *Let* $E_{2n}(s)$, v_n *and* S_n *be defined by* (2.1), (4.2) *and* (4.1) *with the roots* $a_k = r_k e^{i\beta_k}$ *satisfying Condition B. Then there exist two strictly positive constants* c *and* δ *such that for any* u *in* $-\delta \leqq u \leqq \delta$ *we have*

$$\frac{1}{E_{2n}(ir_{n+1}u)} = \exp(-r_{n+1}^2 v_n u^2) + O[r_{n+1}^2 S_n u^4 \exp(-c r_{n+1}^2 S_n u^2)] .$$

The O-term denotes a function of both n *and* u *such that for some constant* M *and all* u *and* n *the absolute value of this function does* *not exceed M times the quantity inside the O-symbol.*

Proof. Let $J_n(u)$ be the function

$$J_n(u) = \frac{1}{E_{2n}(ir_{n+1}u)} - \exp(-r_{n+1}^2 v_n u^2) \qquad (n = 0, 1, 2, \cdots) .$$

Let δ be arbitrary in $0 < \delta < 1/2$ and assume that u is restricted to $|u| \leqq \delta$ throughout the proof. If $c_{2p}(n)$ is defined as

$$c_{2p}(n) = (-1)^p(1/p)r_{n+1}^{2p} \sum_{k=n+1}^{\infty} a_k^{-2p} \qquad (n = 0, 1, 2, \cdots; p = 1, 2, \cdots) ,$$

then

$$(1) \qquad J_n(u) = \exp(-r_{n+1}^2 v_n u^2)\left\{\exp\left[\sum_{p=2}^{\infty} c_{2p}(n)u^{2p}\right] - 1\right\} \qquad (|u| \leqq \delta) .$$

It is interesting to observe that $\lim_{n\to\infty} |c_{2p}(n)| = \infty$ for all p, if $r_k = k^\alpha$ with α in $1/2 < \alpha < 3/2$. Next it is shown that $c_{2p}(n)$ satisfies the inequality

$$|c_{2p}(n)| \leqq \frac{1}{p} r_{n+1}^2 S_n \qquad (n = 0, 1, 2, \cdots; p = 1, 2, \cdots) .$$

If $N(t)$ and $\theta(t)$ are the functions

$$N(t) = \sum_{r_k < t} 1, \ \theta(t) = \int_t^\infty \lambda^{-2} dN(\lambda) \qquad (0 \leqq t < \infty) ,$$

then $|c_{2p}(n)|$ is given by

$$|c_{2p}(n)| = -\frac{1}{p} r_{n+1}^{2p} \int_{r_{n+1}}^{\infty} t^{-2p+2} d\theta(t) .$$

An integration by parts gives the required inequality

$$(2) \qquad |c_{2p}(n)| = \frac{1}{p} r_{n+1}^2 S_n - (p-1) r_{n+1}^{2p} \int_{r_{n+1}}^{\infty} t^{-2p+1}\theta(t)dt \leqq \frac{1}{p} r_{n+1}^2 S_n .$$

Use of the inequality $Re\, v_n \geqq S_n \sin 2\eta$ and (2) in equation (1) gives

$$(3) \qquad |J_n(u)| \leqq \exp(-r_{n+1}^2 S_n u^2 \sin 2\eta)\{\exp[(1-\delta^2)^{-1} r_{n+1}^2 S_n u^4] - 1\} .$$

Choose any δ_1 in $0 < \delta_1 < 1$ and consider the two cases:

Case 1. $(1-\delta^2)^{-1} r_{n+1}^2 S_n u^4 \leqq \delta_1$,

(4) *Case* 2. $(1-\delta^2)^{-1} r_{n+1}^2 S_n u^4 > \delta_1$.

In Case 1, an application of two geometric sum estimates to (3) give the conclusion of the lemma, i.e.

$$(5) \qquad |J_n(u)| \leqq (1-\delta^2)^{-1}(1-\delta_1)^{-1}r_{n+1}^2 S_n u^4 \exp(-u^2 r_{n+1}^2 S_n \sin 2\eta) .$$

For the proof in Case 2, the inequality (3) gives

$$(6) \qquad |J_n(u)| \leqq \exp(-u^2 r_{n+1}^2 S_n \sin 2\eta) + \exp\{r_{n+1}^2 S_n u^2[(1-\delta^2)^{-1}u^2 - \sin 2\eta]\} .$$

Now choose δ as $\delta = (1/2)(\sin 2\eta)^{1/2}$. Then using the inequality

$$(1-\delta^2)^{-1}u^2 - \sin 2\eta \leqq -(2/3)\sin 2\eta$$

and by multiplying (6) by (4), we obtain the conclusion of the lemma for Case 2:

$$(7) \qquad |J_n(u)| \leqq 2(1-\delta^2)^{-1}\delta_1^{-1}r_{n+1}^2 S_n u^4 \exp[-u^2 r_{n+1}^2 S_n (2/3)\sin 2\eta] .$$

Thus (5) and (7) together prove the lemma.

