PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

ON A CLASS OF CAUCHY EXPONENTIAL SERIES

J. A. ANDERSON AND G. H. FULLERTON

This paper was received before the synoptic introduction became a requirement.

1. **Introduction.** Let $Q(z)$ be a meromorphic function with poles $z_1, z_2, z_3, \cdots$, the notation being so chosen that $|z_1| \leqq |z_2| \leqq |z_3| \leqq \cdots$. If $f \in L(0, 1)$, define

$$c_\nu e^{z_\nu x} = \text{res}_{z_\nu} Q(z) \int_0^1 f(t) e^{z(x-t)} dt .$$

Then, the series $\Sigma c_\nu e^{z_\nu x}$ is called the Cauchy Exponential Series (CES) of f with respect to $Q(z)$. If z_ν is of multiplicity m, then c_ν is a polynomial in x of degree at most $m - 1$; if the poles are all simple, with residue λ_ν at z_ν, we may write

$$c_\nu = \lambda_\nu \int_0^1 f(t) e^{-z_\nu t} dt \tag{1}$$

and $\{c_\nu\}$, independent of x, are called the CE constants.

Let $C_p: |z| = r_p$ be an expanding sequence of contours, none of which passes through a pole of $Q(z)$. Suppose C_p contains n_p poles of $Q(z)$. Then,

$$\begin{aligned} \sum_{\nu=1}^{n_p} c_\nu e^{z_\nu x} &= \frac{1}{2\pi i} \int_{C_p} Q(z) dz \int_0^1 f(t) e^{z(x-t)} dt , \\ &= I_p , \quad \text{say} . \end{aligned}$$

Denote by C_p^+, C_p^- the parts of C_p lying in the right, left half-planes respectively. If $Q(z)$ is approximately unity on C_p^+, and is small on C_p^-, in the sense that

$$\int_{C_p^+} (Q(z) - 1) dz \int_0^1 f(t) e^{z(x-t)} dt = o(1) \tag{2}$$

$$\int_{C_p^-} Q(z) dz \int_0^1 f(t) e^{z(x-t)} dt = o(1) \tag{3}$$

as $p \to \infty$, uniformly for $x \in [0, 1]$, then

$$\begin{aligned} I_p &= \frac{1}{2\pi i} \int_{C_p^+} dz \int_0^1 f(t) e^{z(x-t)} dt + o(1) \\ &= \frac{1}{\pi} \int_0^1 \frac{f(t) \sin r_p(x-t)}{x-t} dt + o(1) \end{aligned}$$

Received March 4, 1963.

uniformly in $[0, 1]$, and so the sums I_p behave somewhat like the partial sums of a Fourier series (F.s.). Indeed, when

$$Q(z) = e^z/e^z - 1$$

the CES is the F.s. of f.

In this paper, we shall suppose that

$$Q(z) = \frac{e^z a(z)}{e^z a(z) + b(z)} = \frac{e^z a(z)}{G(z)} \tag{4}$$

where $a(z), b(z)$ are relatively prime polynomials of degree n, and that all the poles are simple. This case was investigated first by Fullerton ([1], 1–34), using a less convenient notation.

The large zeros of $G(z)$ approximate to those of $e^z - c$, where

$$c = -\lim_{|z|\to\infty} b(z)/a(z) \tag{5}$$

i.e. to the points $\{\zeta + 2\pi p i\}$, ζ being the principal value of $\log c$. Hence there is a δ, $0 < \delta < 2\pi$, such that if $r_p = 2p\pi + \delta$, each point of C_p is at a distance greater than a positive constant from the zeros of $G(z)$ and of $e^z - c$. This enables us to prove

THEOREM 1. *Let $f \in L(0, 1)$. Then, as $p \to \infty$,*

$$\sum_{\nu=1}^{n_p} c_\nu e^{z_\nu x} - e^{\zeta x} s_p(x) \to 0$$

uniformly for $x \in [0, 1]$, where $s_p(x)$ is the pth *partial sum of the F.s. of $f(t)e^{-\zeta t}$.*

We next show that there are n relations connecting the CE constants.

