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# ON EXCEPTIONAL JORDAN DIVISION ALGEBRAS

A. A. ALBERT

**In 1957 the author gave a construction of a class, of central simple exceptional Jordan algebras $\mathfrak{H}$, over any field $\mathfrak{F}$ of characteristic not two, called cyclic Jordan algebras. The principal ingredients of this construction were the following:**

**( I ) A cyclic cubic field $\mathfrak{K}$ with generating automorphism $S$ over $\mathfrak{F}$.**

**( II ) A Cayley algebra $\mathfrak{C}$, with $\mathfrak{K}$ as center, so that $\mathfrak{C}$ has dimension eight over $\mathfrak{K}$, and dimension 24 over $\mathfrak{F}$.**

**(III) A nonsingular linear transformation $T$ over $\mathfrak{F}$ of $\mathfrak{C}$, which induces $S$ in $\mathfrak{K}$, and commutes with the conjugate operation of $\mathfrak{C}$.**

**(IV) An element $g$ in $\mathfrak{C}$, and a nonzero element $\gamma$ of $\mathfrak{K}$, such that $g = gT$ and $g\bar{g} = [\gamma(\gamma S)(\gamma S^2)]^{-1}$. Thus $g$ is nonsingular. Also the polynomial algebra $\mathfrak{G} = \mathfrak{F}[g]$ is either a quadratic field over $\mathfrak{F}$ or is the direct sum, $\mathfrak{G} = e_1\mathfrak{F} \oplus e_2\mathfrak{F}$, of two copies $e_i\mathfrak{F}$ of $\mathfrak{F}$.**

**( V ) The properties $[g(xy)T] = [g(xT)](yT)$ and $xT^3 = g^{-1}xg$, for every $x$ and $y$ of $\mathfrak{C}$.**

**In the present paper we shall give a general solution of the equations of (V), and shall determine $T$ in terms of two parameters in $\mathfrak{L} = \mathfrak{K}[g]$ satisfying some conditions of an arithmetic type. We shall also provide a special set of values of all of the parameters of our construction, and shall so provide a proof of the existence of cyclic Jordan division algebras with attached Cayley algebra $\mathfrak{C}$ a division algebra.**

**The existence of a *transformation* $T$ *with the two properties* of (V) for some element $g = gT$, in the Cayley algebra $\mathfrak{C}$ which satisfies (IV), was demonstrated by the author in the 1957 paper[1] only in the case where $\mathfrak{G}$ is not a field, and consequently $\mathfrak{C}$ is a split algebra. In that case it was proved that cyclic Jordan division algebras do exist, for certain kinds[2] of fields $\mathfrak{F}$. Thus the case where $\mathfrak{G}$ is a field, and $\mathfrak{C}$ may possibly be a division algebra, remained.**

If $k$ is any element of an exceptional Jordan division algebra $\mathfrak{H}$, and $k$ is not in $\mathfrak{F}$, the subalgebra $\mathfrak{F}[k]$ is a cubic field over $\mathfrak{F}$. The algebra $\mathfrak{H}$ is then a cyclic Jordan division algebra if and only if an

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[1] See our *A construction of exceptional Jordan division algebras*, Annals of Math. **67** (1958), 1-28.

[2] The actual result is that $\mathfrak{F}$ *cannot be a finite field, an algebraic number field*, or indeed any field such that, if $\mathfrak{D}$ is an associative division algebra of degree three over its center $\mathfrak{F}$ then every nonzero element of $\mathfrak{F}$ is the norm of an element of $\mathfrak{D}$. Moreover, if $\mathfrak{F}_0$ is such that there exists such a division algebra $\mathfrak{D}$ over $\mathfrak{F}_0$, and $\eta$ is an indeterminate over $\mathfrak{F}_0$, then there exists a cyclic exceptional Jordan division algebra over $\mathfrak{F} = \mathfrak{F}_0(\eta)$.

element $k$ of $\mathfrak{H}$ exists such that $\mathfrak{F}[k]$ is a *cyclic* cubic field. Indeed the field $\mathfrak{R}$ of our construction is then isomorphic to the field $\mathfrak{F}[k]$. It is not known whether such an element $k$ always exists, but we shall be able to construct $\mathfrak{H}$ even when a cyclic $\mathfrak{F}[k]$ may not exist. Indeed, let $k$ be any element of $\mathfrak{H}$ not in $\mathfrak{F}$, $\mathfrak{F}[k]$ be noncyclic, and $\mathfrak{R}$ be a scalar extension of $\mathfrak{F}$ isomorphic to $\mathfrak{F}[k]$. Then there is a quadratic extension field $\mathfrak{F}[\omega]$ of $\mathfrak{F}$, such that $\mathfrak{M} = \mathfrak{R} \times \mathfrak{F}[\omega] = \mathfrak{R}(\omega)$ is a *splitting field* of the cubic minimum function of $k$ over $\mathfrak{F}$. Then $\mathfrak{M}$ is a normal field of degree six over $\mathfrak{F}$, and the automorphism group of $\mathfrak{M}$ over $\mathfrak{F}$ is generated by automorphisms $S$ and $J$ such that $S^3 = J^2 = I$, and $JS = S^2J$. The field $\mathfrak{R}$ is the *fixed field* of $\mathfrak{M}$ *under* $J$, and $\omega$ can be selected so that $\omega J = -\omega$, $\omega^2$ is in $\mathfrak{F}$. The field $\mathfrak{F}[\omega]$ is the fixed field of $\mathfrak{M}$ under $S$, and $S$ generates the automorphism group of the cyclic field $\mathfrak{M}$ of degree three over $\mathfrak{F}[\omega]$.

The ingredients of our cyclic construction may then be taken as a portion of the ingredients of a construction of what may now be called *bicyclic* Jordan division algebras. We shall also call $\mathfrak{M}$ a *bicyclic* field.

We replace $\mathfrak{R}$ by the field $\mathfrak{M}$ as the center of $\mathfrak{C}$, and $T$ will induce $S$ in $\mathfrak{M}$. Then we will have the following additional properties:

( VI ) There is a nonsingular linear transformation $P$ over $\mathfrak{F}$ of $\mathfrak{C}$ which induces $J$ in the center $\mathfrak{M}$ of $\mathfrak{C}$, and commutes with the conjugate operation of $\mathfrak{C}$. Then $g = gP$, and so $\mathfrak{F}[g]$ is a quadratic algebra over $\mathfrak{F}$.

(VII) *The property* $g[(xy)P] = [g(yP)](xP)$ *holds for every* $x$ *and* $y$ *of* $\mathfrak{C}$.

(VIII) *The transformations* $P$ *and* $T$ *are related by* $TP = PT^{-1}$, *and* $P^2 = I$ *is the identity transformation.*

We shall give a complete determination of $P$. Indeed our determination of $T$ depends on a normalization of a basis of $\mathfrak{C}$. As usual $\mathfrak{C}$ has a basis of elements generated by basal elements $u, v, w$ where $\mathfrak{F}[u] = \mathfrak{F}[g] = \mathfrak{G}$, $u^2 = \rho \neq 0$ in $\mathfrak{F}$, $v$ is selected so that $uv + vu = 0$ and $v^2 = \varphi \neq 0$ in $\mathfrak{M}$, $w$ is selected so that $uw + wu = vw + wv = 0$ and $w^2 = \psi \neq 0$ in $\mathfrak{M}$ (where $\mathfrak{M} = \mathfrak{R}$ in the cyclic case). In the normalization it is shown that $v$ can be selected so that $vT = av + w$ for $a$ in $\mathfrak{M}[u] = \mathfrak{L}^*$ (where $\mathfrak{L}^*$ becomes $\mathfrak{L} = \mathfrak{R}[u]$ in the cyclic case). Since $P^2 = I$ we show that $v$ can actually be selected so that $vP = v$. With this choice the equations determining $T$ also determine $P$ completely and no new parameters are introduced. With this determination of $T$ and $P$ *our results provide a construction*[3] *of all exceptional Jordan*

[3] Our construction certainly does provide a *form* for all exceptional Jordan division algebras and so provides a means of studying the properties of such algebras. It does not settle the question as to whether there are any bicyclic algebras which are not also cyclic, nor does it give a general solution of the arithmetic restrictions on the defining parameters. The latter problem is not really an algebraic problem and can hardly be expected to be solved without exact specification of the field $\mathfrak{F}$.

*division algebras.*

**2. The algebra $\mathfrak{H}$ and a subalgebra $\mathfrak{C}$.** A cyclic exceptional Jordan algebra $\mathfrak{H}$ can be described in terms of the ingredients given in (I)–(V). Let $\xi$ range over all elements of the cyclic field $\mathfrak{K}$, and $x$ range over all elements of the Cayley algebra $\mathfrak{C}$. Then $\mathfrak{H}$ consists of all three-rowed square matrices

$$A = A(\xi, x) = \begin{pmatrix} \xi & x & g(\gamma\bar{x})T^2 \\ \gamma\bar{x} & \xi S & xT \\ xT^2g^{-1} & (\gamma\bar{x})T & \xi S^2 \end{pmatrix}. \tag{1}$$

This set is closed with respect to the operation $A \cdot B$ defined in terms of the ordinary matrix product $AB$ by $2A \cdot B = AB + BA$. Then it may be seen that $\mathfrak{H}$ is a Jordan algebra. Let

$$\mathfrak{G} = \mathfrak{F}[g]\,, \quad \mathfrak{L} = \mathfrak{K}[g] = \mathfrak{K} \times \mathfrak{G}\,, \tag{2}$$

so that $\mathfrak{L}$ is a commutative associative algebra of dimension six over $\mathfrak{F}$. When $\mathfrak{G}$ is a field so is $\mathfrak{L}$, and $\mathfrak{L}$ is a cyclic field over $\mathfrak{G}$ with a generating automorphism $T$ induced by $gT = g$, $\xi T = \xi S$ for every element of the cyclic cubic field $\mathfrak{K}$ over $\mathfrak{F}$.

The set $\mathfrak{B}$, of all matrices $A(\xi, \eta)$ with $\xi$ in $\mathfrak{K}$ and $\eta$ in $\mathfrak{L}$, is a subalgebra of $\mathfrak{H}$. It is a *special* Jordan algebra of dimension $3 + 6 = 9$ over $\mathfrak{F}$. Let

$$D = A(\theta, 0)\,, \quad E = A(0, 1)\,, \tag{3}$$

where $\theta$ is any element generating $\mathfrak{K} = \mathfrak{F}(\theta)$ over $\mathfrak{F}$. We first derive the following result.

THEOREM 1. *The subalgebra of $\mathfrak{H}$ generated by $D$ and $E$ is $\mathfrak{B}$. The cyclic associative algebra $\mathfrak{B}^* = (\mathfrak{L}, S, g^{-1})$ over $\mathfrak{G}$, has center $\mathfrak{G}$, and $\mathfrak{B}^*$ has an involution $J$ such that $J$ fixes every element of $\mathfrak{K}$ and $gJ = \bar{g}$. Then $\mathfrak{B}^*$ is the associative envelope of $\mathfrak{B}$, and $\mathfrak{B}$ is isomorphic to the set of all elements $A = AJ$ of $\mathfrak{B}^*$.*

For the mapping $\xi \rightarrow A(\xi, 0)$ is an isomorphism of $\mathfrak{K}$ onto the set $\mathfrak{K}_0$ of all matrices $A(\xi, 0)$. Then $\mathfrak{K}_0$ is a cyclic field of degree three over $\mathfrak{F}$ isomorphic to $\mathfrak{K}$, and it has a generating isomorphism $S_0$ over $\mathfrak{F}$ defined by $[A(\xi, 0)]S_0 = A(\xi S, 0)$. Let $\mathfrak{B}_0$ be the *special* Jordan subalgebra of $\mathfrak{H}$ generated by $D$ and $E$ so that $\mathfrak{B}_0$ contains $\mathfrak{K}_0 = \mathfrak{F}[D]$. Then $\mathfrak{B}_0$ contains

$$(4)\qquad 2A(\xi, 0)\cdot E = \begin{pmatrix} 0 & \eta & g(\gamma S^2)(\eta S^2) \\ \gamma\eta & 0 & \eta S \\ g^{-1}\eta S^2 & (\gamma\eta)S & 0 \end{pmatrix} = A(0, \eta)$$

where $\eta = \xi + \xi S$. If $\xi \neq \xi S$ then $\eta \neq 0$. For otherwise $\eta S^2 = \xi S^2 + \xi = 0$ and so $\eta - \eta S^2 = \xi S - \xi S^2 = (\xi - \xi S)S = 0$ contrary to our hypothesis. But then the enveloping associative algebra $\mathfrak{B}_0^*$ of $\mathfrak{B}_0$ contains the matrix

$$(5)\qquad [A(\eta S^2, 0)]^{-1}[A(0, \eta)] = \begin{pmatrix} 0 & \mu & g(\gamma S^2) \\ \gamma & 0 & \mu S \\ g^{-1}(\mu S^2) & \gamma S & 0 \end{pmatrix} = C ,$$

where $\mu = \eta(\eta S^2)^{-1} \neq 1$. Thus $\mathfrak{B}_0^*$ contains

$$(6)\qquad Y = [A(1-\mu, 0)]^{-1}(E - C) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ g^{-1} & 0 & 0 \end{pmatrix},$$

where $Y^3 = g^{-1}I$. It follows that $\mathfrak{B}_0^*$ contains $gI$ and so contains $\mathfrak{G}$ and $\mathfrak{L}_0 = \mathfrak{K}_0[g]$. Thus $\mathfrak{B}_0^*$ contains the cyclic associative algebra $\mathfrak{B}_1 = \mathfrak{L}_0 + \mathfrak{L}_0 Y + \mathfrak{L}_0 Y^2 = (\mathfrak{L}_0, T_0, g^{-1})$, where the generating automorphism $T_0$ of $\mathfrak{L}_0$ over $\mathfrak{F}[g]$ is defined by

$$(7)\qquad YA(a, 0) = Y\begin{pmatrix} a & 0 & 0 \\ 0 & aT & 0 \\ 0 & 0 & aT^2 \end{pmatrix} = A(a, 0)T_0 Y = \begin{pmatrix} aT & 0 & a \\ 0 & aT^2 & 0 \\ 0 & 0 & a \end{pmatrix} Y ,$$

for every $a$ in $\mathfrak{L}_0$. Let us also define a mapping $J$ by

$$[A(a, 0)]J = \begin{pmatrix} a & 0 & 0 \\ 0 & aT & 0 \\ 0 & 0 & aT^2 \end{pmatrix} J = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{a}T & 0 \\ 0 & 0 & \bar{a}T^2 \end{pmatrix},$$

$$(Y)J = \begin{pmatrix} 0 & 0 & \gamma S^2 g \\ \gamma & 0 & 0 \\ 0 & \gamma S & 0 \end{pmatrix}$$

and we see that

$$(9)\qquad (Y)J = \begin{pmatrix} 0 & 0 & g \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma S & 0 \\ 0 & 0 & \gamma S^2 \end{pmatrix} = Y^{-1}A(\gamma, 0) .$$