Next Theorem 4.1 is proved.

Proof. The change of variable $y = r_{n+1}u$ in the integral

$$G_{2n}(t) = \frac{1}{\pi}\int_0^\infty \frac{\cos yt}{E_{2n}(iy)}dy$$

and Lemma 4.2 imply that

$$(1) \qquad G_{2n}(t) = \frac{r_{n+1}}{\pi}\int_0^\delta \cos(r_{n+1}tu)\{\exp(-r_{n+1}^2 v_n u^2) + O[r_{n+1}^2 S_n u^4 \exp(-c r_{n+1}^2 S_n u^2)]\}du + \int_{\delta r_{n+1}}^\infty \frac{\cos ty}{E_{2n}(iy)}dy .$$

The hypothesis that $\lim_{n\to\infty} r_{n+1}^{4/3} S_n = \infty$ guarantees that for all n sufficiently large we have $S_n - r_k^{-2} > (1/2)S_n$. Hence for all large n the second lower bound of part A of Lemma 2.2 satisfies

$$|E_{2n}(iy)| \geqq 1 + \frac{1}{2}y^4 \sin^2\eta \sum_{n+1}^\infty \frac{1}{r_k^2}\left(S_n - \frac{1}{r_k^2}\right) \geqq 1 + \left(\frac{1}{2}y^2 S_n \sin\eta\right)^2 .$$

The latter inequality shows that

$$(2) \qquad \int_{\delta r_{n+1}}^\infty \frac{1}{|E_{2n}(iy)|}dy = O(r_{n+1}^{-3}S_n^{-2}) \qquad (n\to\infty) .$$

Note that

$$r_{n+1}^{-3}S_n^{-2} = O(r_{n+1}^{-2}S_n^{-3/2}) \qquad (n\to\infty) .$$

For any v with $Re\, v > 0$, the function $k(t, v)$ has the representation

$$(3) \qquad k(t, v) = \frac{1}{\pi}\int_0^\infty e^{-vu^2}\cos tu\, du \qquad (-\infty < t < \infty) .$$

Use of (2) and (3) in equation (1) together with some elementary power series estimates of the exponential function give the conclusion of the theorem.

We saw in Theorem 2.1 that the essential step in the proof of the inversion formula was to show that

$$\int_{-\infty}^{\infty} | G_{2n}(t) | \, dt = O(1) \qquad (n \to \infty) .$$

The next theorem gives a more precise asymptotic formula for the L^1-norms of the kernels $G_{2n}(t)$.

If η and v_n are as in Condition A and in equation (4.2), let φ_n be defined by

$$v_n = | v_n | e^{-2i\varphi_n} \tag{4.4}$$

with $| \varphi_n | \leqq \pi/4 - \eta$. The latter implies that in the next corollary we have

$$(\cos^2 \varphi_n - \sin^2 \varphi n)^{-1/2} \leqq (\sin 2\eta)^{-1/2} .$$

COROLLARY 4.3. *Let* $G_{2n}(t)$, φ_n, *and* η *be as in Theorem* 4.1, *equation* (4.4) *and Condition A respectively. Then*

$$\int_{-\infty}^{\infty} | G_{2n}(t) | \, dt \sim (\cos^2 \varphi_n - \sin^2 \varphi_n)^{-1/2} \qquad (n \to \infty) .$$

Proof. Our first estimate of $G_{2n}(t)$ from Lemma 2.4,

$$| G_{2n}(t) | < M S_n^{-1/2} \exp(-K S_n^{-1/2} | t |) ,$$

shows that

$$\int_{-\infty}^{\infty} | G_{2n}(t) | \, dt \sim \int_{-1}^{1} | G_{2n}(t) | \, dt \qquad (n \to \infty) .$$

