PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 3, 1965

TRANSLATION-INVARIANT FUNCTION ALGEBRAS ON COMPACT GROUPS

JOSEPH A. WOLF

Let X be a compact group. $\mathfrak{C}(X)$ denotes the Banach algebra (point multiplication, sup norm) of continuous complex-valued functions on X. A is any closed subalgebra of $\mathfrak{C}(X)$ which is stable under right and left translations and contains the constants. It is shown, by means of the Peter-Weyl Theorem and some multilinear algebra, that the condition (*) *every representation of degree* 1 *of X has finite image* is necessary and sufficient that every possible A be self-adjoint. If X is connected, then (*) means that X is a projective limit of semisimple Lie groups; if X is a Lie group, then (*) means that X is semisimple. The Stone-Weierstrass Theorem then gives a quick classification of all possible algebras A on an arbitrary connected semisimple Lie group X.

In an earlier paper we characterized the compact groups on which every closed translation-invariant(1) function space is self adjoint [1, Theorem 4.1]. An application of the Stone-Weierstrass Theorem [1, § 7] resulted in a classification of the closed translation-invariant function algebras on the connected Lie groups which satisfied the conditions of the characterization. Those Lie groups are the compact connected semisimple Lie groups with no simple component locally isomorphic to $SU(n)\,(n > 2)$, to $SO(4n + 2)$, nor to E_6.

In this paper we give a direct characterization of the compact groups on which every closed translation-invariant function algebra is self adjoint. For compact connected Lie groups, the characterization is that the group be semisimple. Many of these groups have closed translation-invariant function spaces which are not self adjoint. Finally we classify the closed translation-invariant function algebras on compact simple simply connected Lie groups, as an example of a general enumeration procedure.

2. Notation. Let X be a compact group. D denotes the set of all (equivalence classes of) irreducible finite dimensional representations of X. Let $\beta \in D$; then W^β denotes the representation space, and β^* is the representation of X on the dual space of W^β induced by β. β^* is called the *contragredient* of β. $E^\beta = W^\beta \otimes W^{\beta^*}$ is identified with the space of matrix functions (coefficients) of β. Let $C(X)$ denote the

Received April 20, 1964. Partially supported by National Science Foundation Grant GP-812.

(1) Stable under both right and left translations.

set of continuous complex-valued functions on X with sup norm and point multiplication. Then $C(X)$ is a commutative Banach algebra. The Peter-Weyl Theorem says that every element of $C(X)$ has a Fourier development along $\{E^\beta\}_{\beta\in D}$, i.e.,

$$(*) \qquad C(X) \sim \sum_{\beta\in D} E^\beta .$$

The *translation group* $T(X)$ of X is the group consisting of all transformations

$$(u, v): x \to uxv^{-1}, \qquad u, v, x \in X ,$$

of X. A subset $S \subset C(X)$ is called *translation-invariant* if $T(X)\cdot S = S$. The closed translation-invariant subspaces of $C(X)$ are just the subspaces spanned by sets of the form $\{E^\beta\}_{\beta\in F}$ with $F \subset D$.

If X is a compact connected Lie group endowed with a Riemannian metric invariant under both right and left translations, then X is a Riemannian symmetric space and $T(X)$ is the largest connected group of isometries.

A subspace of $C(X)$ is called *self-adjoint* if it is closed under complex conjugation of functional values.

Let α and β be representations of a group Y, with finite dimensional representation spaces V and W. Then $V \otimes W$ denotes the tensor product. If $\{v_i\}$ is a basis of V and $\{w_j\}$ is a basis of W, then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$. We have matrix functions defined by

$$\alpha(y)\cdot v_i = \sum_u \alpha_{iu}(y)\cdot v_u$$
$$\beta(y)\cdot w_i = \sum_t \beta_{jt}(y)\cdot w_t .$$

Now let $\alpha \otimes \beta$ be the representation of Y on $V \otimes W$ given by

$$(\alpha \otimes \beta)(y)\cdot v \otimes w = \{\alpha(y)\cdot v\} \otimes \{\beta(y)\cdot w\} .$$

$\alpha \otimes \beta$ is given on the basis $\{v_i \otimes w_j\}$ by

$$(\alpha \otimes \beta)(y)\cdot v_i \otimes w_j = \sum_{u,t} \alpha_{iu}(y)\beta_{jt}(y) v_u \otimes w_t .$$