It is easily verified that $J$ is an involution of $\mathfrak{B}_1^*$ and we note that $(YJ)^3 = g\gamma(\gamma S)(\gamma S^2)I = (\bar{g})^{-1}I = (Y^3)J$. It is also easy to verify that $\mathfrak{G}$ is the center of the algebra $\mathfrak{B}_1^*$. Note that, when $\mathfrak{G}$ is a field $\mathfrak{B}_1^*$ is contral simple over $\mathfrak{G}$. However, in the $g$-split case, where $\mathfrak{G}$ is not a field, $\mathfrak{B}_1^*$ is a direct sum of two cyclic central simple algebras over $\mathfrak{F}$.

The general element $X$ of $\mathfrak{B}_1^*$ can be expressed uniquely in the form $X = A(a, 0) + A(b, 0)Y + (Y)JA(c, 0)$ for $a, b, c$ in $\mathfrak{L}_0$. Then $X = XJ$ if and only if $a = \bar{a}$ and $b = \bar{c}$. Thus $a = \bar{a} = \xi$ is in $\mathfrak{K}$,

$$(10)\qquad A(b, 0)Y = \begin{pmatrix} b & 0 & 0 \\ 0 & bT & 0 \\ 0 & 0 & bT^2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ g^{-1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & bT \\ bT^2g^{-1} & 0 & 0 \end{pmatrix},$$

and we see that

$$(11)\qquad \begin{aligned} (YJ)A(\bar{b}, 0) &= \begin{pmatrix} 0 & 0 & g\gamma S^2 \\ \gamma & 0 & 0 \\ 0 & \gamma S & 0 \end{pmatrix}\begin{pmatrix} \bar{b} & 0 & 0 \\ 0 & \bar{b}T & 0 \\ 0 & 0 & \bar{b}T^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & g(\gamma S^2)\bar{b}T^2 \\ \gamma\bar{b} & 0 & 0 \\ 0 & (\gamma\bar{b})T & 0 \end{pmatrix}. \end{aligned}$$

Hence $A(b, 0)Y + YJA(\bar{b}, 0) = A(0, b)$. *We have shown that the set of all elements $X = XJ$ of $\mathfrak{B}_1^*$ is $\mathfrak{B}$.* But $\mathfrak{B}^* \supseteq \mathfrak{B}_0^* \supseteq \mathfrak{B}_1^* \supseteq \mathfrak{B}$, and so $\mathfrak{B}_1^* \supseteq \mathfrak{B}^*$. It follows that $\mathfrak{B}^* = \mathfrak{B}_0^* = \mathfrak{B}_1^*$ is our cyclic algebra. This proves the theorem.

THEOREM 2. *If $\mathfrak{G}$ is a field and $\mathfrak{H}$ is a division algebra the algebra $\mathfrak{B}^*$ is a division algebra.*

It is well known[4] that the enveloping associative algebra $\mathfrak{B}^*$ of a Jordan algebra $\mathfrak{B}$ of dimension 9 over a field $\mathfrak{F}$ is either $(\mathfrak{B}^*)^{(+)}$, where $\mathfrak{B}^*$ is an associative division algebra, or is the set of $J$-symmetric elements of an associative division algebra $\mathfrak{B}^*$ of degree three over a quadratic field $\mathfrak{G}$. Here $J$ is an involution over $\mathfrak{F}$ of $\mathscr{B}^*$ which does *not* leave the center $\mathfrak{G}$ of $\mathfrak{B}^*$ elementwise fixed. When $\mathfrak{H}$ is a Jordan division algebra so is $\mathfrak{B}$ and, when $\mathfrak{G}$ is a field, the envelope $\mathfrak{B}^*$ is known to be a division algebra. However, when $\mathfrak{G}$ is not a field, the algebra $\mathfrak{B}$ has a representation by the set of all matrices

$$(12)\qquad \begin{pmatrix} A & 0 \\ 0 & AJ \end{pmatrix},$$

for matrices $A$ in an associative division algebra $\mathfrak{D}$ with an involution $J$ moving some element of its quadratic center over $\mathfrak{F}$.

We know that if $\mathfrak{B}^* = (\mathfrak{L}, S, g^{-1})$ is a division algebra so is its

[4] See Section 3 of the reference in 2.

square $\mathfrak{B}_1^* = (\mathfrak{L}, S, g^{-2}) = (\mathfrak{L}, S, h)$, where $h = [\gamma(\gamma S)(\gamma S^2)g^2]^{-1} = (g\bar{g})g^{-2} = \bar{g}g^{-1}$. Hence $\bar{g} \neq g$. Also $\mathfrak{B}_1^* (\mathfrak{L}, S, -h)$ since $-h$ is the product of $h$ by the norm $(-1)^3$ of $-1$. Thus $-\bar{g}g^{-1} \neq 1$ and we can state this result as follows.

LEMMA 1. *Let $\mathfrak{H}$ be a Jordan division algebra, and $\mathfrak{G} = \mathfrak{F}[u]$ be a field, so that $\mathfrak{G}$ is the center of the cyclic associative algebra $\mathfrak{D} = (\mathfrak{L}, S, h)$, where $h = \bar{g}g^{-1}$. Then $\mathfrak{D}$ is a division algebra and $g = \lambda + u$ for $u^2 = \rho \neq 0$ in $\mathfrak{F}$ and $\lambda \neq 0$ in $\mathfrak{F}$.*

Note, in closing, that $\mathfrak{D}$ is isomorphic to the square of the cyclic algebra $\mathfrak{B}^*$, and hence $(\mathfrak{L}, S, g \cdot \bar{g}^{-1})$ is isomorphic to $\mathfrak{B}^*$.

3. **Some properties of $\mathfrak{C}$.** The structure theory for Cayley algebras $\mathfrak{C}$ implies that, if $\mathfrak{L} = \mathfrak{K}[u]$ is a quadratic subalgebra over $\mathfrak{L}$ of $\mathfrak{C}$ generated by an element $u$ such that $u^2 = \rho \neq 0$ in $\mathfrak{L}$, then the space $\mathfrak{Z} = \mathfrak{C}_u$, of all elements $z$ of $\mathfrak{C}$ such that $zu + uz = 0$, contains elements $v$ and $w$ such that

$$v^2 = \varphi\,, \qquad w^2 = \psi\,, \qquad vw + wv = 0 \tag{13}$$

for elements $\varphi \neq 0, \psi \neq 0$ in $\mathfrak{K}$. Then $\mathfrak{Z} = v\mathfrak{L} + w\mathfrak{L} + (vw)\mathfrak{L} = \mathfrak{L}v + \mathfrak{L}w + \mathfrak{L}(vw)$, and $\mathfrak{C} = \mathfrak{L} + \mathfrak{Z}$. Indeed $za = \bar{a}z$ for every $a$ of $\mathfrak{L}$ and $z$ of $\mathfrak{Z}$ where, if $a = \alpha + \beta u$ for $\alpha$ and $\beta$ in $\mathfrak{K}$, then $\bar{a} = \alpha - \beta u$. For all such selections of $v$ and $w$ the space $\mathfrak{Q} = \mathfrak{L} + \mathfrak{L}v$ is a quaternion subalgebra of $\mathfrak{C}$ with an involution $x = a + bv \to \bar{x} = \bar{a} - bv$. Then *multiplication in $\mathfrak{C}$ is given by*

$$(x + yw)(x_1 + y_1 w) = xx_1 + \bar{y}_1 y\psi + (y_1 x + y\bar{x}_1)w\,, \tag{14}$$

for all $x, x_1, y, y_1$ in $\mathfrak{Q}$. We shall use this relation frequently in our computations.

We are assuming that $u^2 = \rho \neq 0$ is in $\mathfrak{F}$. Let $\Gamma = \mathfrak{F}(\sqrt{\rho})$ so that $\mathfrak{G}_\Gamma = \Gamma[u] = e_1\Gamma + e_0\Gamma$ for pairwise orthogonal idempotents $e_1$ and $e_0$ whose sum is the unity element of $\mathfrak{C}$. Let $\Omega = \mathfrak{K}(\sqrt{\rho})$, so that $\mathfrak{L}_\Omega = e_1\Omega + e_0\Omega$, where we know that $\bar{e}_1 = e_0$, $\bar{e}_0 = e_1$ and thus $e_1 z = ze_0$, $e_0 z = ze_1$ for every $z$ in $\mathfrak{Z}_\Omega$. Then $\mathfrak{Z}_\Omega = \mathfrak{Z}_{10} + \mathfrak{Z}_{01}$, where $\mathfrak{Z}_{10} = e_1\mathfrak{Z}_\Omega = \mathfrak{Z}_\Omega e_0$ and $\mathfrak{Z}_{01} = e_0\mathfrak{Z}_\Omega = \mathfrak{Z}_\Omega e_1$. We also know that, if $x_{10}$ is *any* nonzero element of $\mathfrak{Z}_{10}$, this space has a basis $x_{10}$, $y_{10} = x_{10}T$, $z_{10} = x_{10}T^2$ over $\Omega$. Similarly, if $x_{01}$ is any element not zero in $\mathfrak{Z}_{01}$ the elements $x_{01}$, $y_{01} = x_{01}T$, $z_{01} = x_{01}T^2 = y_{01}T$ form a basis of $\mathfrak{Z}_{01}$ over $\Omega$. Moreover, if we select any $x_{10}$ we can take

$$\begin{aligned} x_{01} &= y_{10}z_{10}\,, \qquad y_{01} = x_{01}T = z_{10}x_{10}\,, \\ z_{01} &= x_{01}T^2 = x_{10}y_{10}\,. \end{aligned} \tag{15}$$

Then we have the properties given by

$$\begin{aligned} x_{10}x_{01} &= y_{10}y_{01} = z_{10}z_{01} = \sigma e_1 \\ x_{01}x_{10} &= y_{01}y_{10} = z_{01}z_{10} = \sigma e_0 \end{aligned} \tag{16}$$

$$x_{10}y_{01} = x_{10}z_{01} = y_{10}z_{01} = y_{10}x_{01} = z_{10}x_{01} = z_{10}y_{01} = 0\ , \tag{17}$$

$$x_{01}y_{10} = x_{01}z_{10} = y_{01}z_{10} = y_{01}x_{10} = z_{01}x_{10} = z_{01}y_{10} = 0\ , \tag{18}$$

$$x_{01}y_{01} = -\sigma z_{10}\ , \qquad y_{01}z_{01} = -\sigma x_{10}\ , \qquad z_{01}x_{01} = -\sigma y_{10}\ . \tag{19}$$

$$a_{ij}^2 = 0\ , \qquad e_i a_{ij} = a_{ij}e_j = a_{ij}\ , \qquad e_j a_{ij} = a_{ij}e_i = 0 \tag{20}$$

for $i \neq j$, $i$ and $j$ equal to 0 or 1 and $a_{ij}$ in $\mathfrak{Z}_{ij}$. We now have the following result.[5]

THEOREM 3. *Let $\mathfrak{H}$ be the cyclic exceptional Jordan algebra over $\mathfrak{F}$ consisting of the matrices of* (1) *where (I)–(V) hold, and $g$ be the element of Property IV so that $g = \lambda + u$ for $\lambda \neq 0$ in $\mathfrak{F}$, $u^2 = \rho \neq 0$ in $\mathfrak{F}$ and $uT = u$. Suppose that $\Omega = \mathfrak{K}(\sqrt{\rho})$, so that $\mathfrak{L}_\Omega = \Omega[u] = e_1\Omega + e_0\Omega$ for orthogonal idempotents $e_1$ and $e_0$, and that $\mathfrak{Z}$ is the set of all elements $z$ in $\mathfrak{C}$ such that $zu + uz = 0$, from which $\mathfrak{Z}_\Omega = \mathfrak{Z}_{10} + \mathfrak{Z}_{01}$, where $\mathfrak{Z}_{10} = e_1\mathfrak{Z}_\Omega = \mathfrak{Z}_\Omega e_0$ and $\mathfrak{Z}_{01} = e_0\Omega_\Omega = \mathfrak{Z}_\Omega e_1$. Then, if $z$ is any element of $\mathfrak{Z}$ which has nonzero components $z_{10}$ and $z_{01}$, the space $\mathfrak{Z}$ has a basis $z, zT, xT^2, zu, (zu)T, (zu)T^2$ over $\mathfrak{K}$.*

For $z = z_{10} + z_{01}$, $\mathfrak{L}_\Omega$ contains $e_0$, and the space over $\Omega$ spanned by the six elements given contains $z_{10}$, $(zT)e_0 = (ze_0)T = z_{10}T$ and $(zT^2)e_0 = (ze_0)T^2 = z_{10}T^2$. Similarly our space contains $z_{01}, z_{01}T, z_{01}T^2$. But then $\mathfrak{Z}_\Omega$ is spanned over $\Omega$ by the six given elements and so they span $\mathfrak{Z}$. Since the dimensions of $\mathfrak{Z}$ over $\mathfrak{K}$ and $\mathfrak{Z}_\Omega$ over $\Omega$ are six these elements form a basis for each set.

