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ON THE GENERALIZED F. AND M. RIESZ THEOREM

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Let X be a compact Hausdorff space, $C(X)$ the algebra of all continuous complex valued functions on X, and let A be a sup-norm algebra on X, that is, A is a uniformly closed algebra of continuous complex valued functions on X that contains the constants and separates the points. If ϕ is a complex homomorphism of A then let $M(\phi)$ be the set of all positive, regular, Borel measures on X that represent ϕ. If μ is a finite, (complex), regular, Borel measure on X then we write $\mu \perp A$ if $\int f d\mu = 0$ for all $f \in A$. Let ϕ be a complex homomorphism of A and $m \in M(\phi)$, then we say that m satisfies the Riesz theorem if whenever μ is a finite, (complex), regular, Borel measure on X and $\mu \perp A$ then $\mu_a \perp A$ and $\mu_s \perp A$ where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m. It is quite easy to see that if $m \in M(\phi)$ and m satisfies the Riesz theorem then for all $\rho \in M(\phi)$ we have ρ is absolutely continuous with respect to m. We will show that this condition is also sufficient. This is done by means of a theorem which says that if $F \subseteqq X$ is a compact G_δ such that $m(F) = 0$ for all $m \in M(\phi)$ then there exists a sequence f_n in A such that $|f_n| \leqq 1$ on X, $\phi(f_n) \to 1$, and $f_n \to 0$ uniformly on F.

The proof given is not a generalization of the modern proof of the F. and M. Riesz theorem as given in [4], for instance, but is closer in form to the original proof of F. and M. Riesz. If $X = S_1 \cup S_2$ is the decomposition of X corresponding to the decomposition $\mu = \mu_a + \mu_s$, then by means of Theorem 1 we find a bounded sequence in A that converges to the characteristic function of S_1 almost everywhere with respect to the total variation of the measure μ. It is known (see Hoffman [4] and Lumer [5]) that if $M(\phi) = \{m\}$ then the Riesz theorem holds for the measure m. It is known that $M(\phi)$ is not empty [4].

It what follows, all measures are assumed to be finite, regular, Borel measure, and ϕ is a fixed complex homomorphism of A.

LEMMA 1. *Let $\{\nu_n\}$ be a sequence of positive measures on X having the measure m as a weak-star accumulation point. Suppose $F \subseteqq Y$*

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is compact and that $\nu_n(F) \geqq \varepsilon_0 > 0$ *for all* n. *Then* $m(F) \geqq \varepsilon_0$.

Proof. There exists a decreasing sequence of open sets $\mathscr{O}_n \supseteqq F$ such that $m(\mathscr{O}_n - F) \to 0$. There exists a sequence u_n of continuous real valued functions such that $u_n = 1$ on F, $u_n = 0$ on $X - \mathscr{O}_n$ and $0 \leqq u_n \leqq 1$ elsewhere. From the construction, $u_n \to \chi_F$ a.e. (m), where χ_F is the characteristic function of F. So we have,

$$m(F) = \int (\chi_F - u_k)dm + \int u_k d\nu_n + \int u_k(dm - d\nu_n) \, .$$

Note that $\int u_k d\nu_n \geqq \nu_k(F) \geqq \varepsilon_0$ for all n and k. Now, $\int(\chi_F - u_k)dm$ can be made small by choosing k large, and once k is fixed $\int u_k(dm - d\nu_n)$ can be made small by proper choice of n. This proves the lemma.

The proof of the next lemma can be found in [1], Theorem 3.b.