THEOREM 2. *Let $f \in L(0, 1)$. If c_ν is defined by* (1), *for $\nu = 1, 2, \cdots$, then*

$$\sum_{\nu=1}^{\infty} \frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)} = 0 \tag{6}$$

$(r = 0, 1, \cdots, n - 1)$, *where $F(z) = e^{-z}G(z)$.*

This naturally leads to the following question: if a sequence of numbers $\{\beta_\nu\}$ satisfies $\sum_{\nu=1}^{\infty} c_\nu \beta_\nu = 0$, what is the nature of the β_ν? The answer is given by

THEOREM 3. *Let $\{\beta_\nu\}$ be a sequence of numbers such that $\sum_{\nu=1}^{\infty} c_\nu \beta_\nu = 0$ for every* CES *$\Sigma c_\nu e^{z_\nu x}$. Then, there are constants*

$\alpha_0, \cdots, \alpha_{n-1}$ *such that*

$$\beta_\nu = \sum_{r=0}^{n-1} \frac{\alpha_r z_\nu^r}{\lambda_\nu F'(z_\nu)} .$$

Because of the relations (6), we cannot expect that, given a sequence $\{c_\nu\}$ with $\sum_{\nu=1}^{\infty} |c_\nu|^2 < \infty$, there is a function $f \in L^2(0, 1)$ such that (1) is true for each ν. However, we can prove

THEOREM 4. *If* $\{c_\nu\}$, $\nu > n$, *is a sequence with* $\sum_{\nu>n} |c_\nu|^2 < \infty$, *there is a function* $f \in L^2(0, 1)$ *such that* (1) *is true for each* $\nu > n$, *and upon defining* $c_1, \cdots, c_n$ *by* (1), *such that* $\sum_{\nu=1}^{\infty} c_\nu e^{z_\nu x}$ *converges in mean to* f.

Alternatively, we can alter every c_ν and so obtain a Riesz-Fischer analogue. We have

THEOREM 5. *Let* $\{c_\nu\}$ *be a sequence with* $\sum_{\nu=1}^{\infty} |c_\nu|^2 < \infty$. *Then, there are constants* $\gamma_0, \cdots, \gamma_{n-1}$ *such that if*

$$d_\nu = c_\nu + \sum_{r=0}^{n-1} \frac{\gamma_r z_\nu^r}{G'(z_\nu)} ,$$

the numbers d_ν *are the* CE *constants of a function* $f \in L^2(0, 1)$.

We next investigate the problem of the uniqueness of CES. We prove

THEOREM 6. *If* $\sum_{\nu=1}^{\infty} d_\nu e^{z_\nu x} = f(x)$ *almost everywhere in* $[0, 1]$, *then there are constants* $\sigma_0, \cdots, \sigma_{n-1}$ *such that*

$$d_\nu = \lambda_\nu \int_0^1 f(t) e^{-z_\nu t} dt + \sum_{r=0}^{n-1} \frac{\sigma_r z_\nu^r}{G'(z_\nu)} \tag{7}$$

Finally, the question arises whether it is possible to generalise the function $Q(z)$ given by (4), so that the CES of f is uniformly equi-convergent with a F.s. The functions

$$P(z) = \frac{e^z \alpha(z) + \beta(z)}{e^z a(z) + b(z)}$$

where $\alpha(z)$, $\beta(z)$ are polynomials of degree n, are obvious generalisations. As $\boldsymbol{Re}\, z \to \infty$, $P(z)$ tends to a number $\omega_1 \neq 0$; as $\boldsymbol{Re}\, z \to -\infty$, to $\omega_2 \neq 0$. Suppose $\omega_1 \neq \omega_2$, and define

$$Q_1(z) = \frac{1}{\omega_1 - \omega_2} \{P(z) - \omega_2\} ;$$

then $Q_1(z)$ satisfies (2), (3). If the CES of f with respect to $Q_1(z)$ is uniformly equiconvergent in $[0, 1]$ with $e^{\zeta x}$ multiplied by the F.s. of $f(t)e^{-\zeta t}$, for each $f \in L(0, 1)$, then

$$\alpha(z) = \omega_1 a(z) \qquad \text{and} \qquad \beta(z) = \omega_2 b(z) ,$$

so that $P(z) = (\omega_1 - \omega_2)Q(z) + \omega_2$. We omit the proof.