In the case where $\mathfrak{S}$ is a field we can strengthen our result as follows:

THEOREM 4. *Let $\mathfrak{S}$ be a field and $z$ be any nonzero element of $\mathfrak{Z}$. Then $z, zT, zT^2, zu, (zu)T, (zu)T^2$ form a basis of $\mathfrak{Z}$ over $\mathfrak{K}$.*

For it suffices to show that if $z \neq 0$ is in $\mathfrak{Z}$ then $z$ is not in $\mathfrak{Z}_{10}$ or in $\mathfrak{Z}_{01}$ where we are assuming that $e_1$ and $e_0$ are not in $\mathfrak{S}$. By symmetry it suffices to show that no element $z_{10}$ of $\mathfrak{Z}_{10}$ is in $\mathfrak{Z}$. Let $x_{10}$ be in $\mathfrak{Z}$ so that $x_{10}T = y_{10}$ and $x_{10}T^2 = z_{10}$ are in $\mathfrak{Z}$. But $y_{10}z_{10} = x_{01}$ is in $\mathfrak{C}$ and in $\mathfrak{Z}_{01}$ and so must be in $\mathfrak{Z}$. Hence $x_{10}x_{01} = \sigma e_1$ must be in $\mathfrak{C}$ and in $\mathfrak{L}_\Omega$ and so must be in $\mathfrak{L}$. It follows that $\overline{\sigma e_1} = \sigma e_0$ is in $\mathfrak{L}$

[5] The discussion up to this point is that of Section 13 of the reference in 2.

and so $\sigma e_1 + \sigma e_0 = \sigma$ is in $\mathfrak{L}$ and must be in $\mathfrak{K}$, $e_1$ is in $\mathfrak{L}$ contrary to our hypothesis that $\mathfrak{L}$ is a field and $e_1$ is a singular idempotent of $\mathfrak{L}_\Omega$. This completes our proof.

**4. A normalized basis of $\mathfrak{C}$.** Let $x = x_{10} + x_{01}$, where $x_{01} = (x_{10}T)(x_{10}T^2)$ and $x_{10}$ is any nonzero element of $\mathfrak{Z}_{10}$. Then we have

$$x^2 = x_{10}x_{01} + x_{01}x_{10} = \sigma e \neq 0 \qquad (\sigma \text{ in } \mathfrak{K}_\Omega) . \tag{21}$$

Also $xT = y_{10} + y_{01}$ and $(xT)^2 = \sigma S$. But

$$\begin{aligned} x(xT) + (xT)x &= (x_{10} + x_{01})(y_{10} + y_{01}) + (y_{10} + y_{01})(x_{10} + x_{01}) \\ &= x_{10}y_{10} + y_{10}x_{10} + x_{01}y_{01} + y_{01}x_{01} = 0 . \end{aligned}$$

We have thus proved the existence of an element $z$ in $\mathfrak{Z}_\Omega$ such that

$$z^2 \neq \varphi e \neq 0 , \qquad zT = a_z z + w_z , \tag{22}$$

for $a_z$ in $\mathfrak{L}_\Omega$ and $w_z$ in $\mathfrak{Z}_\Omega$ with the property that $zw_z + w_z z = 0$, and $w_z^2 = \psi_z \neq 0$. When $\Omega = \mathfrak{F}$ we actually have $a_z = 0$ and take $v = z$, $w = vT = w_z$, and have

$$wT = z_{10} + z_{01} = x_{10}y_{10} + y_{01}x_{01} = (x_{10} + x_{01})(y_{10} + y_{01})vw .$$

We now turn to the case where $\mathfrak{G}$ is a field.

We are assuming that $\mathfrak{H}$ is a division algebra, and that $\mathfrak{G} = \mathfrak{F}[g]$ is a quadratic field. Let $q_1, \cdots, q_m$ be a basis of $\mathfrak{Z}$ over $\mathfrak{K}$, so that $m = 18$. The general element of $\mathfrak{Z}$ is $z = \eta_1 q_1 + \cdots + \eta_m q_m$ for independent indeterminates $\eta_i$ over $\mathfrak{K}$ (and *hence over* $\mathfrak{K}_\Omega$). Then

$$z^2 = \varphi(\eta_1, \cdots, \eta_m) \tag{23}$$

is a polynomial in $\eta_1, \cdots, \eta_m$ with coefficients in $\mathfrak{F}$. This polynomial is not identically zero since, in fact, there exist values $\bar{\eta}_i$ of the $\eta_i$ in $\mathfrak{K}_\Omega$ such that $z = x, z^2 = \sigma \neq 0$, where $x$ is given by (21). We now observe that there must be a basis of the Cayley algebra $\mathfrak{C}_\Lambda$ over $\Lambda = \mathfrak{K}(\eta_1, \cdots, \eta_m)$, consisting of $1, u, z, uz, w_0, uw_0, zw_0, (uz)w_0$, and $\mathfrak{Z}_\Lambda = z\Lambda + (uz)\Lambda + w_0\Lambda + (uw_0)\Lambda + (zw_0)\Lambda + [(uz)w_0]\Lambda$. Then the elements $h$ of $w_0\Lambda + (uw_0)\Lambda + (zw_0)\Lambda + [(uz)w_0]\Lambda$ all have the property that $hz + zh = hu + uh = h(uz) + (uz)h = 0$. Hence

$$zT = a_z z + w_z , \tag{24}$$

where $a_z$ is in $\Lambda[u]$, $w_z$ is in $\mathfrak{Z}_\Lambda$, $w_z z + zw_z = 0$. The coefficients of $a_z$ are in $\Lambda$, and so are the quotients of two fixed polynomials of $\mathfrak{K}[\eta_1, \cdots, \eta_m]$ by a polynomial $\varphi_1(\eta_1, \cdots, \eta_m)$. The coordinates of $w_z$ are also in $\Lambda$ and so are the quotients of polynomials in $\mathfrak{K}[\eta_1, \cdots, \eta_m]$ by a polynomial $\varphi_2(\eta_1, \cdots, \eta_m)$. Then $w_z^2 = \psi(\eta_1, \cdots, \eta_m)[\varphi_2(\eta_1, \cdots, \eta_m)]^{-2}$. We know that there exist values of $\eta_1, \cdots, \eta_m$ in $\mathfrak{K}_\Omega$ such that the corresponding element $z_0 \neq 0$ has the property $z_0 T = w_0$ for $w_0 z_0 + z_0 w_0 = 0$, and $w_0^2 = \psi_0 \neq 0$. But then $\varphi\varphi_1\varphi_2\psi$ is a polynomial in $\mathfrak{K}(\eta_1, \cdots, \eta_m)$ not identically zero, and we can find values $\eta_{i0}$ of the $\eta_i$ in $\mathfrak{K}$ such that, if $v = \eta_{10}q_1 + \cdots + \eta_{m0}q_m$, then

$$v^2 = \varphi \neq 0 , \qquad w^2 = \psi \neq 0 , \qquad vw + wv = 0 , \qquad vT = av + w \qquad (a \text{ in } \mathfrak{L}; \varphi, \psi \text{ in } \mathfrak{K}) . \tag{25}$$

We state this result as follows:

LEMMA 2. *If properties (I)–(V) hold the Cayley algebra $\mathfrak{C}$ is generated by nonsingular square roots $u, v, w$ such that* (25) *holds and the product formula* (14) *is valid for every $x, x_1, y, y_1$ of $\mathfrak{D} = \mathfrak{K} + u\mathfrak{K} + v\mathfrak{K} + (uv)\mathfrak{K}$.*

This normalization will permit us to determine the transformation $T$.

**5. A determination of $T$.** We are assuming that $\bar{x}T = \overline{xT}$ for every $x$ of $\mathfrak{C}$, and that[6] $[(xy)T]\bar{g} = (xT)[(yT)\bar{g}]$ for every $x$ and $y$ of $\mathfrak{C}$. This latter relation is equivalent to $[(\bar{y}\bar{x})T]\bar{g} = (\bar{y}T)[(\bar{x}T)\bar{g}]$ and thus our pair of relations is equivalent to the pair

$$\overline{xT} = \bar{x}T , \qquad g[(xy)T] = [g(xT)](yT) \quad (x, y \text{ in } \mathfrak{C}) . \tag{26}$$

Since $g$ is nonsingular, and $gT = g$ it follows that, if $d$ is any element of $\mathfrak{L} = \mathfrak{K}[g] = \mathfrak{K}[u]$, then

$$(dx)T = (dT)(xT) , \qquad (xd)T = (xT)(dT) , \tag{27}$$

*for every $x$ in $\mathfrak{C}$ and $d$ in $\mathfrak{L}$.* Thus we may write the second relation of (26) as

$$g[(xy)T] = [g(xy)]T = [(gx)T](yT) , \tag{28}$$

and also as

$$[(xy)T]\bar{g} = (xT)[(yT)\bar{g}] = (xT)[(y\bar{g})T] = [(xy)\bar{g}]T . \tag{29}$$

Our algebra $\mathfrak{C}$ is a Cayley algebra over the cubic extension field $\mathfrak{K}$ of $\mathfrak{F}$, and $\mathfrak{Q} = \mathfrak{L} + \mathfrak{L}v$ is a quaternion subalgebra of $\mathfrak{C}$ over $\mathfrak{K}$. The set $\mathfrak{Z}$ of all elements $z$ in $\mathfrak{C}$ such that $zu + uz = 0$ is the space

$$\mathfrak{Z} = \mathfrak{L}v + \mathfrak{L}w + \mathfrak{L}(vw) = v\mathfrak{L} + w\mathfrak{L} + (vw)\mathfrak{L} = \mathfrak{C}_u \tag{30}$$

of dimension 6 over $\mathfrak{K}$. Then our multiplication table (14) implies that

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[6] In 2 our product relation of (V) was given on page 15 in what we see here is the equivalent form $[(xy)T\bar{f}] = (xT)[yT)\bar{f}]$, where $f = (\gamma S^2)g$.

$$(31)\qquad \begin{gathered}(hv)w = \bar{h}(vw)\ ,\\ vw + wv = v(vw) + (vw)v = w(vw) + (vw)v = 0\ ,\end{gathered}$$

for every element $h$ in $\mathfrak{L}$. We now have the following trivial result.

LEMMA 3. *Let $T$ be defined so that $\alpha T = \alpha S$ for every $\alpha$ of $\mathfrak{K}$, $uT = u$, $xT$ is in $\mathfrak{Z}$ for every $x$ of $\mathfrak{Z}$. Then $\overline{xT} = \bar{x}T$ for every $x$ of $\mathfrak{C}$.*

This result is an immediate consequence of the fact that $\bar{x} = -x$, $\overline{xT} = -xT = \bar{x}T$ for every $x$ in $\mathfrak{Z}$.

We have already assumed that (25) holds and we also know from Theorem 4 that

$$(32)\qquad vT^2 = (aTa)v + (aT)w + wT\ ,$$

and that $\mathfrak{Z} = \mathfrak{L}v + \mathfrak{L}(vT) + \mathfrak{L}(vT^2)$. It follows that

$$(33)\qquad wT = bv + cw + d(vw)\ ,$$

*where $b, c, d$ are in $\mathfrak{L}$* and $d \neq 0$. We shall now derive the following principal result.