LEMMA 2. *Let* $u \in C(X)$ *be real valued and suppose*

$$\sup \{Re\, \phi(g) \mid Re\, g \leqq u, g \in A\} \leqq \gamma \leqq \inf \{Re\, \phi(g) \mid Re\, g \geqq u, g \in A\}$$

then there exists $\rho \in M(\phi)$ *such that* $\int u d\rho = \gamma$. *In particular, there exists* $\rho_u \in M(\phi)$ *such that*

$$\sup \{Re\, \phi(g) \mid Re\, g \leqq u, g \in A\} = \int u d\rho_u \, .$$

THEOREM 1. *Let* $F \subseteqq X$ *be a compact* G_δ *such that* $m(F) = 0$ *for all* $m \in M(\phi)$, *then there exists a sequence* $f_n \in A$ *such that*

(1) $|f_n| \leqq 1$ *on* X.

(2) $\phi(f_n) \geqq e^{-2/n}$.

(3) $|f_n| \leqq e^{-n}$ *on* F.

Proof. Since F is a compact G_δ, there is a sequence of open sets $\{\mathscr{O}_n\}$ such that $\bar{\mathscr{O}}_{n+1} \subseteqq \mathscr{O}_n$ and $\bigcap_n \mathscr{O}_n = F$. Let $\varepsilon > 0$ be given, then there exists an integer N such that for all $n \geqq N$, $\rho(\mathscr{O}_n) < \varepsilon$ for all $\rho \in M(\phi)$. For suppose this were not true, then there would exist $\varepsilon_0 > 0$ and sequences $\rho_k \in M(\phi)$ and $\mathscr{O}_{n_k}$ such that $\rho_k(\mathscr{O}_{n_k}) \geqq \varepsilon_0$ Let $U_k = \mathscr{O}_{n_k}$ then we have $\rho_k(U_k) \geqq \varepsilon_0 > 0$ and $\bar{U}_{k+1} \subseteqq U_k$. The sequence ρ_k has a weak-star limit point ρ, and it is well known that $\rho \in M(\phi)$ hence $\rho(F) = 0$. Fix k, then $\rho(U_k) \geqq \rho(\bar{U}_{k+1})$, now $\rho_n(\bar{U}_{k+1}) \geqq \rho_n(U_{k+1}) \geqq \rho_n(U_n) \geqq \varepsilon_0 > 0$ for all $n \geqq k + 1$. Therefore by Lemma 1 we have $\rho(U_k) \geqq \varepsilon_0 > 0$ for all k. But this contradicts the fact that $\rho(F) = 0$. Hence by proper choice of subsequence we may assume that $\rho(\mathscr{O}_n) < (1/n^2)$

for all $\rho \in M(\phi)$. Now for each n there exists $u_n \in C(X)$ such that $u_n = -n$ on F, $u_n = 0$ on $X - \mathscr{O}_n$ and $-n \leqq u_n \leqq 0$ elsewhere. By Lemma 2, there exists $\rho_n \in M(\phi)$ such that

$$\sup \{Re\, \phi(g) \mid Re\, g \leqq u_n,\, g \in A\} = \int u_n d\rho_n ,$$

and hence for each n there exists $g_n \in A$ such that $Re\, g_n \leqq u_n$ and

$$\int Re\, g_n dm \geqq \int u_n d\rho_n - \frac{1}{n} \geqq -n\rho_n(\mathscr{O}_n) - \frac{1}{n} \geqq -\frac{2}{n} .$$

We may also assume that $\int Im g_n dm = 0$. If we now define $f_n = e^{g_n}$ it follows that

(1) $|f_n| = e^{Re\, g_n} \leqq e^{u_n} \leqq 1$

(2) $\int f_n dm = \exp\left[\int g_n dm\right] = \exp\left[\int Re\, g_n dm\right] \geqq e^{-2/n}$

(3) $|f_n| = e^{Re\, g_n} \leqq e^{-n}$ on F.

The sequence $\{f_n\}$ of Theorem 1 is bounded in norm by 1, yet $\phi(f_n) \rightarrow 1$. We show that this implies that $\psi(f_n) \rightarrow 1$ for all ψ in the same part as ϕ. For definition of part see [4]. For this we use a result of Bishop [2]: if ϕ, ψ are in the same part and m_ϕ is a representing measure for ϕ, then there exists a representing measure m_ψ for ψ such that $m_\phi \leqq A m_\psi$ for some constant A.