2. Proof of Theorem 1. In (4), write

$$Q(z) = \frac{e^z}{e^z - c} + R(z) ;$$

then

$$R(z) = \frac{-e^z\{ca(z) + b(z)\}}{(e^z - c)G(z)} .$$

By the choice of C_p, there is a positive constant A such that, on C_p,

$$|e^z - c| > A \max(|e^z|, 1)$$
$$|G(z)| > A \max(|e^z z^n|, |z^n|) .$$

Further, by (5),

$$ca(z) + b(z) = O(|z^{n-1}|)$$

as $|z| \to \infty$. Hence,

$$\begin{aligned}\int_{C_p^+} R(z)dz \int_0^1 f(t)e^{z(x-t)}dt &= O\left(\int_{C_p^+}\left|\frac{e^{z(x-1)}}{z}dz\int_0^1 f(t)e^{-zt}dt\right|\right)\\ &= o\left(\int_{C_p^+}\left|\frac{e^{z(x-1)}}{z}dz\right|\right)\\ &= o(1)\end{aligned}$$

as $p \to \infty$, uniformly for $x \leqq 1$. Similarly,

$$\begin{aligned}\int_{C_p^-} R(z)dz \int_0^1 f(t)e^{z(x-t)}dt &= O\left(\int_{C_p^-}\left|\frac{e^{zx}}{z}dz\int_0^1 f(t)e^{z(1-t)}dt\right|\right)\\ &= o\left(\int_{C_p^-}\left|\frac{e^{zx}}{z}dz\right|\right)\\ &= o(1)\end{aligned}$$

as $p \to \infty$, uniformly for $x \geqq 0$.

Since, for large p, the number of zeros of $e^z - c$ inside C_p differs from $2p + 1$ by at most 1 and

$$\int_0^1 f(t)e^{(\zeta+2p\pi i)(x-t)}dt = o(1) ,$$

it follows that

$$\sum_{\nu=1}^{n_p} c_\nu e^{z_\nu x} = \frac{1}{2\pi i}\int_{C_p} \frac{e^z}{e^z - c} dz \int_0^1 f(t) e^{z(x-t)} dt + o(1)$$
$$= \sum_{\nu=-p}^{p} \int_0^1 f(t) e^{(\zeta + 2\pi i \nu)(x-t)} dt + o(1)$$
$$= e^{\zeta x} s_p(x) + o(1)$$

as $p \to \infty$, uniformly in $[0, 1]$, and this completes the proof.

3. The proof of Theorem 2 will depend upon

LEMMA 1. *For* $r = 0, 1, \cdots, n-1$,

$$\int_{C_p} \frac{z^r e^{-zt}}{F(z)} dz = o(1)$$

as $p \to \infty$, *boundedly for* $0 < t < 1$.

Proof. Define C_p^+, C_p^- as in § 1; then, for $r = 0, 1, \cdots, n-1$,

$$\int_{C_p^+} \frac{z^r e^{-zt}}{F(z)} dz = O\Big(\int_{C_p^+} | z^{r-n} e^{-zt} dz |\Big)$$
$$= O\Big(\int_{-\pi/2}^{\pi/2} \exp(-t\rho\cos\theta) d\theta\Big) \quad (\rho = r_p)$$
$$= O\Big(\int_0^{\pi/2} \exp(-t\rho\sin\theta) d\theta\Big)$$
$$= O\Big(\int_0^{\pi/2} \exp\Big(-\frac{2t\rho\theta}{\pi}\Big) d\theta\Big)$$

which is $o(1)$ as $p \to \infty$, boundedly for $t > 0$. Similarly,

$$\int_{C_p^-} \frac{z^r e^{-zt}}{F(z)} dz = o(1)$$

boundedly for $t < 1$. Hence the result.

4. Proof of Theorem 2. Since the zeros of $F(z)$ are simple,

$$\operatorname{res}_{z_\nu} \frac{z^r e^{-zt}}{F(z)} = \frac{z_\nu^r e^{-z_\nu t}}{F'(z_\nu)} ;$$

hence, by Lemma 1, for $r = 0, 1, \cdots, n-1$,

$$\sum_{\nu=1}^{n_p} \frac{z_\nu^r e^{-z_\nu t}}{F'(z_\nu)} = o(1)$$

as $p \to \infty$, boundedly for $0 < t < 1$. By the choice of C_p, $n_{p+1} - n_p = 2$

for large p, and so, since the terms are $o(1)$ as $\nu \to \infty$, we may replace n_p by p in the above summation. If we multiply by $f(t)$ and integrate over $[0, 1]$, we have (6).