THEOREM 5. *Let $vT$ and $wT$ be defined by* (25) *and* (33) *for $a$, $b, c, d$ in $\mathfrak{L}$ and $d \neq 0$. Then* (28) *holds only if*

$$(34)\qquad (vw)T = g^{-1}\bar{g}[-(\bar{d}\psi)v + (\bar{d}\bar{a}\varphi)w - \bar{b}(\varphi S)\psi^{-1}(vw)]\ ,$$

*where the relations*

$$(35)\quad \psi = \varphi S - a\bar{a}\varphi\ ,\qquad \psi(\psi S) = (b\bar{b}\varphi S - d\bar{d}\psi^2)\varphi\ ,\qquad c = -\bar{a}b\varphi\psi^{-1}\ .$$

*Conversely, let $T$ be defined by $uT = u$, $\alpha T = \alpha S$ for every $\alpha$ of $\mathfrak{K}$,* (25), (33), (34) *and by*

$$(36)\qquad \begin{aligned}&h_0 + h_1v + h_2w + h_3(vw)T\\ &\quad = h_0T + h_1T(vT) + (h_2T)(wT) + (h_3T)[(vw)T]\ ,\end{aligned}$$

*so that $\overline{xT} = \bar{x}T$ for every $x$ of $\mathfrak{C}$. Then the multiplicative relation* (28) *holds in $\mathfrak{C}$.*

For proof we compute $[g(vT)](wT) = (gav + gw)[bv + cw + (\bar{d}v)w]$ by the use of (14) to obtain

$$(37)\qquad [g(vT)](wT) = g(a\bar{b}\varphi + \bar{c}\psi) - (\bar{d}\bar{g}\psi)v + (\bar{d}\bar{g}\bar{a}\varphi)w + [g(acb)v]w\ .$$

If (26) holds we have $[g(vT)](wT) = g[(vw)T]$, and so the right member of (37) must be an element of $\mathfrak{Z}$. Thus $a\bar{b}\varphi + \bar{c}\psi = 0 = c\psi + \bar{a}b\varphi$ and our formula in (35) for $c$ has been verified. Equation (37) is the result of a computation which remains valid if we replace $g$ throughout by unity and the result is

$$(38) \qquad (vT)(wT) = -\bar{d}\psi v + \bar{d}\bar{a}\varphi w + (\bar{a}\bar{c} - \bar{b})(vw) .$$

We now observe that the square of any element of $\mathfrak{Z}$ is computed by the use of the formula

$$(39) \qquad h_1v + h_2w + h_3(vw)^2 = h_1\bar{h}_1\varphi + h_2\bar{h}_2\psi = h_3\bar{h}_3\varphi\psi .$$

Hence $(vT)^2 = a\bar{a}\varphi + \psi = v^2T = \varphi T = \varphi S$. Also

$$\begin{aligned}(wT)^2 = b\bar{b}\varphi + c\bar{c}\psi = \ d\bar{d}\varphi\psi &= b\bar{b}\varphi + a\bar{a}b\bar{b}\varphi^2\psi^{-1} - d\bar{d}\varphi\varphi \\ &= b\bar{b}\psi^{-1}\varphi(\psi + a\bar{a}\varphi) - d\bar{d}\varphi\psi = b\bar{b}\varphi\varphi S\psi^{-1} - d\bar{d}\varphi\psi \\ &= (b\bar{b}\varphi S - d\bar{d}\psi^2)\varphi\psi^{-1} = w^2T = \psi S .\end{aligned}$$

This completes our proof of (35). We also compute $ac - b = -a\bar{a}b\varphi\psi^{-1} - b = -b\psi^{-1}(\varphi a\bar{a} + \psi)$, and we use (35) to obtain

$$(40) \qquad b - ac = b\psi(\varphi S) .$$

This yields the formula

$$(41) \qquad [g(vT)](wT) = \bar{g}[-(\bar{d}\psi)v + (\bar{d}\bar{a}\varphi)w - \bar{b}(\varphi S)\psi(vw)] .$$

If (26) holds we have $g[(vw)T] = [g(vT)](wT)$, and we use (41) to obtain (34).

Conversely, let $T$ be defined as in the statement of our theorem. Then (41) holds and (34) implies that $g[(vw)T] = [g(vT)](wT)$. We compute $(wT)(vT)$ and see that the relation

$$(42) \qquad (vT)(wT) = -(wT)(vT)$$

is a consequence of the properties we have assumed. It follows immediately that every element of $\mathfrak{C}$ is uniquely expressible in the form $x = h_0 + h_1(vT) + h_2(wT) + h_3[(vT)(wT)]$ for $h_0, h_1, h_2, h_3$ in $\mathfrak{L}$. Moreover $[h(vT)](wT) = \bar{h}[(vT)(wT)]$ for every $h$ in $\mathfrak{L}$. Also the space $\mathfrak{Z} = \mathfrak{L}(vT) + \mathfrak{L}(wT) + \mathfrak{L}[(vT)(wT)]$ and (36) defined $T$ uniquely. If $x = h_1v + h_2w + h_3(vw)$ is in $\mathfrak{Z}$ then (39) yields $x^2$, and

$$xT = (h_1T)(vT) + (h_2T)(wT) + (h_3T)[(vw)T]$$

implies that $(xT)^2 = [(h_1\bar{h}_1)T](v^2T) + [(h_2\bar{h}_2)T](w^2T) + [(h_3\bar{h}_3)T](vw)^2T$. We also observe that

$$(43) \qquad q[(vw)T] = [g(vT)](wT) = \bar{q}[(vT)(wT)] ,$$

from which

$$(44) \qquad \begin{aligned}(vw)T &= (q^{-1}\bar{g})[(vT)(wT)] ,\\ [(vw)T]^2 &= [(vT)(wT)]^2 = -(vT)^2(wT)^2 .\end{aligned}$$

Thus $[(vw)T]^2 = -(\varphi S)(\psi S) = [(vw)^2]T$. We have shown that $(xT)^2 = x^2T$ for every $x$ in $\mathfrak{Z}$.

It should be clear that the linearity of the second relation in (26) in $x$ and $y$ implies that, to prove (26), we need only derive it for basal elements of $\mathfrak{C}$. Let $xy + yx = 0$ for elements $x$ and $y$ of $\mathfrak{Z}$ so that $g[(xy)T] = [g(xT)](yT)$. Then if $h$ is in $\mathfrak{K}$ we have

$$g[(hx)y]T = g[\bar{h}(xy)]T = g[\bar{h}T(xy)] = (\bar{h}T)[g(xy)T] = \bar{h}T[(gxT)(yT)]$$
$$= [hTg(xT)](yT) = [(hx)T](yT) .$$

Similarly, if $k$ is in $\mathfrak{K}$, the relation $g[(xy)T] = [g(xT)](yT)$ implies that $g[x(ky)]T = [g(xT)](ky)T$. The relation $g[(hx)(kx)]T = [g(hx)T][(kx)T]$ follows from the fact that $g[(hx)(kx)]T = g[h\bar{k}x^2]T = g(h\bar{k}Tx^2T)$ and $[g(hx)T](kx)T = [(ghT)(xT)][(kT)(xT)] = ghT\bar{k}T(xT)^2 = g(h\bar{k})Tx^2T$. Thus it suffices to prove (26) for $x$ and $y$ elements selected as a distinct pair of the elements $v, w$ and $vw$. We have already derived this result for $x = v$ and $y = w$. The relation $g[v(vw)]T = [g(vT)](vw)T$ follows since $g[v(vw)]T = g(\varphi w)T = (g\varphi S)wT$ while

$$[g(vT)][(vw)T] = [g(vT)]\bar{g}g^{-1}\cdot(vT)(wT) = g(vT)[(g\bar{g}^{-1})(vT)\cdot(wT)]$$
$$= [g\bar{g}^{-1}(vT)g(vT)](wT) = [g(vT)^2](wT) = g\varphi S(wT)$$

as desired. The remaining verifications are of a similar nature and will not be given here.

Let us note that we have really shown that the definition of $vT$, and of $vT^2$ via the definition of $wT$, have determined $T$ uniquely. Thus we really only need to find the effect of the fact that $xT^3 = g^{-1}\bar{g}x$ to complete our conditions on the defining parameters $a, b, d$. We shall see here that the property $vT^3 = g^{-1}vg$ will imply that $xT^3 = g^{-1}xg$ for every $x$ of $\mathfrak{C}$.

## 6. The property $xT^3 = g^{-1}xg$.

We have already seen that

$$vT^2 = (aTa + b)v + (aT + c)w + d(vw) . \tag{45}$$

Then

$$\begin{aligned} vT^3 &= g^{-1}vg = g^{-1}\bar{g}v \\ &= (aT^2aT + bT)(av + w) + (aT^2) + (cT)[bv + cw + d(vw)] \\ &\quad + dT\bar{g}g^{-1}[-\bar{d}(\psi v - \bar{a}\varphi w) = \bar{b}\psi^{-1}\varphi S(vw)] . \end{aligned} \tag{46}$$

Equate coefficients of $v, w$ and $vw$ respectively to obtain the conditions

$$(aT^2aT + bT)a + (aT^2 + cT)b = \bar{g}g^{-1}[1 + \bar{d}(dT)\psi] , \tag{47}$$

$$aT^2aT + bT + (aT^2 + cT)c + \bar{d}\bar{a}\bar{g}g^{-1}(dT)\varphi = 0 , \tag{48}$$

and

(49) $$aT^2 + cT = d^{-1}(dT)\bar{b}\bar{g}g^{-1}\psi^{-1}\varphi S .$$

Thus $(aT^2 + cT)c = -\bar{a}b\bar{b}d^{-1}dT\bar{g}g^{-1}\psi^{-2}\varphi S\varphi$, and (48) becomes $aT^2aT + bT = \psi^{-2}\bar{a}dT\bar{g}g^{-1}\varphi d^{-1}(b\bar{b}\varphi S - d\bar{d}\psi^2)$, while (35) yields

(50) $$aT^2aT + bT = \psi^{-1}\psi S\bar{a}\bar{g}g^{-1}d^{-1}(dT) .$$

Then (47) becomes

$$\begin{aligned} &a\bar{a}\psi^{-1}\psi S\bar{g}g^{-1}d^{-1}dT + d^{-1}dTb\bar{b}\bar{g}g^{-1}\psi^{-1}\varphi S \\ &\quad = \bar{g}g^{-1}d^{-1}dT\psi^{-1}(a\bar{a}\psi S + b\bar{b}\varphi S) = \bar{g}g^{-1}(1 + \bar{d}dT\psi) , \end{aligned}$$

and this is equivalent to $dT(a\bar{a}\psi S + b\bar{b}\varphi S - d\bar{d}\psi^2) = d\psi$. Replace $b\bar{b}\varphi S - d\bar{d}\psi^2$ by $\varphi^{-1}\psi\psi S$ to get

$$dTa\bar{a}\psi S - \psi\psi S\varphi^{-1} = (dT)\psi S\varphi^{-1}(a\bar{a}\varphi + \psi) = (dT)\psi S\varphi S\varphi^{-1} = d\psi .$$

We have proved the important property that

(51) $$s = d\psi\varphi = sT .$$

Equation (51) is an invariance property imposing a condition on $d$. We shall now show that (50) determines $b$ in terms of $a$ and $\varphi$. In fact

$$\begin{aligned} bT &= \bar{a}\bar{g}^{-1}d^{-1}dT\psi S\psi^{-1} - (aT^2)(aT) \\ &= \bar{g}g^{-1}\psi\varphi[(\psi\varphi)^{-1}]S\bar{a}\psi S\psi^{-1} - (aT^2)(aT) = \bar{g}g^{-1}\varphi(\varphi S)^{-1}\bar{a}(aT)(aT^2) . \end{aligned}$$

Hence

(52) $$b = \bar{g}g^{-1}\varphi S^2\varphi^{-1}\bar{a}T^2 - a(aT) ,$$

and we obviously have

(53) $$\bar{b} = g\bar{g}^{-1}\varphi S^2\varphi^{-1}aT^2 - \bar{a}(\bar{a}T) .$$

We form

$$\begin{aligned} \varphi^2b\bar{b} &= [\bar{g}g^{-1}\varphi S^2aT^2 - \varphi a(aT)][g\bar{g}^{-1}\varphi S^2aT^2 - \varphi\bar{a}(\bar{a}T)] \\ &= (\varphi S^2)^2(a\bar{a})T^2 + a\bar{a}(a\bar{a})T\varphi^2 - \varphi\varphi S^2[\bar{g}g^{-1}\bar{a}(\bar{a}T)(\bar{a}T^2) + g\bar{g}^{-1}a(aT)(aT^2)] . \end{aligned}$$

Thus

$$\begin{aligned} \varphi^2\varphi Sb\bar{b} &= \varphi S\varphi S^2(\varphi - \psi S^2) + \varphi(\varphi S - \psi)(\varphi S^2 - \psi S) \\ &\quad - \varphi\varphi S\varphi S^2[\bar{g}g^{-1}\bar{a}(\bar{a}T)(\bar{a}T^2) + g\bar{g}^{-1}a(aT)(aT^2)] \\ &= 2\varphi\varphi S\varphi S^2 - (\varphi S\varphi S^2\psi S^2 + \varphi S^2\varphi\psi + \varphi\varphi S\psi S) \\ &\quad + \varphi\psi\psi S - [\varphi\varphi S\varphi S^2\bar{g}g^{-1}\bar{a}\bar{a}T\bar{a}T^2 + g\bar{g}^{-1}a(aT)(aT)^2] . \end{aligned}$$

But

$$\varphi\psi(\psi S) = \varphi^2(\varphi S)b\bar{b} - (d\varphi\psi)(\overline{d\varphi\psi}) = \varphi^2\varphi Sb\bar{b} - s\bar{s} .$$

Hence $s\bar{s} = \varphi^2(\varphi S)b\bar{b} - \varphi\psi\psi S$ and we obtain

$$(54)\qquad \begin{aligned} s\bar{s} &= 2\varphi\varphi S\varphi^2 - (\varphi\varphi S\psi S + \varphi S\varphi S^2\psi S^2 + \varphi S^2\varphi\psi) \\ &\quad - \varphi(\varphi S)(\varphi S^2)[g\bar{g}^{-1}a(aT)(aT^2) + \bar{g}g^{-1}\overline{a(aT)(aT^2)}] \,. \end{aligned}$$

Let us write

$$(55)\qquad t = \frac{g}{\bar{g}}\,, \qquad \nu(h) = h(hT)(hT^2)\,, \qquad \mu(h) = h + hT + hT^2$$

for every element $h$ in $\mathfrak{L}$ so that $\nu(h)$ is the norm and $\mu(h)$ the trace in $\mathfrak{L}$ over $\mathfrak{G}$. Now $\psi = \varphi S - a\bar{a}\varphi$, and $\psi\varphi(\varphi S^2) = \varphi(\varphi S^2) - a\bar{a}\varphi^2(\varphi S^2)$. Then we obtain the formula

$$(56)\qquad -s\bar{s} = \nu(\varphi) + t\nu(\varphi a)T^2 + t^2\nu[\varphi(\bar{a}T)t^{-1}] - t\mu[\varphi\varphi a(\varphi S^2\bar{a}t^{-1})] \,.$$