COROLLARY 1. *If $\{f_n\}$ is the sequence of Theorem* 1 *and ψ is in the same part as ϕ, then $\psi(f_n) \rightarrow 1$.*

Proof. Let m be a representing measure for ψ, and ρ be a representing measure for ϕ such that $m \leqq A\rho$ for some constant A. Then we have $m = g\rho$ where g is bounded. Since $\psi(f_n) \rightarrow 1$ we have $\int f_n d\rho \rightarrow 1$. This, together with the fact that $|f_n| \leqq 1$ implies that $f_n \rightarrow 1$ in measure, with respect to the measure ρ. Since g is bounded it follows that $f_n g \rightarrow g$ in measure with respect to the measure ρ. The fact that $|f_n g| \leqq |g|$ now implies that $\psi(f_n) = \int f_n g d\rho \rightarrow \int g d\rho = \int dm = 1$.

COROLLARY 2. *Suppose there is a measure $m \in M(\phi)$ such that $\rho \ll m$ for all $\rho \in M(\phi)$, and suppose $F \subseteqq X$ is compact and $m(F) = 0$. Then there exists a sequence $f_n \in A$ satisfying* (1), (2), (3) *of Theorem* 1.

Proof. There exists a sequence $\{\mathscr{O}_n\}$ of open sets such that $F \subseteqq \mathscr{O}_{n+1} \subseteqq \mathscr{O}_n$ and $m(\mathscr{O}_n) \rightarrow 0$. For each n, there exists a set F_n which is a compact G_δ such that $F \subseteqq F_n \subseteqq \mathscr{O}_n$. Let $F_1 = \bigcap_n F_n$, then $F \subseteqq F_1$,

F_1 is a compact G_δ and $m(F_1) = 0$. It follows that $\rho(F_1) = 0$ for all $\rho \in M(\phi)$. Now apply Theorem 1 to the set F_1.

THEOREM 2. *Suppose there exists $m \in M(\phi)$ such that $\rho \ll m$ for all $\rho \in M(\phi)$. Let $\mu \perp A$ and let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m. Then $\mu_a \perp A$ and $\mu_s \perp A$.*

Proof. Let S be a Borel set that carries μ_s and $m(S) = 0$. Then there exists an increasing sequence $F_n \subseteqq S$ of compact sets such that $|\mu_s\ (F_n) \to |\mu_s|(S)$, where $|\mu_s|$ denotes the total variation of μ_s. For each F_n we have a sequence $f_{n,k} \in A$ such that

(1) $|f_{n,k}| \leqq 1$.

(2) $\int f_{n,k} dm \geqq e^{-2/k}$.

(3) $|f_{n,k}| \leqq e^{-k}$ on F_n.

Define $h_n = f_{nn}$ then we have:

(1′) $|h_n| = |f_{n,n}| \leqq 1$.

(2′) $\int h_n dm = \int f_{n,n} dm \geqq e^{-2/n}$.

(3′) $|h_n| = |f_{n,n}| \leqq e^{-n}$ on F_n.

From 1′ and 2′ it follows that $h_n \to 1$ in measure with respect to m and hence we have a subsequence $h_{n_k} \to 1$ a.e. (m). From 3′ we have $h_{n_k} \to 0$ a.e. $(|\mu_s|)$. Hence $g_k = h_{n_k} \to \chi_{X-S}$ a.e. $(|\mu|)$. So if $f \in A$ then for each k, $g_k f \in A$ and we have $0 = \int g_n f d\mu \to \int_{X-S} f d\mu = \int f d\mu_a$. This proves the theorem.

We point out that if the homomorphism ϕ has a representing measure m such that $\rho \in M(\phi)$ implies $\rho \ll m$ then it follows easily from the result of Bishop mentioned earlier that every ψ that lies in the same part as ϕ has a representing measure with this same property.

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