5. We now prove

LEMMA 2. *Let* $a(z) = \sum_{k=0}^{n} a_k z^k$, and $b(z) = \sum_{k=0}^{n} b_k z^k$. *Then,*

$$\sum_{r=0}^{n-1} z_\mu^r \sum_{k=r+1}^{n} (b_k + a_k e^{z_\nu}) z_\nu^{k-r-1} + a(z_\mu) e^{z_\mu} \int_0^1 e^{(z_\nu - z_\mu)t} dt = \begin{cases} 0 & \nu \neq \mu \\ G'(z_\mu) & \nu = \mu \end{cases} \tag{8}$$

Proof. Write the left-hand side of (8) as

$$\mathscr{L} + \mathscr{M}; \tag{9}$$

then,

$$\mathscr{L} = \sum_{k=1}^{n} (b_k + a_k e^{z_\nu}) \sum_{r=0}^{k-1} z_\mu^r z_\nu^{k-r-1} .$$

If $\nu \neq \mu$,

$$\mathscr{L} = \frac{b(z_\nu) - b(z_\mu) + e^{z_\nu}\{a(z_\nu) - a(z_\mu)\}}{z_\nu - z_\mu} .$$

$$\mathscr{M} = a(z_\mu) \frac{e^{z_\nu} - e^{z_\mu}}{z_\nu - z_\mu} ;$$

since $G(z_\nu) = G(z_\mu) = 0$, (9) is zero. If $\nu = \mu$, (9) is

$$\begin{aligned} \sum_{k=1}^{n} k(b_k + a_k e^{z_\mu}) z_\mu^{k-1} &+ a(z_\mu) e^{z_\mu} \\ &= b'(z_\mu) + e^{z_\mu}(a'(z_\mu) + a(z_\mu)) \\ &= G'(z_\mu) . \end{aligned}$$

This proves the lemma.

6. **Proof of Theorem 3.** We have $\sum_{\nu=1}^{\infty} c_\nu \beta_\nu = 0$ for every sequence $\{c_\nu\}$ of CE constants, i.e.

$$\sum_{\nu=1}^{\infty} \beta_\nu \lambda_\nu \int_0^1 f(t) e^{-z_\nu t} dt = 0$$

for every $f \in L(0, 1)$. Hence, by a well-known theorem ([2], § 279),

$$\int_{1-x}^{1} \sum_{\nu=1}^{p} \beta_\nu \lambda_\nu e^{-z_\nu t} dt \to 0 \tag{10}$$

as $p \to \infty$, boundedly for $x \in [0, 1]$. We recall (8); if we multiply by $\beta_\nu \lambda_\nu e^{-z_\nu}$ and sum from $\nu = 1$ to $\nu = p$, where p is greater than an assigned integer μ, we obtain

$$\beta_\mu\lambda_\mu e^{-z_\mu}G'(z_\mu) = \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=1}^{p} \beta_\nu\lambda_\nu \sum_{k=r+1}^{n} (b_k e^{-z_\nu} + a_k) z_\nu^{k-r-1} + a(z_\mu)e^{z_\mu}\int_0^1 e^{-z_\mu t}\sum_{\nu=1}^{p}\beta_\nu\lambda_\nu e^{z_\nu(t-1)}dt = \sum_{r=0}^{n-1} L_{r,p} z_\mu^r + \mathscr{N}_p , \qquad \text{say.}$$

Let

$$\phi_p(t) = \sum_{\nu=1}^{p} \beta_\nu\lambda_\nu e^{z_\nu(t-1)} ,$$

$$\Phi_p(x) = \int_0^x \phi_p(t)dt = \int_{1-x}^1 \sum_{\nu=1}^{p} \beta_\nu\lambda_\nu e^{-z_\nu t}dt .$$