Let α be a representation of a group Y on a finite dimensional space V. The symmetric group S_r on r letters acts on $V \otimes \cdots \otimes V$ (r times) by permutation of the factors. The *alternation* $\Lambda^r(V)$ consists of all elements $p \in V \otimes \cdots \otimes V$ (r times) such that $\sigma(p) = (\text{sign}\,\sigma)p$ for all $\sigma \in S_r$. If $\{v_1, \cdots, v_n\}$ is a basis of V, then $\Lambda^r(V)$ has a basis consisting of all $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r}$ where $1 \leqq i_1 < i_2 < \cdots < i_r \leqq n$ and

$$v_{i_1} \wedge \cdots \wedge v_{i_r} = \sum_{\sigma\in S_r} (\text{sign}\,\sigma) v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(r)}} .$$

For example, if $r = 2$ then $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1$; $v_1 \wedge \cdots \wedge v_n$ spans $\Lambda^n(V)$. The action of $\alpha \otimes \cdots \otimes \alpha$ on $V \otimes \cdots \otimes V$ commutes with the action of S_r. Thus $\alpha \otimes \cdots \otimes \alpha$ (r times) preserves the subspace $\Lambda^r(V)$ of $V \otimes \cdots \otimes V$. The representation on that subspace, the so-called rth *alternation of* α, is denoted $\Lambda^r(\alpha)$. If $\alpha(y)\cdot v_i = \sum_j \alpha_{ij}(y)\cdot v_j$, then one checks that $\Lambda^r(\alpha y)\cdot v_{(i)} = \sum_{(j)} A_{(i)(j)}(y)\cdot v_{(j)}$ where $v_{(i)} = v_{i_1} \wedge \cdots \wedge v_{i_r}$ and

$$A_{(i)(j)} = \begin{vmatrix} \alpha_{i_1 j_1} \, \alpha_{i_1 j_2} \cdots \alpha_{i_1 j_r} \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{i_r j_1} \, \alpha_{i_r j_2} \cdots \alpha_{i_r j_r} \end{vmatrix} .$$

3. The characterization.

Our main result is:

3.1. THEOREM. *Let X be a compact topological group. Then every closed translation-invariant subalgebra of $C(X)$ is self adjoint, if and only if, every representation of degree* 1 *of X has finite image.*

3.2. COROLLARY. *Let X be a compact connected group. Then every closed translation-invariant subalgebra of $C(X)$ is self adjoint, if and only if, X is an inverse limit of semisimple Lie groups.*

3.3. COROLLARY. *Let X be a compact connected Lie group. Then every closed translation-invariant subalgebra of $C(X)$ is self adjoint, if and only if, X is semisimple.*

The proof of Theorem 3.1 is based on two simple lemmas.

3.4. LEMMA. *Let λ be a finite dimensional representation of a group Y. Then λ^* is equivalent to the representation*

$$y \longrightarrow (\det\cdot\lambda(y))^{-1}\cdot\Lambda^n(\lambda(y))$$

where $n + 1$ is the degree of λ.

Proof. Let V be the representation space and choose a nonzero element $e \in \Lambda^{n+1}(V)$. If g is a linear transformation of V, then the determinant of g is given by $\Lambda^{n+1}(g)\cdot e = (\det\cdot g)e$. If $w \in \Lambda^n(V)$, we define a linear functional f_w on V by $w \wedge v = f_w(v)\cdot e$; $w \longrightarrow f_w$ is an isomorphism of $\Lambda^n(V)$ onto the dual space V^* of V. Let ν be the representation $y \longrightarrow (\det\cdot\lambda(y))^{-1}\cdot\Lambda^n(\lambda y)$. Then

$$\begin{aligned} \{f_{\nu(y)\cdot w}(\lambda(y)\cdot v)\}e &= \nu(y)w \wedge \lambda(y)v \\ &= (\det\cdot\lambda(y))^{-1}\cdot\Lambda^{n+1}(\lambda y)(w \wedge v) \\ &= w \wedge v = f_w(v)\cdot e \,. \end{aligned}$$

Thus the isomorphism $w \to f_w$ induces an equivalence of ν with λ^*.

3.5. LEMMA. *Let λ and μ be finite dimensional representations of a group Y. Then every matrix function of $\Lambda^r(\lambda)$ is a linear combination of products of r matrix functions of λ, and every matrix function of $\lambda \otimes \mu$ is a linear combination of products of matrix functions of λ with matrix functions of μ.*

Proof. Obvious from the definitions of $\Lambda^r(\lambda)$ and $\lambda \otimes \mu$.

3.6. *Proof of Theorem* 3.1. Suppose first that X has a representation π of degree 1 with infinite image. Then $\pi(X)$ is a circle group consisting of the unimodular complex numbers, and $\pi^n: x \to \pi(x)^n$ is another representation of degree 1. Let A be the subalgebra of $C(X)$ spanned by the E^{π^n}, $n \geqq 0$. Then A is a closed translation-invariant subalgebra of $C(X)$; $\pi \in A$ and $\bar{\pi} \notin A$, so A is not self adjoint.