But the norm form of the cyclic algebra $\mathfrak{D} = (\mathfrak{L}, T, t)$ over $\mathfrak{G}$ is the function

$$(57)\qquad \begin{aligned} \Delta(x) &= \Delta(h_0 + h_1y + h_2y^2) \\ &= \nu(h_0) + t\nu(h_1) + t^2\nu(h_2) - t\mu[h_0(h_1T)(h_2T^2)] \,. \end{aligned}$$

Hence (56) implies that

$$(58)\qquad s\bar{s} = \Delta(x)\,, \qquad -\psi = \varphi[(\varphi a)T^2]y + \varphi(\bar{a}T)t^{-1}y^2 \,.$$

Conversely, let $d$ be an element such that $s = d\varphi\psi = sT$, and let $s\bar{s} = \Delta(x)$ where (58) defines $x$. Then we have seen that (50) defines $b$ so that (52) holds. Also (50) is the result of using (49) in (48) and, if (48) and (49) hold, then (47) is equivalent to $s = sT$. It follows that the condition $vT^3 = g^{-1}vg$ is equivalent to the definition (52) of $b$, to $s = d\varphi\psi = (d\varphi\psi)T$, and to $s\bar{s} = \Delta(x)$ providing that we can show that (49) holds. Replace $dTd^{-1}\varphi S\psi^{-1} = (d\varphi\psi)T(d\varphi\xi)^{-1}\varphi(\psi S)^{-1}$ by the value $\varphi(\psi S)^{-1}$ in (49) and we have the relation

$$(59)\qquad (\psi S)(aT^2 + cT) = \frac{\bar{g}}{g}(\varphi\bar{b}) \,.$$

Use (35) to see that

$$(60)\qquad (\psi S)(cT) = -(\bar{a}T)(bT)(\varphi S)\,,$$

and so (59) becomes

$$(61)\qquad \psi SaT^2 - (\bar{a}T)(bT)(\varphi S) = \frac{\bar{g}}{g}(\varphi\bar{b}) \,.$$

By (52) and (53) this relation is equivalent to

$$\begin{aligned} &\psi S(aT^2) - (\bar{a}T)(\varphi S)[\bar{g}g^{-1}\varphi(\varphi S)^{-1}\bar{a} - (aT)(aT^2)] \\ &\quad = \bar{g}g^{-1}\varphi[g\bar{g}^{-1}\varphi S^2\varphi^{-1}aT^2 - \bar{a}(\bar{a}T)] = \varphi S^2aT^2 - \bar{g}g^{-1}\varphi\bar{a}(\bar{a}T) \,. \end{aligned}$$

This is then equivalent to $\psi SaT^2 + [(a\bar{a})T\varphi S]aT^2 = \varphi S^2aT^2$, that is, to $[\psi S + (a\bar{a})T\varphi S - \varphi S^2]aT^2 = [\psi + a\bar{a}\varphi - \varphi S]TaT^2 = 0$. This is true by (35).

The definition of $T$ given by (36) implies that $xT^3 = g^{-1}xg$ if and only if $vT^3 = g^{-1}vg$, $wT^3 = g^{-1}wg$, and $(vw)T^3 = g^{-1}(vw)g$. But $vT = av + w$ implies that $vT^2 = aTvT + wT$, $vT^3 = aT^2vT^2 + wT^2 = g^{-1}\bar{g}v$, $wT^3 = (g^{-1}\bar{g}v)T = (aT^2vT^2)T = g^{-1}\bar{g}(av + w) - a(g^{-1}\bar{g}v) = g^{-1}\bar{g}w$ as desired. Then

$$\begin{aligned} g^3(vw)T^3 &= g^2[g(vw)]TT^2 = g^2[g(vT)(wT)]T^2 \\ &= g[g^2(vT^2)\cdot wT^2]T = [g^3(vT^3)]wT^3 \end{aligned}$$

and

$$\begin{aligned} (vw)T^3 &= g^{-3}(\bar{g})^3[(g^{-1}vg)(g^{-1}wg)] = g^{-3}(\bar{g})^3[(g^{-1}\bar{g}v)(g^{-1}\bar{g}w)] \\ &= g^{-3}(\bar{g})^3[(g^{-2}\bar{g}^2v)w] = g^{-3}(\bar{g})^3\bar{g}^{-2}g^2(vw) = g^{-1}\bar{g}(vw) \end{aligned}$$

as desired. We have proved the following result.

THEOREM 6. *Let the conditions of Theorem* 5 *hold, and let* $b = \bar{g}g^{-1}\varphi S^2\varphi^{-1}\bar{a}T^2 - a(aT)$, $d\psi\varphi = (d\psi\varphi)T = s$ *for an element* $s$ *in* $\mathfrak{G}$ *such that* $s\bar{s}$ *is the norm* $\Delta(x)$ *in the cyclic algebra* $(\mathfrak{L}, T, g\bar{g}^{-1})$ *with* $x = -\varphi - [(\varphi a)T^2]y - [\varphi(\bar{a}T)]y^2$. *Then Property V holds in* $\mathfrak{C}$.

7. **The norm condition.** Let us begin with some properties of associative division algebras. Let $\mathfrak{D}$ be an associative division algebra whose center is a quadratic field $\mathfrak{F}[g] = \mathfrak{G}$. We form a quadratic extension $\Gamma = \mathfrak{F}[g^*]$, where $g^* = \lambda + \sqrt{\rho}$, so that the mapping $a \to a^*$ of $\mathfrak{G}$ onto $\Gamma$ determined by $g \to g^*$ is an isomorphism leaving $\mathfrak{F}$ elementwise fixed. We then take the direct product $\mathfrak{D} \times \Gamma$ which is an algebra over $\Gamma$. The algebra $\mathfrak{G}_\Gamma = \mathfrak{G} \times \Gamma$ is the direct sum $\mathfrak{G}_\Gamma = e_1\Gamma \oplus e_0\Gamma$, where the mapping $x \to xe_1$ is an isomorphism over $\mathfrak{F}$ of $\mathfrak{D}$ onto $\mathfrak{D}e_1$, and $e_1$ and $e_0$ are orthogonal idempotents such that $e_1 + e_0 = 1$ is the unity element of $\mathfrak{D}$ and of $\mathfrak{G}$. If $h$ is any element of $\mathfrak{G}$ its image $he_1$ is in $e_1\mathfrak{G}_\Gamma = e_1\Gamma$. The algebra $\mathfrak{D}$ has a norm form $\Delta(x)$ on $\mathfrak{D}$ to $\mathfrak{G}$, and $\mathfrak{D}_1$ has a corresponding norm form $\Delta(xe_1)$ on $\mathfrak{D}e_1$ to $\mathfrak{G}e_1$, and indeed $\Delta(xe_1) = \Delta(x)e_1$. Then $h$ in $\mathfrak{G}$ is the norm $h = \Delta(x)$ of an element $x$ of $\mathfrak{D}$, if and only if $he_1 = \Delta(xe_1)$.

We now consider our exceptional Jordan division algebra $\mathfrak{H}$ with attached Cayley algebra $\mathfrak{C}$ containing the subfield $\mathfrak{G} = \mathfrak{F}[g]$. We form $\mathfrak{H} \times \Gamma$, and have split $\mathfrak{G}$ and $\mathfrak{C}$. We have already selected a basis of $\mathfrak{C}$ with $v$, $vT$, $vT^2$, $uv$, $u(vT)$, $u(vT^2)$ a basis of $\mathfrak{Z}$ over $\mathfrak{K}$. Then $e_1v$, $e_1(vT)$, $e_1(vT^2)$ are left linearly independent in the quadratic extension $\mathfrak{L}e_1 = \mathfrak{L}_\Gamma e_1$, and so are $e_0b$, $e_0(vT)$, $e_0(vT^2)$. We take $x_{10} = e_1v$, $y_{10} = x_{10}T = e_1(vT)$, $z_{10} = x_{10}T^2 = e_1(vT^2)$ and have

$$
\begin{aligned}
x_{01} = y_{10}z_{10} &= [e_1(vT)][e_1(vT^2)] = (e_1av + e_1w)(e_1b)v + (e_1c)w + (e_1d)(vw) \\
&= [e_1(ac - b)v]w + (e_1av)[(e_0\bar{d}v)w] + (e_1w)[(e_0\bar{d}v)w] \\
&= [e_1(ac - b)v]w + (e_0a\bar{d}\varphi)w - e_0(\psi\bar{d}w) .
\end{aligned}
$$

Since $e_0 = \bar{e}_1$ and $e_1e_0 = e_0e_1 = 0$ we see that

$$
\begin{aligned}
x_{10}x_{01} &= (e_1v)[e_1(ac - b)v \cdot w] + (e_1v)[(e_0a\bar{d}\varphi)w] - (e_1v)[(e_0\bar{d}\psi)v] \\
&= -e_1\bar{d}\psi\varphi = -e_1\bar{s} = \sigma e_1 ,
\end{aligned}
$$

where $\sigma$ is the parameter of (16). It is known[7] that $\mathfrak{H}$ is a division algebra if and only if $\mathfrak{D}_1 = (\mathfrak{L}, T, g\cdot\bar{g}^{-1})e_1$ is a division algebra, and $\sigma \neq \Delta(x_1)$ for any $x_1$ in $\mathfrak{D}_1$. Also $\mathfrak{H}$ is a division algebra if and only if $\mathfrak{H}_\Gamma$ is a division algebra. Hence we may state the condition that $\mathfrak{H}$ be a division algebra *in all cases* as follows.

THEOREM 7. *Let the conditions of Theorems* 5 *and* 6 *be satisfied, so that the exceptional Jordan algebra is defined over* $\mathfrak{F}$ *as the set of matrices in* (1) *where* $\mathfrak{D} = (\mathfrak{L}, T, g\bar{g}^{-1})$ *is a cyclic associative algebra, and* $\mathfrak{D}$ *is either a division algebra or a direct sum of two division algebras. Then* $\mathfrak{H}$ *is a division algebra if and only if* $s = d\varphi\psi$ *is not the norm of any element of* $\mathfrak{D}$.

We shall pass on now to a construction of a class of *noncyclic* exceptional Jordan division algebras.

**8. Bicyclic algebras.** Let $\mathfrak{H}$ be an exceptional Jordan division algebra over its center $\mathfrak{F}$, and $k$ be an element of $\mathfrak{H}$ not in $\mathfrak{F}$. The Jordan subalgebra of $\mathfrak{H}$ generated by $k$ is a field $\mathfrak{K}_0 = \mathfrak{F}[k]$. We have already considered the case where $\mathfrak{K}_0$ is cyclic over $\mathfrak{F}$. Assume then that $\mathfrak{K}$ is not a cyclic field. Then there is a normal splitting field $\mathfrak{M} = \mathfrak{F}[\theta, \omega]$ of degree six over $\mathfrak{F}$ of $\mathfrak{K}$. The automorphism group of $\mathfrak{M}$ over $\mathfrak{F}$ is isomorphic to the symmetric group on three letters, and is generated by two automorphisms $S$ and $J$, where $S^3 = J^2 = I$ and $JS = S^2J$. The fixed field of $\mathfrak{M}$ under $S$ is a quadratic field $\Omega = \mathfrak{F}[\omega]$, where

$$
\omega^2 = \zeta , \qquad \omega S = \omega , \qquad \omega J = -\omega \qquad (\zeta \text{ in } \mathfrak{F}) . \tag{62}
$$

The fixed field of $\mathfrak{M}$ under $J$ is a cubic field $\mathfrak{K} = \mathfrak{F}[\theta]$, and $\mathfrak{K}$ isomorpic over $\mathfrak{F}$ to $\mathfrak{K}_0$ under the mapping induced by $k \to \theta$. Then $\mathfrak{M} = \Omega[\theta]$ is cyclic over $\Omega$, and its galois group over $\Omega$ is generated by the automorphism $S$.

We now form the algebra $\mathfrak{H}_\Omega$ and consider it as an algebra over $\Omega$. It is cyclically generated and all of our properties of such algebras hold. It has an automorphism $J$ induced by the automorphism $J$ of $\Omega$, and

[7] See Theorem 9 of 2.

$J$ is defined so that it leaves $\mathfrak{H}$ elementwise fixed. Then $\mathfrak{H}_{\mathfrak{M}} = \mathfrak{H} \times \mathfrak{M}$ is reduced, and $\mathfrak{H}_{\mathfrak{M}}$ has automorphisms $S$ and $J$ such that *$\mathfrak{H}$ is the set of all elements of $\mathfrak{H}_{\mathfrak{M}}$ fixe by $S$ and $J$.*

The algebra $\mathfrak{H}$ has three pairwise orthogonal idempotents $e_1, e_2, e_3$, such that

$$(63) \qquad e_1S = e_2 , \quad e_2S = e_3 , \quad e_3S = e_1 , \quad e_1J = e_1 , \quad e_2J = e_3 , \quad e_3J = e_2 .$$

Then

$$(64) \qquad k = \theta e_1 + (\theta S)e_2 + (\theta S^2)e_3 ,$$

and its easy to verify that the element $k$ of $\mathfrak{H}$ has the required property that $k = kS = kJ$.