By (10), $\Phi_p(x) \to 0$ as $p \to \infty$, boundedly for $x \in [0, 1]$. Thus,

$$\mathscr{N}_p = a(z_\mu)e^{z_\mu}\int_0^1 e^{-z_\mu t}\phi_p(t)dt = a(z_\mu)e^{z_\mu}\left\{\Phi_p(1)e^{-z_\mu} + z_\mu\int_0^1 e^{-z_\mu t}\Phi_p(t)dt\right\} = o(1) \qquad \text{as } p \to \infty .$$

Hence, since $e^{-z}G(z) = F(z)$,

$$\sum_{r=0}^{n-1} L_{r,p}\, z_\mu^r = \beta_\mu\lambda_\mu F'(z_\mu) + \varepsilon_\mu \tag{11}$$

where the numbers $\{L_{r,p}\}$ are independent of μ, and $\varepsilon_\mu \to 0$ as $p \to \infty$.

Giving μ distinct values $\mu_1, \cdots, \mu_n$, (11) yields a regular system of n linear equations for $L_{0,p}, \cdots, L_{n-1,p}$. The solution is

$$L_{r,p} = \frac{\sum_{i=1}^{n} \{\beta_{\mu_i}\lambda_{\mu_i}F'(z_{\mu_i}) + \varepsilon_{\mu_i}\}\Delta_i^{(r)}}{\det (z_{\mu_i}^{j-1})}$$

where $\Delta_i^{(r)}$ are cofactors of elements in the $(r + 1)$th column of the matrix $(z_{\mu_i}^{j-1})$, $(i, j = 1, 2, \cdots, n)$. The only nonconstant terms in this expression for $L_{r,p}$ are ε_{μ_i}, which are $o(1)$ as $p \to \infty$. Hence, for $r = 0, 1, \cdots, n - 1$, $\{L_{r,p}\}$ converges, to α_r say. Letting $p \to \infty$ in (11), we have the result.

7. To prove Theorem 4, we require three lemmas.

LEMMA 3. *If $p > n$, there are numbers $d_1, \cdots, d_n$ such that*

$$e^{z_p x} + \sum_{k=1}^{n} d_k e^{z_k x}$$

is its own CES.

Proof. We shall show that there are numbers $d_1, \cdots, d_n$ such that, if

$$S(x) = e^{z_p x} + \sum_{k=1}^{n} d_k e^{z_k x} ,$$

then, for $\mu \notin \{1, \cdots, n, p\}$,

$$\int_0^1 S(x) e^{-z_\mu x} dx = 0 . \tag{12}$$

Since the functions $e^{z_1 x}, \cdots, e^{z_n x}, e^{z_p x}$ are linearly independent, and by Theorem 1, the CES of $S(x)$ converges everywhere in $(0, 1)$ to $S(x)$, it will then follow that $S(x)$ is its own CES.

For $\mu \neq k$,

$$\begin{aligned}\int_0^1 e^{(z_k - z_\mu)x} dx &= \frac{e^{-z_\mu}}{z_k - z_\mu}\{e^{z_k} - e^{s}_{\mu}\} \\ &= \frac{e^{-z_\mu}\{a(z_k)b(z_\mu) - a(z_\mu)b(z_k)\}}{a(z_k)a(z_\mu)(z_k - z_\mu)} \\ &= \frac{e^{-z_\mu}\sigma(z_k, z_\mu)}{a(z_k)a(z_\mu)} , \qquad \text{say.}\end{aligned}$$

Thus, if $\mu \notin \{1, \cdots, n, p\}$, and $d_1, \cdots, d_n$ are any n numbers, the left-hand side of (12) is

$$\begin{aligned}&\frac{e^{-z_\mu}}{a(z_\mu)}\left\{\frac{\sigma(z_p, z_\mu)}{a(z_p)} + \sum_{k=1}^{n} \frac{d_k \sigma(z_k, z_\mu)}{a(z_k)}\right\} \\ &\qquad = \frac{e^{-z_\mu}}{a(z_\mu)a(z_p)}\left\{\sigma(z_p, z_\mu) + \sum_{k=1}^{n} \delta_k \sigma(z_k, z_\mu)\right\} \\ &\qquad = I_\mu \qquad \text{say, where } \delta_k = \frac{a(z_p)d_k}{a(z_k)}.\end{aligned}$$