An elementary integration shows that

$$\int_{-\infty}^{\infty} | k(t, | v_n | e^{-2i\varphi_n}) | \, dt = (\cos^2 \varphi_n - \sin^2 \varphi_n)^{-1/2} ,$$

and that

$$\lim_{n \to \infty} \int_{1 < |t| < \infty} | k(t, v_n) | \, dt = 0 .$$

Finally, our second estimate of $G_{2n}(t)$ from Theorem (4.1),

$$G_{2n}(t) = k(t, v_n) + O(| a_{n+1} |^{-2} S_n^{-3/2}) ,$$

together with the assumption B that $| a_{n+1} |^{-2} S_n^{-3/2}$ goes to zero with $1/n$, gives the conclusion of the theorem,

$$\int_{-\infty}^{\infty} |G_{2n}(t)|\,dt \sim (\cos^2\varphi_n - \sin^2\varphi_n)^{-1/2} \qquad (n \to \infty)\ .$$

REMARKS 1. If the roots a_k defining the kernels $G_{2n}(t)$ are of the form $a_k = r_k e^{i\beta}$ for some $|\beta| < \pi/4$, then $\varphi_n = \beta$ for all n, and the asymptotic formula of the previous corollary becomes infinite as $\beta \to \pi/4$. The latter fact suggests that our present methods cannot be used to generalize the inversion theorem 3.5 in order to allow the roots to lie in any angular sector about the real axis exceeding or even equal to forty five degrees.

2. It is an open question whether all the results of this section are valid if the hypothesis that $\lim_{n\to\infty} r_{n+1}^2 S_n^{3/2} = \infty$ is replaced by the weaker assumption that $\lim_{n\to\infty} r_{n+1}^2 S_n = \infty$.

3. It is also an open question whether under some assumption similar to Condition B the integral

$$\int_{-\infty}^{\infty} |tG'_{2n}(t)|\,dt$$

is asymptotic to a constant times $(\cos^2\varphi_n - \sin^2\varphi_n)^{-3/2}$.

5. An explicit example. In this section the sequence of kernels $G_{2n}(t)$ is explicitly evaluated corresponding to $E(s) = \cos(\pi e^{-i\beta}s)$ where β is some number in $|\beta| < \pi/4$.

If $E_{2n}(s)$ is the function

$$E_{2n}(s) = \prod_{n+1}^{\infty}\left(1 - \frac{s^2}{(k-1/2)^2 e^{2i\beta}}\right) \qquad (n = 0, 1, 2, \cdots)\ ,$$

then as in equation (2.3), the kernel $G_{2n}(t)$ is given by

$$G_{2n}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{E_{2n}(s)}\,ds \qquad (-\infty < t < \infty;\ n = 0, 1, 2, \cdots)\ .$$

Let a and w be $a = e^{i\beta}$ and $w = e^{at}$. For $k > n$, the residue of the integrand $e^{st}/E_{2n}(s)$ at $s = (k-1/2)a$ is

$$\frac{a}{\pi}\, w^{(2k-1)/2}(-1)^k \prod_{j=1}^{n}\left(1 - \frac{(k-1/2)^2}{(j-1/2)^2}\right) = cw^{k-1}(-1)^{n+k}\frac{(k+n-1)!}{(k-n-1)!}\ ,$$

where c is defined as

$$c = \frac{a \exp(at/2) 2^{4n} n!^2}{\pi (2n)!^2}\ .$$

The kernel $G_{2n}(t)$ is easily seen to be the sum of the residues in the right half plane $Re\, s > 0$, i.e.

$$G_{2n}(t) = cw^n \left(\frac{d}{dw}\right)^{2n} \frac{w^{2n}}{1+w} \qquad (-\infty < t < \infty;\ n = 0, 1, 2, \cdots)\ .$$

By use of the Leibnitz rule for differentiation of products, we obtain

$$(5.1)\quad G_{2n}(t) = \frac{2^{2n-1}n!^2}{\pi(2n)!} a\left[\operatorname{sech} \frac{at}{2}\right]^{2n+1} \quad (-\infty < t < \infty;\ n = 0, 1, 2, \cdots)\ .$$

REMARKS 1. Although the above computation is also valid for any β with $\pi/4 < |\beta| < \pi/2$ it can be shown that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |G_{2n}(t)|\, dt = \infty$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |tG'_{2n}(t)|\, dt = \infty$$

for such a β.

2. Perhaps the inversion Theorem 2.1 remains valid if the roots a_k are allowed to lie in an angular sector of exactly forty-five degrees provided the function $\varphi(t)$ is continuous and of bounded variation at the point $t = x$ at which its value is to be recovered. The latter has been shown to be true in [1] for the special kernel $G(t)$ given by (5.1) with $\beta = \pi/4$.

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HARVARD UNIVERSITY