It is known[8] that $\mathfrak{H}_{\Omega}$ is a division algebra if and only if $\mathfrak{H}$ is a division algebra. We recall that $\mathfrak{H}_{\Omega}$ is the algebra of all matrices $A = A(\xi, x)$ of (1), and that $\xi$ ranges over all elements of $\mathfrak{M}$, $x$ over all elements of a Cayley algebra $\mathfrak{C}$ over $\mathfrak{M}$. Then $A = \xi e_1 + (\xi S)e_2 + (\xi S^2)e_3 + x_{12} + (xT)_{23} + (xTU)_{13}$ for linear transformations $T$, $U$, $V$ over $\Omega$ of $\mathfrak{C}$, where $T$, $U$, $V$ all induce $S$ in $\mathfrak{M}$. Also $x_{12}S = (xT)_{23}$, $y_{23}S = (yU)_{13}$, $z_{13}S = (zV)_{12}$, so that $x_{12}S^3 = x_{12} = (xTUV)_{12}$. Thus

$$(65) \qquad TUV = I , \qquad V = \delta U , \qquad xU = f(\bar{x}T) , \qquad \overline{xT} = \bar{x}T ,$$

and we also know that

$$(66) \qquad \begin{gathered} g = gT , \qquad f = (\gamma S^2)g , \qquad \delta = \gamma S , \\ (g\bar{g})^{-1} = \gamma(\gamma S)(\gamma S^2) , \qquad f\bar{f} = \frac{\gamma S^2}{\gamma(\gamma S)} , \end{gathered}$$

We have, of course, already derived the properties of $T$.

The transformation $J$ determines linear transformations $W$, $P$, $W_0$ on $\mathfrak{C}$ over $\mathfrak{F}$, all inducing $J$ in $\mathfrak{M}$, and so mapping $e_2$ onto $e_3$, $e_3$ onto $e_2$ and $e_1$ onto itself. Then

$$(67) \qquad x_{12}J = (xW)_{13} , \qquad y_{23}J = (yP)_{23} , \qquad z_{13}J = (zW_0)_{12} .$$

Since $J^2$ is the identity we clearly have the properties

$$(68) \qquad W_0 = W^{-1} , \qquad P^2 = I .$$

We next compute $x_{12}JS = (xW)_{13}S = (xWV)_{12} = (x_{12}S^2)J = (xTU)_{13}J = (xTUW^{-1})_{12}$ and we have shown that

$$(69) \qquad TUW^{-1} = WV .$$

We also have $y_{23}JS = (yP)_{23}S = (yPU)_{13} = y_{23}S^2J = (yUV)_{12}J = (yUVW)_{13}$, from which

[8] See Theorem 2 of 2.

(70) $$PU = UVW = T^{-1}W\,,$$

a result equivalent to the main result connecting $W$ and $P$, that is, the relations

(71) $$W = TPU\,.$$

Finally, $z_{13}JS = (zW^{-1})_{12}S = (zW^{-1}T)_{23} = z_{13}S^2J = (zVT)_{23}J = (zVTP)_{23}$, and we have shown that

(72) $$W^{-1}T = VTP\,.$$

In view of (65) formula (69) is equivalent to $V^{-1}W^{-1} = (WV)^{-1} = WV$, and (65) also implies that $V^{-1}W^{-1}T = TP$, $V^{-1}W^{-1} = TPT^{-1}$. Thus (69) follows from (65), (72), and the fact that $P^2 = I$. By (71) we see that (72) is equivalent to $(TPU)^{-1}T = U^{-1}P = VTP$, $P = UVTP$. This is automatically satisfied since $TUV = UVT = I$.

LEMMA 4. *The relations $x_{ij}JS = x_{ij}S^2$ hold if and only if $W = TPU$.*

We next use the multiplicative formulas

(73) $$2x_{12}\cdot y_{23} = (xy)_{13}\,, \qquad 2x_{12}\cdot y_{13} = \gamma(\bar{x}y)_{23}\,, \qquad 2x_{13}\cdot y_{23} = \delta(x\bar{y})_{12}\,.$$

We have already used the fact that

(74) $$r = A(0, e) = e_{12} + e_{23} + f_{13} = \begin{pmatrix} 0 & e & f \\ \gamma e & 0 & e \\ \gamma\delta\bar{f} & \delta e & 0 \end{pmatrix},$$

where $r$ is in $\mathfrak{H}$, and so $r = rS$ a result equivalent to $eT = e$, $eU = f$. Then $r = rJ = (eW)_{13} + (eP)_{23} + (fW^{-1})_{13}$ and so

(75) $$eW = f\,, \qquad eP = e\,.$$

Apply $J$ to the first relation of (73) to obtain $2(x_{12}\cdot y_{23})J = (xy)_{13}J = [(xy)W^{-1}]_{12} = 2(x_{12}J)$, $(y_{23}J) = 2(xW)_{13}\cdot(yP)_{23} = \delta[(xW)(\overline{yP})]_{12}$, and we have

(76) $$(xy)W^{-1} = \delta(xW)(\overline{yP})\,.$$

Apply $J$ to the second relation of (73) to obtain $2(x_{12}\cdot y_{13})J = [\gamma(\bar{x}y)_{23}]J = (\gamma J)[(\bar{x}y)P]_{23} = 2(xW)_{13}\cdot(yW^{-1})_{12} = \gamma[(\overline{yW^{-1}})(xW)]_{23}$ and we have

(77) $$(\gamma J)[(\bar{x}y)P] = \gamma(\overline{yW^{-1}})(xW)\,.$$

Finally, the application of $J$ to the third relation of (73) yields $2(x_{13}\cdot y_{23})J = \delta(x\bar{y})_{12}J = (\delta J)[(x\bar{y})W]_{13} = 2(xW^{-1})_{12}(yP)_{23} = [(xW^{-1})(yP)]_{13}$, and we have derived the final condition

$$(78)\qquad (xW^{-1})(yP) = \delta J[(x\bar{y})W] .$$

We substitute $x = y = e$ in (76), (77) and (78) to see that

$$(79)\qquad eW^{-1} = \delta f , \qquad \gamma J = \gamma\delta f\bar{f} , \qquad \delta Jf = \delta f .$$

Hence

$$(80)\qquad \gamma J = \gamma S^2 , \qquad \delta = \delta J .$$

We next replace $x$ or $y$ in our relations by $e$ to obtain

$$(81)\qquad yW^{-1} = \delta f(\overline{yP}) , \qquad xW^{-1} = \delta(xW)$$

from (76),

$$(82)\qquad \gamma J(\bar{x}P) = \gamma(\delta\bar{f})(xW) , \qquad (\gamma J)(yP) = \gamma(\overline{yW^{-1}})f$$

from (77), and

$$(83)\qquad xW^{-1} = \delta(xW) , \qquad f(yP) = \bar{y}W$$

from (78).

The relation $xW^{-1} = \delta(xW)$ is equivalent to

$$(84)\qquad W^2 = \delta^{-1}I .$$

The first relation of (81) is then equivalent to

$$(85)\qquad yW = f(\overline{yP})$$

But the second relation of (83) is equivalent to $yW = f(\bar{y}P)$. Hence we have

$$(86)\qquad \overline{yP} = \bar{y}P , \qquad yW = f(\bar{y}P) .$$

Multiply the first relation of (82) by $f$ to obtain $(\gamma S^2)(\gamma\delta)^{-1}f(\bar{x}P) = f\bar{f}(xW)$, a relation satisfied by (86). The second relation of (82) is equivalent to $\gamma\bar{F}(yW^{-1}) = \gamma S^2(\bar{y}P) = \gamma\delta\bar{f}(yW)$, and so all of our relations are satisfied if (84), (85), and (86) all hold.

By (78) we have $(xW^{-1})(yP) = [\delta f(\bar{x}P)](yP) = \delta f(y\bar{x})P$. Replace $\bar{x}$ by $y$ and $\bar{y}$ by $x$ to obtain

$$(87)\qquad f[(xy)P] = [f(yP)](xP) .$$

We also know that

$$(88)\qquad fW = \delta^{-1}e$$

by (79), and (85) yields

$$(89)\qquad \delta^{-1}e = f(\overline{fP}) .$$

Multiply by $\bar{f}$ to obtain $\bar{f} = \delta\bar{f}f\overline{fP}, fP = \delta^{-1}(f\bar{f})^{-1}f = \delta^{-1}\gamma\delta(\gamma S^2)^{-1}f =$

$\gamma(\gamma S^2)^{-1}f$. Then $g = (\gamma S^2)^{-1}f$ implies that $g = \gamma^{-1}fP$, and $gP = (\gamma J)^{-1}f = (\gamma S^2)^{-1}f$. We have derived our *second invariance* property, that is, the relation

$$gP = g \ . \tag{90}$$

Hence $g$ is fixed by both $P$ and $T$.

Relation (76) is equivalent to $\delta f(\overline{xy})P = \delta[f(\bar{x}P)](\bar{y}P)$ a relation which is clearly a form of (87). Relation (77) is equivalent to $\gamma J(\bar{x}y)P = \gamma\delta[(yP)\bar{f}][f(\overline{yP})]$. This is equivalent to $(\gamma S^2)(xy)P = \gamma\delta[(yP)\bar{f}][f(xP)]$. Since $f\bar{f} = \gamma S^2(\gamma\gamma)^{-1}$ our relation is equivalent to

$$\bar{f}[f(xy)P] = \bar{f}\cdot[f(yP)][(xP)] = [(yP)\bar{f}][f(xP)] \ .$$

This result will follow when we derive the following result.

LEMMA 5. *The relation* $\bar{f}[(fy)x] = (y\bar{f})(fx)$ *holds in* $\mathfrak{C}$ *for every* $x$ *and* $y$ *of* $\mathfrak{C}$.

For the relation is trivially satisfied if $x, f$ and $y$ are in an associative subalgebra of $\mathfrak{C}$. Since it is linear in $x$ and $y$ it will clearly hold if it holds for $xu + ux = yu + uy = xy + yx = 0$. Thus it suffices to verify the relation for $y = av$, $x = bw$, where $a$ and $b$ are in $\mathfrak{M}[g]$. But then (14) implies that

$$\bar{f}[f(av\cdot bw)] = \bar{f}[(fa)v\cdot bw] = \bar{f}[(bfa)v\cdot w] = [\bar{f}(\overline{bfa})](vw)$$

and

$$[(av)\bar{f}][f(bw)] = [(af)v][(fb)w] = [(fb)(af)\cdot v]w = (\overline{fbaf})(vw)$$

and we have proved the relation.

We have now shown that the relations imposed by the conditions (63) on the idempotents of $\mathfrak{H}_{\mathfrak{M}}$, and the fact that $\mathfrak{H}$ is the set of all elements of $\mathfrak{H}_{\mathfrak{M}}$ fixed by $S$ and $J$, will imply that there exists a linear transformation $P$ on $\mathfrak{C}$ over $\mathfrak{F}$ inducing $J$ in $\mathfrak{M}$ and such that

$$\bar{x}P = \overline{xP} \ , \qquad gP = g \ , \qquad xP^2 = x \ , \qquad \gamma P = \gamma J = \gamma S^2 \tag{91}$$

for every $x$ of $\mathfrak{C}$. We also have shown that

$$g[(xy)P] = [g(yP)](xP) \tag{92}$$

for every $x$ and $y$ of $\mathfrak{C}$. By (85) we have $yW = f(\bar{y}P)$, and by (71) and (65) we have $yW = yTPU = f(\bar{y}TPT)$. But then $TPT = P$, that is,

$$xTP = xPT^{-1} \ , \tag{93}$$

for every $x$ of $\mathfrak{C}$. We have also seen that $\mathfrak{H}$ consists of all matrices

$A(\xi, x) = A(\xi, 0) + q(x) = [A(\xi, x)]J$ where $q(x) = x_{12} + (xT)_{23} + (xTU)_{13}$ so that $\xi = \xi J$ is in $\mathfrak{K}$ and $q(x) = [q(x)]J = (xTUW^{-1})_{12} + (xTP)_{23} + (xW)_{13}$. This occurs if and only if the first of the relations

$$(94) \qquad xT = xTP\,, \qquad xW = xTPU = xTU\,, \qquad xTUW^{-1} = x\,,$$

holds for every $x$ of $\mathfrak{C}$. The remaining two relations are consequences of $xW = xTPU$.

Conversely, let $x$ range over all elements of $\mathfrak{C}$ such that $xT = xTP$ and let $\mathfrak{H}$ be the set of all corresponding matrices $A(\xi, x)$ for $\xi = \xi J$ in $\mathfrak{K}$. Then $A(\xi, 0)\cdot A(\eta, 0) = A(\xi\eta, 0)$ where $\xi\eta$ is in $\mathfrak{K}$ if $\xi$ and $\eta$ are in $\mathfrak{K}$. Also $2A(\xi, 0)\cdot q(x) = q(y)$, where $y = (\xi + \xi S)x$. Then $yT = (\xi S + \xi S^2)xT$ and $yTP = (\xi SJ + \xi S^2 J)xTP = (\xi S^2 + \xi S)J(xT) = yT$, and so $a(\xi, 0)\cdot q(x)$ is in $\mathfrak{H}$. But then $\mathfrak{H}$ is a Jordan algebra over $\mathfrak{F}$ of $\mathfrak{H}_{\mathfrak{M}}$ if and only if $[q(x)]^2$ is in $\mathfrak{H}$ for every $xT = xTP$, since $2q(x)\cdot q(y) = [q(x + y)]^2 - [q(x)]^2 - [q(y)]^2$. It is easy to show that $[q(x)]^2 = A(\beta, 0) + q(y)$, where $\beta = \gamma(x\bar{x}) + [\gamma(x\bar{x})]S^2 = \gamma(x\bar{x}) + [\gamma(x\bar{x})]J = \beta J$ since our assumption that $xT = xTP = xPT^{-1}$ implies that

$$(95) \qquad xP = xT^2\,, \qquad (x\bar{x})P = (\bar{x}P)(xP) = (\bar{x}T^2)(xT^2) = (x\bar{x})T^2\,.$$

Now $2x_{12}\cdot(xTU)_{13} = (yT)_{23}$ by (73), where

$$(96) \qquad yT = \gamma\bar{x}(xTU)\,.$$

Also

$$2[x_{12}\cdot(xTU)_{13}]J = 2x_{12}J\cdot(xTU)_{13}J = 2(xTUW^{-1})_{13}\cdot(xW)_{12} = (yT)_{23}J = (yTP)_{23}$$

by (67) and (68) and we use (73) to obtain

$$(97) \qquad yTP = \gamma(\overline{xTUW^{-1}})(xW) = yT$$

by (94), and our proof of closure is complete. Since $\mathfrak{C}$ has dimension 8 over $\mathfrak{M}$, and thus has dimension 48 over $\mathfrak{F}$, the dimension of the subspace fixed by $P$ is 24. Hence the space of all elements $x$ of $\mathfrak{C}$ such that $xT = xTP$ is 24 and this confirms the fact that $\mathfrak{H}$ has dimension 27. We state our result as follows.