The symmetric polynomial

$$\sigma(x, y) = \frac{a(x)b(y) - a(y)b(x)}{x - y}$$

can be expressed in the form

$$\sum_{r=0}^{n-1} P_r(x) y^r$$

where $P_r(x)$ is a polynomial in x of degree at most $n - 1$. Then,

$$I_\mu = \frac{e^{-z_\mu}}{a(z_\mu)a(z_p)} \sum_{r=0}^{n-1} z_\mu^r \left\{ P_r(z_p) + \sum_{k=1}^{n} \delta_k P_r(z_k) \right\} .$$

This is zero for each $\mu \notin \{1, \cdots, n, p\}$ if

$$P_r(z_p) + \sum_{k=1}^{n} \delta_k P_r(z_k) = 0 \qquad (r = 0, 1, \cdots, n-1),$$

which happens if

$$z_p^r + \sum_{k=1}^{n} \delta_k z_k^r = 0 \qquad (r = 0, 1, \cdots, n-1).$$

Since this system of n linear equations for the unknowns $\delta_1, \cdots, \delta_n$ is regular, the lemma follows.

COROLLARY. *Given the constants $c_{n+1}, \cdots, c_p$ of Theorem 4, there are numbers $c_1^{(p)}, \cdots, c_n^{(p)}$ such that*

$$T_p(x) = \sum_{k=1}^{n} c_k^{(p)} e^{z_k x} + \sum_{\nu=n+1}^{p} c_\nu e^{z_\nu x}$$

is its own CES.

LEMMA 4. *The numbers $c_1^{(p)}, \cdots, c_n^{(p)}$ are unique and, for $k = 1, 2, \cdots, n$, the sequence $\{c_k^{(p)}\}$ converges.*

Proof. By Theorem 2, the numbers $c_1^{(p)}, \cdots, c_n^{(p)}$ satisfy the regular system of linear equations

$$\frac{c_1^{(p)} z_1^r}{\lambda_1 F'(z_1)} + \cdots + \frac{c_n^{(p)} z_n^r}{\lambda_n F'(z_n)} = -\sum_{\nu=n+1}^{p} \frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)}$$

$(r=0, 1, \cdots, n-1)$, and so are determined uniquely. Since $\sum_{\nu>n} |c_\nu|^2 < \infty$, and

$$|\lambda_\nu F'(z_\nu)| > K |z_\nu^n|$$

where K is a constant,

$$\sum_{\nu=n+1}^{p} \frac{c_\nu z_\nu^r}{\lambda_\nu F'(z_\nu)}$$

converges, for $r = 0, 1, \cdots, n-1$. Hence, by an argument used in the proof of Theorem 3, $\{c_k^{(p)}\}$ converges, for $k = 1, 2, \cdots, n$.

LEMMA 5. *There is a positive constant A such that if $\{a_\nu\}$ is any finite set of numbers, then*

$$\int_0^1 |\Sigma a_\nu e^{z_\nu x}|^2 \, dx \leqq A \Sigma |a_\nu|^2 .$$

This may be proved by an argument similar to that of Lemma 3 of [3].

8. Proof of Theorem 4. Let p, q be integers such that $q > p > n$. Then,

$$T_q(x) - T_p(x) = \sum_{k=1}^{n} (c_k^{(q)} - c_k^{(p)}) e^{z_k x} + \sum_{\nu=p+1}^{q} c_\nu e^{z_\nu x} .$$

By Lemma 5, there is a constant $A > 0$ such that

$$\int_0^1 | T_q(x) - T_p(x) |^2 dx \leqq A \left\{ \sum_{k=1}^{n} | c_k^{(q)} - c_k^{(p)} |^2 + \sum_{\nu=p+1}^{q} | c_\nu |^2 \right\} .$$

Hence, by Lemma 4, $\{T_p(x)\}$ converges in mean to a function $f \in L^2(0, 1)$.

Let $\nu > n$. Since $T_p(x)$ is its own CES,

$$c_\nu = \lambda_\nu \int_0^1 T_p(x) e^{-z_\nu x} dx \qquad (p \geqq \nu).$$

Hence,

$$\begin{aligned} c_\nu &= \lambda_\nu \lim_{p \to \infty} \int_0^1 T_p(x) e^{-z_\nu x}\, dx \\ &= \lambda_\nu \int_0^1 f(x) e^{-z_\nu x} dx . \end{aligned}$$

Define $c_1, \cdots, c_n$ by this formula; then,

$$c_k = \lim_{p \to \infty} c_k^{(p)} \qquad (k = 1, 2, \cdots, n) ,$$

and $\sum_{\nu=1}^{\infty} c_\nu e^{z_\nu x}$ converges in mean to f. This completes the proof.

9. Proof of Theorem 5. If we multiply (8) by c_ν and sum from $\nu = 1$ to $\nu = p$, where p is greater than an assigned integer μ, we obtain

$$\begin{aligned} c_\mu G'(z_\mu) &= \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=1}^{p} c_\nu \sum_{k=z+1}^{n} (a_k e^{r_\nu} + b_k) z_\nu^{k-r-1} \\ &\quad + a(z_\mu) e^{z_\mu} \int_0^1 e^{-z_\mu t} \sum_{\nu=1}^{p} c_\nu e^{z_\nu t} dt \qquad (13) \\ &= \mathscr{L}_p + \mathscr{M}_p , \qquad \text{say.} \end{aligned}$$

Since $\sum_{\nu=1}^{\infty} | c_\nu |^2 < \infty$, $\sum_{\nu=1}^{\infty} c_\nu e^{z_\nu t}$ converges in mean to a function $f \in L^2(0, 1)$. Hence,

$$\mathscr{M}_p \to d_\mu G'(z_\mu) \qquad \text{as } p \to \infty$$

where

$$d_\mu = \lambda_\mu \int_0^1 f(t) e^{-z_\mu t} dt ,$$

Next,

$$\mathscr{L}_p = \sum_{r=0}^{n-1} \delta_r z_\mu^r - \sum_{r=0}^{n-1} z_\mu^r \sum_{\nu=2}^{p} c_\nu \sum_{k=0}^{z} (a_k e^{r_\nu} + b_k) z_\nu^{k-r-1} \tag{14}$$

where

$$\delta_r = c_1 \sum_{k=r+1}^{n} (a_k e^{r_1} + b_k) z_1^{k-r-1} .$$

Since

$$\sum_{k=0}^{r} (a_k e^{z_\nu} + b_k) z_\nu^{k-r-1} = O(\nu^{-1})$$

the summation over ν in (14) converges, as $p \to \infty$, to η_r say. The result now follows upon writing

$$\eta_r + \delta_r = \gamma_r .$$

10. Before establishing the uniqueness theorem, we prove two lemmas.

LEMMA 6. *If* $\sum_{\nu=1}^{\infty} d_\nu e^{z_\nu x} = f(x)$ *almost everywhere in* $[0, 1]$, *and* $d_\nu = O(\nu^{-2})$, *there are constants* $\sigma_0, \cdots, \sigma_{n-1}$ *such that* (7) *is satisfied for* $\nu = 1, 2, \cdots$.

Proof. We have (13), with c_ν replaced by d_ν. We may write this as

$$d_\mu G'(z_\mu) = \sum_{r=0}^{n-1} M_{r,p} z_\mu^r + \lambda_\mu G'(z_\mu) \int_0^1 e^{-z_\mu t} \Big\{ f(t) - \sum_{\nu=p+1}^{\infty} d_\nu e^{z_\nu t} \Big\} dt .$$

Since

$$\begin{aligned} \int_0^1 e^{-z_\mu t} \sum_{\nu=p+1}^{\infty} d_\nu e^{z_\nu t} dt &= O\Big(\sum_{\nu=p+1}^{\infty} | d_\nu | \Big) \\ &= o(1) \qquad \text{as } p \to \infty , \end{aligned}$$

and $\{M_{r,p}\}$ converges, to σ_r say, for $r = 0, 1, \cdots, n-1$, we obtain (7).

LEMMA 7. *If the series* $\sum_{\nu=2}^{\infty} b_\nu$ *is convergent, then*

$$\sum_{\nu=2}^{\infty} b_\nu \left(\frac{\sinh z_\nu h}{z_\nu h} \right)^2 \to \sum_{\nu=2}^{\infty} b_\nu$$

as $h \downarrow 0$.