THEOREM 8. *Let the relations of* (91), (92) *and* (93) *hold for every $x$ in the Cayley algebra $\mathfrak{C}$ over the field $\mathfrak{M} = \mathfrak{K}[\omega]$, and $\mathfrak{H}$ be the set of all matrices $A(\xi, x)$ for $\xi$ in $\mathfrak{K}$ and $xT = xTP$ in $\mathfrak{C}$, so that $\mathfrak{H}$ is a subspace over $\mathfrak{F}$ of the algebra $\mathfrak{H}_{\mathfrak{M}}$ of all threerowed J-Hermitian matrices with elements in $\mathfrak{C}$. Then $\mathfrak{H}$ is a Jordan subalgebra of $\mathfrak{H}_{\mathfrak{M}}$ and $\mathfrak{H}_{\mathfrak{M}} = \mathfrak{H} \times \mathfrak{M}$.*

We now pass on the determination of all transformations $P$ with the properties we have found as a consequence of the fact that $\mathfrak{H}$ is the set of all elements in $\mathfrak{H} \times \mathfrak{M}$ fixed by $S$ and $J$.

**9. Determination of $P$.** The relation (92) implies that

$$(ax)P = (xP)(aP) , \qquad (xa)P = (aP)(xP) , \tag{98}$$

for every $x$ of $\mathfrak{C}$ and $a$ in

$$\mathfrak{L}^* = \mathfrak{M}[g] = \mathfrak{M}[u] . \tag{99}$$

But (98) implies that, if $\mathfrak{C}_u$ is the set of all elements $x$ of $\mathfrak{C}$ such that $xu + ux = 0$, then $u(xP) + (xP)u = 0$. Hence

$$\mathfrak{C}_u P = \mathfrak{C}_u . \tag{100}$$

We seek to determine $P$ so that (91), (92), (93) all hold. Let us first derive a certain normalized basis of $\mathfrak{C}$. We have already seen that $P$ induces $J$ in $\mathfrak{M}$, and so leaves $\mathfrak{K}$ elementwise fixed, and also leaves $u$ fixed. Also $\mathfrak{M} = \mathfrak{K}[\omega]$, where $\omega^2 = \zeta$ in $\mathfrak{F}$; and $\omega J = -\omega$. Thus the effect of $P$ on $\mathfrak{L}^* = \mathfrak{M}[u] = \mathfrak{K}[u, \omega]$ is completely known. But every element $x$ of $\mathfrak{C}_u$ has the form $x = x_1 + x_2\omega$, where $2x_1 = x + xP$ and $2x_2 = (x - xP)\omega^{-1} = (x - xP)\zeta^{-1}\omega$, and we see that $x_1 = x_1P$ and $x_2 = x_2P$. It follows immediately that $\mathfrak{C}_u$ has a basis of elements $u_1, \cdots, u_6$ over $\mathfrak{M}$ where $u_i = u_iP$ for $i = 1, \cdots, 6$. We also saw in (24) that the general element $x = \xi_1u_1 + \cdots + \xi_6u_6$ of $\mathfrak{C}_u$ has the property that

$$xT = a_x x + w_x , \qquad w_x x + x w_x = 0 . \tag{101}$$

Here the $\xi_i$ are independent indeterminates over $\mathfrak{M}$, $a_x$ is in $\mathfrak{L}^*(\xi_1, \cdots, \xi_6)$ and

$$\begin{aligned} x^2 &= \varphi(x) = \varphi(\xi_1, \cdots, \xi_6) , \\ w_x^2 &= \psi(x) = \psi(\xi_1, \cdots, \xi_6) \end{aligned} \tag{102}$$

for $\varphi(x)$ and $\psi(x)$ in $\mathfrak{M}(\xi_1, \cdots, \xi_6)$. There is thus a polynomial $\pi(\xi_1, \cdots, \xi_6)$ in $\mathfrak{M}(\xi_1, \cdots, \xi_6)$ which is the product of the numerator and denominator polynomials of $\varphi(x)$ and $\psi(x)$ and which is not identically zero by the argument used to derive (25). But if we select values $\alpha_i$ in $\mathfrak{F}$ of the $\xi_i$ such that $\pi(\alpha_1, \cdots, \alpha_6) \neq 0$ as we can always do we obtain an element $v = \alpha_1u_1 + \cdots + \alpha_6u_6$ such that

$$\begin{aligned} v = vP , \qquad vT = av + w , \\ vw + wv = 0 , \qquad v^2 = \varphi , \qquad w^2 = \psi , \end{aligned} \tag{103}$$

where $a$ is in $\mathfrak{M}[u]$, $\varphi$ and $\psi$ are nonzero elements of $\mathfrak{M}$. Then the multiplicative formula of (14) holds for products in our Cayley algebra $\mathfrak{C}$ over $\mathfrak{M}$ where, as before, the quaternion algebra $\mathfrak{D} = \mathfrak{M} + \mathfrak{M}u + \mathfrak{M}v + \mathfrak{M}(uv)$.

Relation (92) implies that

$$(xP)^2 = x^2P \,, \tag{104}$$

for every $x$ of $\mathfrak{C}$ and, in particular, for all elements of $\mathfrak{C}_u$ where in fact $x^2$ is in the center $\mathfrak{M}$ of $\mathfrak{C}$. Hence our normalization implies that $v^2P = (vP)^2 = v^2$, that is,

$$\varphi = \varphi J \,. \tag{105}$$

Let us now utilize the relations derived in our determination of the transformation $T$. We can use (32) and (33) to write

$$vT^2 = hv + kw + d(vw) \,, \tag{106}$$

where (52) implies that

$$h = b + a(aT) = \frac{\bar{g}}{g}\frac{(\varphi\bar{a})T^2}{\varphi} \,, \tag{107}$$

and that

$$k = aT + c \,. \tag{108}$$

From (50) we have

$$\begin{aligned} kT &= aT^2 + cT = (dT)d^{-1}(\bar{b}\bar{g})g^{-1}\psi^{-1}(\varphi S) \\ &= (d\varphi\psi)^{-1}(d\varphi\psi)T\varphi(\psi S)^{-1}(\bar{b}\bar{g})g^{-1} = \bar{g}g^{-1}\varphi(\psi S)^{-1}\bar{b} \,, \end{aligned}$$

since $s = d\varphi\psi = sT$. Hence we have derived the consequence

$$k = (kT)T^2 = \frac{\bar{g}}{g}\frac{\varphi S^2}{\psi}(\bar{b}T^2) \,. \tag{109}$$

We now apply $P$ to $vT$ to get $vTP = vPT^{-1} = vT^{-1} = vT^2T^{-3} = g(vT^2)g^{-1} = (av + w)P = (\bar{a}P)v + wP$. Hence

$$wP = \frac{g}{\bar{g}}[hv + kw + d(vw)] - (\bar{a}P)v \,. \tag{110}$$

However, we may actually show that

$$wP = \frac{g}{\bar{g}}[kw + d(vw)] \,. \tag{111}$$

for $vw$ is in $\mathfrak{C}_u$ and so is $(vw)P$. Then $q[(vw)P] = g[(wP)v]$ is in $\mathfrak{C}_u$ and so the term in $v$ of $wP$ must vanish. Thus $g(\bar{g})^{-1}h - \bar{a}P = 0 = (\varphi\bar{a})T^2\varphi^{-1} - \bar{a}P$ and so we have

$$(a\varphi)P = (a\varphi)T^2 \,. \tag{112}$$

By (104) we square the value of $wP$ in (111) and obtain

(113) $$\psi J = (k\bar{k} - d\bar{d}\varphi)\psi \,.$$

Moreover, $g[(vw)P] = [(g^2\bar{g}^{-1}k)w]v + [(g^2\bar{g}^{-1}d)(vw)]v$. However, if $t$ is in $\mathfrak{M}[u]$ we know that $(tw)v = -(tv)w = -\bar{t}(vw)$, and $[t(vw)]v = [(\bar{t}v)w]v = -(\bar{t}\varphi)w$. Thus $g[(vw)P] = -\bar{g}^2g^{-1}\bar{k}(vw) - \bar{g}^2g^{-1}\bar{d}\varphi w$ and so we have found that

(114) $$(vw)P = -\frac{\bar{g}^2\bar{k}}{g^2}(vw) - \frac{\bar{g}^2\bar{d}\varphi w}{g^2} \,.$$

Let us now derive a consequence of (112). We first see that

(115) $$\varphi = \varphi J \,, \qquad \varphi S = \varphi JS = \varphi S^2 J \,, \qquad \varphi S^2 = \varphi SJ \,.$$

Then (112) implies that

(116) $$aP = aT^2\frac{\varphi S^2}{\varphi} \,, \qquad aTP = aPT^2 = aT\frac{\varphi S}{\varphi S^2} \,, \qquad aT^2P = aPT = a\frac{\varphi}{\varphi S} \,.$$

Also (107) implies that $\varphi b = \bar{g}g^{-1}(\varphi\bar{a})T^2 = (\varphi a)(aT)$, and so $(\varphi b)P = \bar{g}g^{-1}\varphi\bar{a} - (\varphi a)PaT\varphi S(\varphi S^2)^{-1} = \bar{g}g^{-1}\varphi\bar{a} = (\varphi a)T^2(\varphi S^2)^{-1}(a\varphi)T$, from which we have

(117) $$(\varphi b)P = (\varphi b)T \,.$$

We also use (109) to see that

$$\psi k = \bar{g}g^{-1}(\varphi\bar{b})T^2 = \bar{g}g^{-1}(\varphi\bar{b}P)T = \bar{g}g^{-1}(\varphi\bar{b}T^2)P \,,$$

that is,

(118) $$(\psi k)P = \psi k \,, \qquad kP = \psi(\psi J)^{-1}k \,.$$

We now apply $P$ to (111) and use $P^2 = I$ to see that

$$w = [\bar{g}g^{-1}(\bar{k}P)](wP) - (\bar{g}g^{-1}\bar{d}P)[(vw)P] = (\bar{g}g^{-1}\bar{k}P)[(g\bar{g}^{-1}k)w + (g\bar{g}^{-1}d)(vw)] - (\bar{g}g^{-1}\bar{d}P)[(\bar{g}g^{-1})^2(\bar{d}\varphi)w + (\bar{g}g^{-1})^2\bar{k}(vw)] \,.$$

Then

(119) $$(\bar{k}P)d = (\bar{g}g^{-1})^3\bar{k}\bar{d}P \,.$$

Use (118) to obtain $\psi(\psi J)d = (\bar{g}g^{-1})^3\bar{d}P$, $g^3d\psi = \bar{g}^3(\bar{d}\psi)P$. Since $s = d\varphi\psi$ and $\varphi = \varphi P$ we have shown that

(120) $$(g^3s)P = \overline{g^3}s$$

We also see that $(wP)P = w$ implies that

(121) $$1 = k(\bar{k}P) - \bar{d}\bar{d}P(\bar{g}g^{-1})^3\varphi \,.$$

But $(\bar{k}P)k = (\psi J)^{-1}\psi(k\bar{k})$ and $\bar{d}P = (\bar{g}g^{-1})^3 d\psi(\psi J)^{-1}$, so that our relations imply that (121) is equivalent to (113). We have proved that our defining formulas for $wP$ and $(vw)P$ imply that $wP^2 = w$ holds if and only if the condition (112) on $a$, (113) on $\psi J$, and (120) on $s$ all hold.

Conversely, let the element $x$ defined by

$$x = a_0 + a_1 v + a_2 w + a_3(vw) \qquad (a_i \text{ in } \mathfrak{L}^*) \tag{122}$$

be the general element of $\mathfrak{C}$, and define the transformation $P$ by

$$xP = \bar{a}_0 P + (\bar{a}_1 P)v + (\bar{a}_2 P)(wP) + (\bar{a}_3 P)[(vw)P] , \tag{123}$$

where $wP$ is given by (110), and $(vw)P$ by (114). Assume also that the conditions $\varphi = \varphi J$, (112), (113), and (120), hold, so that $wP^2 = w$ and $vP^2 = v$. Also, from $\varphi = \varphi J$, we have $(vP)^2 = v^2 P$, we have $(wP)^2 = w^2$ from (113), and $[(vw)P]^2 = (vw)^2 P$ from (114) and (113). Since $x$ is in $\mathfrak{C}_u$ if and only if $a_0 = 0$ it follows from (123) that $\mathfrak{C}_u P = \mathfrak{C}_u$.

Since (92) is linear in $x$ and $y$, it will hold if and only if it holds for $x = qx_0$, $y = ry_0$, where $q$ and $r$ are in $\mathfrak{L}^*$, and $x_0$ and $y_0$ are any of the elements $1, v, w,$ or $vw$. If $x_0 = y_0 = z$ then $[(qz)(rz)]P = [(q\bar{r})z^2]P = [(q\bar{r})P](z^2 P)$, since $z^2$ is in $\mathfrak{L}^*$ when $z$ is in $\mathfrak{C}_u$. But $[(rz)P][(qz)P] = (\bar{r}P)(zP)(\bar{q}P)(zP) = (\bar{r}P)(qP)(zP)^2 = [(q\bar{r})P](z^2P)$ for $z = v, w,$ or $vw$, and so (92) holds for $x_0 = y_0$. It also holds for $x_0 = 1$ or $y_0 = 1$ since it then becomes (98), which is a consequence of (123).