Proof. By a classical result, it is sufficient to show that

(i) $\left(\dfrac{\sinh z_\nu h}{z_\nu h} \right)^2 \to 1$ as $h \downarrow 0$, for $\nu = 2, 3, \cdots$

(ii) $\sum_{\nu=2}^{\infty}\left|\left(\frac{\sinh z_{\nu+1}h}{z_{\nu+1}h}\right)^2-\left(\frac{\sinh z_\nu h}{z_\nu h}\right)^2\right|$

is bounded as $h \downarrow 0$. It is evident that (i) is satisfied; (ii) may be established by the method of Theorem 1 of [4].

11. Proof of Theorem 6. The hypothesis of convergence implies that $d_\nu = o(1)$. If we define

$$\Psi(x)=\sum_{\nu=2}^{\infty}\frac{d_\nu e^{z_\nu x}}{z_\nu^2} \tag{15}$$

this series is uniformly and absolutely convergent, in $[0, 1]$. Now

$$\frac{\Psi(x+2h)+\Psi(x-2h)-2\Psi(x)}{4h^2}=\sum_{\nu=2}^{\infty}d_\nu e^{z_\nu x}\left(\frac{\sinh z_\nu h}{z_\nu h}\right)^2$$

and hence, by Lemma 7, the second generalised derivative of $\Psi(x)$ equals $f(x)-d_1e^{z_1x}$ almost everywhere in $[0, 1]$. It follows that

$$\Psi(x)=\int_0^x dt\int_0^t (f(u)-d_1e^{z_1u})du+lx+m$$

where l, m are constants. Since

$$d_\nu/z_\nu^2=o(\nu^{-2}) ,$$

we may apply Lemma 6 to the series (15). Thus, there are constants $\alpha_0, \cdots, \alpha_{n-1}$ such that

$$\frac{d_\nu}{z_\nu^2}=\lambda_\nu\int_0^1\Psi(t)e^{-z_\nu t}dt+\sum_{z=0}^{n-1}\frac{\alpha_r z_\nu^r}{G'(z_\nu)} \tag{16}$$

for $\nu = 2, 3, \cdots$.

If we integrate by parts twice, we can write (16) in the form

$$d_\nu=\lambda_\nu\int_0^1 f(t)e^{-z_\nu t}dt+\sum_{r=0}^{n+1}\frac{\sigma_r z_\nu^r}{G'(z_\nu)} ,$$

where $\sigma_0, \cdots, \sigma_{n+1}$ are constants. Since $G'(z_\nu)\sim -b_n z_\nu^n$,

$$d_\nu=o(1) \qquad \text{and} \qquad \lambda_\nu\int_0^1 f(t)e^{-z_\nu t}dt=o(1) ,$$

we have

$$\sigma_n=\sigma_{n+1}=0 ,$$

and for $\nu = 2, 3, \cdots$, we have (7). Finally, by Theorem 1 and Lemma 1,

$$\sum_{\nu=1}^{\infty}\left\{\lambda_\nu\int_0^1 f(t)e^{-z_\nu t}dt+\sum_{r=0}^{n-1}\frac{\sigma_r z_\nu^r}{G'(z_\nu)}\right\}e^{z_\nu x}$$

is summable $(C, 1)$ almost everywhere in $[0, 1]$ to

$$f(x) - \left\{\lambda_1 \int_0^1 f(t) e^{-z_1 t} dt + \sum_{r=0}^{n-1} \frac{\sigma_r z_1^r}{G'(z_1)}\right\} e^{z_1 x}$$

so that we have (7) for $\nu = 1$, and the proof is complete.

In conclusion, the authors wish to express their gratitude to Professor S. Verblunsky of Belfast, for his helpful criticism and advice.

References

1. G. H. Fullerton, *Expansions of a function in a series of exponentials*, Ph. D. Thesis (Belfast, 1959)

2. E. W. Hobson, *The theory of functions of a real variable*, vol. II, (Cambridge, 1926)

3. S. Verblunsky, *On an expansion in exponential series* (II), Quart, J. Math. (Oxford) (2), **10** (1959), 99-109.

4. ———, *A uniqueness theorem for the exponential series of Herglotz*, Proc. Camb. Phil. Soc. **56** (1960), 220-232.

THE UNIVERSITY, NOTTINGHAM