Let us then turn to the cases where $x_0 \neq y_0$ and $x_0$ and $y_0$ are selected to be $v, w$ or $vw$. We shall let $q$ and $r$ be in $\mathfrak{L}^*$ in all cases, and begin by computing $g[(qv)(rw)]P = g[(rqv)w]P = g[\overline{rq}(vw)]P = [(grq)P](vw)P$. We also compute

$$\begin{aligned}[(g(rw)P)](qv)P = \{(g\bar{r})Pg\bar{g}^{-1}[kw + d(vw)]\}(\bar{q}Pv) = (g^2\bar{g}^{-1}\bar{r}Pkw)(\bar{q}P)v \\ + [(g^2\bar{g}^{-1}\bar{r}Pd)(vw)][(\bar{q}P)v] = -[\bar{g}^2g^{-1}(rP)\bar{k}(qP)](vw) \\ - [\bar{g}^2g^{-1}(rP)\bar{d}(qP)\varphi]w = g(rq)P[(vw)P] ,\end{aligned}$$

and have verified (92) for $x_0 = v$ and $y_0 = w$.

We next compute $g[(qv)\cdot r(vw)]P = g[(qv)\cdot(\bar{r}v)w]P = g(\bar{p}\bar{q}\varphi wP) = g(rq)P\varphi(wP)$. We also compute

$$\begin{aligned}[g\bar{r}P(vw)P][(\bar{q}P)v] &= [(\bar{q}P)v][(\bar{r}P)(\bar{g}^2g^{-1}\bar{d}\varphi)w + (\bar{r}P\bar{g}^2g^{-1}\bar{k})(vw)] \\ &= (\bar{g}^2g^{-1}\bar{d}\varphi\bar{r}P\bar{q}Pv)w + (\bar{q}Pv)[(rPg^2\bar{g}^{-1}kv)w] \\ &= (rPg^2\bar{g}^{-1}kqP\varphi)w + g^2\bar{g}^{-1}d\varphi rPqP(vw) = [g(rq)P]wP\end{aligned}$$

and have shown that (92) holds for $x_0 = v$, $y_0 = vw$.

Our third stage is the computation of

$$g[qw\cdot r(vw)]P = g[qw\cdot(\bar{r}v)w]P = g[-\bar{r}vq\psi]P = -g[(rq)P](\psi J)v .$$

Also

$$
\begin{aligned}
-g[r(vw)]P\cdot(qw)P &= -[g(\bar{r}P)(vw)P][(\bar{q}P)(wP)] \\
&= [\bar{g}^2g^{-1}(\bar{r}P)\bar{d}\varphi w + (g^2\bar{g}^{-1}\cdot rP\cdot kv)w][(\bar{q}P\cdot g\bar{g}^{-1}k)w \\
&\quad + (qP\cdot\bar{g}g^{-1}\bar{d}v)w] = \bar{g}^2g^{-1}(\bar{r}P)\bar{d}\varphi(qP)\bar{g}g^{-1}\bar{k}\psi \\
&\quad - \bar{g}^2g^{-1}(\bar{r}P)\bar{k}(qP)\bar{g}g^{-1}\bar{d}\varphi\psi \\
&\quad - (qP)\bar{g}g^{-1}\bar{d}g^2\bar{g}^{-1}(rP)d\varphi v\psi \\
&\quad + (qP)\bar{g}g^{-1}\bar{k}g^2\bar{g}^{-1}(rP)kv\psi \\
&= -(qr)Pg[d\bar{g}\varphi - k\bar{k}]v\psi = g(qr)P(\psi J)v
\end{aligned}
$$

by (113), and (92) holds for $x_0 = w$ and $y_0 = vw$.

If $x_0 = w$ and $y_0 = v$ we have $g[(qw)(rv)]P = -g[(rv)(qw)]P = -g(rq)P[(vw)P]$. Also $(g\bar{r}Pv((\bar{q}P)wP = -[(\bar{q}P)(wP)][(g\bar{r}P)v]$. Write $q = q_1\bar{g}$, $r = r_1\bar{g}^{-1}$ so that $\bar{q}P = \bar{q}_1Pg$, and $\bar{r}P = (\bar{r}_1P)g^{-1}$, while

$$
\begin{aligned}
[g(\bar{r}P)v][(\bar{q}P)wP] &= -[g(q_1w)P][(r_1v)P](r_1v)P = -g[(r_1v)(q_1w)]P \\
&= -h(r_1q_1)P(vw)P = -g(rq)P[(vw)P] .
\end{aligned}
$$

The case $x_0 = vw, y_0 = v$ is taken care of similarly. Probably the simplest procedure for the case $x_0 = vw, y_0 = w$ is the type of computation used in the case $x_0 = w, y_0 = vw$.

We now turn to the property $xTP = xPT^{-1}$. We first turn to a rather immediate consequence of the basic property $g(xy)T = g(xT)(yT)$.

LEMMA 6. *If $x$ and $y$ are in $\mathfrak{C}$ then $[(gx)y]T^{-1} = g[(xT^{-1})(yT^{-1})]$. If also $x$ or $y$ is in $\mathfrak{L}^*$ then $(xy)T^{-1} = (xT^{-1})(yT^{-1})$.*

For $g[(xy)T] = [g(xy)]T = [g(xT)](yT)$ and so $[g\cdot(xT^{-1})(yT^{-1})]T = (gx)y$ and our result follows.

Let us now observe that the relation $xTP = xPT^{-1}$ holds for $x$ in $\mathfrak{L}^*$. Since the relation is linear it suffices to derive it for $x = qz$ where $q$ is in $\mathfrak{L}^*$ and $z = v, w, vw$. But $(qz)TP = [(qT)(zT)]P = (zTP)(qTP)$ and $(qz)PT^{-1} = [(zP)(qP)]T^{-1} = (zPT^{-1})(qPT^{-1}) = zPT^{-1}(qTP)$, so then it suffices to prove the result for $x = v, w$ or $vw$. There thus remains only the case $x = vw$. We form

$$
\begin{aligned}
g\bar{g}[(vw)TP] &= g[\bar{g}(vw)TP] = g\cdot[g(vw)T]P = g[(gv)T(wT)]P \\
&= [g(wT)TP][(gv)TP] = [(\bar{g}w)TP][(qv)TP] \\
&= [(\bar{g}w)PT^{-1}][(gv)PT^{-1}] .
\end{aligned}
$$

Thus

$$
\begin{aligned}
g\{(g\bar{g})[(vw)TP]\} &= [g(\bar{g}w)P\cdot(gv)P]T^{-1} = \{g\cdot[(gv)(\bar{g}w)]P\}T^{-1} \\
&= \{g\cdot[(g\bar{g}v)w]P\}T^{-1} = g[g\bar{g}(vw)]PT^{-1} = g(g\bar{g})[(vw)PT^{-1}]
\end{aligned}
$$

and our proof is complete.

The only remaining property is $xP^2 = x$. This holds for $x$ in $\mathfrak{L}^*$, for $x = v$, and for $x = w$ as we have already seen. Also (123) implies that $xP^2 = x$ for all $x$ if and only if the relation holds for $x = v, w, vw$. There thus remains the case $x = vw$, and we compute $g\bar{g}[(vw)P^2] = g[g(vw)P]P = g[(gw)Pv]P = (gv)[(gw)P^2] = (gv)(gw) = (g\bar{g}v)w = g\bar{g}(vw)$, and our proof is complete. We have proved the following fundamental result.

THEOREM 9. *Let a basis of* $\mathfrak{C}$ *be selected so that* $vP = v$. *Then* $P$ *is completely determined by the relation* $vT = av + w$ *as in* (111), (114), (112), (123), *where* (112), (113) *and* (120) *hold. Conversely, if we define* $P$ *by* $vP = v$, (111), (114), (122), (123) *where* (112), (113) *and* (120) *hold then* $P$ *satisfies Properties VI, VII and VIII.*

This completes our determination of $P$. We close our discussion with the proof of the existence theorem referred to in our Introduction.

**10. Construction of a special class of algebras.** We shall now construct a class of cyclic exceptional Jordan division algebras in which $\mathfrak{C}$ is a division algebra. We shall assume that $vT = w$, that is, $a = b = c = 0$ and so

$$vT = w\ , \qquad wT = d(vw)\ . \tag{124}$$

As a consequence our relations become

$$\psi = \varphi S\ , \qquad \psi S = \varphi S^2 = -\varphi\psi d\bar{d} = -\varphi(\varphi S)d\bar{d}\ . \tag{125}$$

Then, if $s = d\varphi\psi$, we have the value

$$s\bar{s} = d\bar{d}(\varphi\psi)^2 = d\bar{d}[\varphi(\varphi S)]^2 = -\varphi(\varphi S)\varphi S^2\ . \tag{126}$$

We need to satisfy the condition that $s$ is not a norm in the cyclic algebra $\mathfrak{C} = (\mathfrak{L}, T, \bar{g}g^{-1})$, where $g$ is in $\mathfrak{F}[u]$, $u^2 = \rho$ in $\mathfrak{F}$, $\mathfrak{F}[u]$ is a field, $\mathfrak{D}$ is a division algebra, and $\mathfrak{C}$ is a division algebra.

Assume first that $\mathfrak{F}_0$ is a real algebraic number field, and that $\mathfrak{K}_0$ is a cyclic cubic extension of $\mathfrak{F}_0$ such that there is a prime ideal $\pi$ of $\mathfrak{K}_0$ for which $\mathfrak{K}_0 \times \mathfrak{F}_{0\pi}$ is unramified over the $\pi$-adic extension $\mathfrak{F}_{0\pi}$ of $\mathfrak{F}_0$. We select a negative element $\rho$ of $\mathfrak{F}_0$ such that $\pi = \pi_1\bar{\pi}_1$ for conjugate prime ideals of $\mathfrak{F}_0(u)$ with $u^2 = \rho$. There is an element $h$ which is in the ldeal $\pi_1$ and is in neither $\pi_1^2$ nor $\bar{\pi}_1$. Take $g = h(\bar{h})^{-1}$ and see that the cyclic algebra $(\mathfrak{L}_0, S, g)$ is a division algebra. So is $(\mathfrak{L}_0, S, g\bar{g}^{-1})$ since $g\bar{g}^{-1} = g^2$. Observe that $g\bar{g} = 1$ is the norm of $\gamma^{-1}$ if $\gamma$ is any element of norm 1 in $\mathfrak{K}_0$. Hence Property $IV$ holds.

Every *cyclic* cubic extension of a real field is totally real. Let $\beta$ be a totally positive element of $\mathfrak{K}_0$ so that $\beta$, $\beta S$ and $\beta S^2$ are all positive, and define

$$\varphi = -\frac{\beta S}{\beta}. \tag{127}$$

Then $\varphi$, $\varphi S$, $\varphi S^2$ are all negative, and

$$\varphi(\varphi S)(\varphi S^2) = -1. \tag{128}$$

The Cayley algebra $\mathfrak{C}_0 = \mathfrak{K}_0 + \mathfrak{K}_0 v + \mathfrak{K}_0 w + \mathfrak{K}_0(vw)$ has negative parameters $u^2 = \rho$, $v^2 = \varphi$ and $w^2 = \varphi S = \psi$, and so has a totally positive norm form over the real field $\mathfrak{K}$. Hence $\mathfrak{C}_0$ is a division algebra.

We shall now select $s$. We assume that $\eta$ is an indeterminate over $\mathfrak{K}_0[u]$, and take $\mathfrak{F} = \mathfrak{F}_0(\eta)$, $\mathfrak{K} = \mathfrak{K}_0(\eta)$, $\mathfrak{L} = \mathfrak{K}(u)$, $\mathfrak{C} = \mathfrak{C}_0 \times \mathfrak{F}$, $\mathfrak{D} = \mathfrak{D}_0 \times \mathfrak{F}$. Then $\mathfrak{D}$ is an associative division algebra, and $\mathfrak{C}$ is a Cayley division algebra. Write

$$s = \frac{\eta + u}{\eta - u}. \tag{129}$$

Since $s$ is not in $\mathfrak{F}_0(u)$ we know that $s + 1 \neq 0, s - 1 \neq 0$. But $(\eta - u)s = \eta + u$, $\eta(s - 1) = u(s + 1)$, and the indeterminate

$$\eta = u\left(\frac{s+1}{s-1}\right) \tag{130}$$

is in $\mathfrak{F}_0(u, s)$. Hence $s$ must alse be an indeterminate over $\mathfrak{K}(u)$. But then it is known[9] that $s$ is not the norm of an element $x$ in $\mathfrak{D}$. Also $s\bar{s} = 1$. But $d = s(\varphi\psi)^{-1}$ where $s\bar{s} = 1$ and so $d\bar{d} = (\varphi\psi)^{-2} = [\varphi(\varphi S)]^{-2} = [\varphi(\varphi S)]^{-1}[\varphi(\varphi S)(\varphi S^2)]^{-1}\varphi S^2$. By (128) we have $\varphi S^2 = \psi S = -\varphi(\varphi S)d\bar{d}$ and (35) holds. This completes our construction of $\mathfrak{H}$ and $\mathfrak{H}$ is a Jordan division algebra, is cyclic, and is such that $\mathfrak{C}$ is a division algebra.

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[9] This is proved on page 27 of 2.