Now suppose that every representation of degree 1 of X has finite image and let A be a closed translation-invariant subalgebra of $C(X)$. Then A is spanned by subspaces $E^\beta \subset C(X)$, say for β running over a subset $D_A \subset D$, and we must prove that A contains the adjoint of E^β for every $\beta \in D_A$. In orthonormal dual bases of the representation spaces, $\overline{\beta(x)} = {}^t\beta(x)^{-1}$ for every $x \in X$; thus E^{β^*} is the adjoint of E^β. Now we need only prove that $\beta \in D_A$ implies $\beta^* \in D_A$.

Let $\alpha \in D_A$ and define $\gamma(x) = \det \cdot \alpha(x)$. Then $\gamma = \Lambda_r(\alpha)$ where r is the degree of α, so $\gamma \in D_A$ by Lemma 3.5. $\gamma(X)$ is finite by hypothesis; thus $\gamma(X)$ is cyclic of some finite order n, consisting of the numbers $e^{2\pi ik/n}$. Now $\bar{\gamma} = \gamma^{-1} = \gamma^* = \gamma^{n-1} \in A$, so $\gamma^{-1} \in D_A$. $\Lambda^{r-1}(\alpha) \in D_A$ and D_A is closed under $\otimes$, by Lemma 3.5; thus $\gamma^{-1} \otimes \Lambda^{r-1}(\alpha) \in D_A$. Lemma 3.4 says $\alpha^* = \gamma^{-1} \otimes \Lambda^{r-1}(\alpha)$; thus $\alpha^* \in D_A$.

3.7. *Proof of corollaries.* If X is compact connected Lie group, then $X = X' \cdot T$ where X' is the derived group, T is the identity component of the center of X, and $T \cap X'$ is finite. T is a product of circle groups, X' is semisimple, and the following conditions are equivalent: (i) X is semisimple, (ii) $X = X'$, and (iii) $T = \{1\}$.

Let X be semisimple and let α be a representation of degree 1. $\alpha(X)$ is commutative, so the kernel of α contains $X' = X$. Thus $\alpha(X)$ is finite.

Let X be not semisimple. Then $S = T/T \cap X'$ is a torus of positive dimension, so there is a nontrivial representation β of S of degree 1. If $\pi: X \to X/X' = S$ is the projection, then $\alpha = \beta \cdot \pi$ is a representation of degree 1 of X and $\alpha(X) = \beta(S)$ is infinite.

Corollary 3.3 is proved.

Let X be a compact connected group. If π is a finite dimensional representation, we have the Lie group $\pi(X)$; $\{\pi(X)\}$ is an inverse system of compact Lie groups with X as inverse limit, and $\{\pi(X)\}$ is maximal for this property. A representation α is of degree 1 with infinite image, if and only if $\alpha(X)$ is a circle group. If each $\pi(X)$ is semisimple, then none is a circle group. If some $\pi(X)$, say $\alpha(X)$, is a circle group, then $\beta(X)$ is not semisimple for $\beta > \alpha$, so X cannot be an inverse limit of a subsystem of $\{\pi(X)\}$ which consists of semisimple groups. Corollary 3.2 follows.

4. Enumeration of self adjoint function algebras.

The enumeration of closed translation-invariant self adjoint function algebras on a compact group is based on the following combination of the Peter-Weyl Theorem and the Stone-Weierstrass Theorem.

4.1. THEOREM. *Let X be a compact topological group, let D be the set of equivalence classes of the irreducible representations of X, and, given $\pi \in D$, let E^π denote the space of matrix functions of π. Then there are one-to-one correspondences between*

(i) *closed self adjoint translation-invariant subalgebras A of $C(X)$ which contain the constants,*

(ii) *subsets $F \subset D$ with the properties* (a) $1_X \in F$, (b) *if $\beta \in F$ then $\beta^* \in F$ and* (c) *if $\alpha, \beta \in F$ then F contains every irreducible summand of $\alpha \otimes \beta$, and*

(iii) *closed normal subgroups $\Gamma \subset X$.*

The correspondences are given by

$$A \sim \sum_{\beta \in F} E^\beta \quad \text{and} \quad A = C(X/\Gamma)$$

where $C(X/\Gamma) \subset C(X)$ by defining $f(x) = f(x\Gamma)$ for $f \in C(X/\Gamma)$.

4.2. COROLLARY. *If X is a compact group for which every representation of degree 1 has finite image, then the closed translation-invariant subalgebras of $C(X)$ which contain the constants, are just the algebras $C(Y)$ where Y is a quotient group of X.*

Corollary 4.2 is an immediate consequence of Theorems 3.1 and 4.1.

4.3. *Proof of Theorem* 4.1. Let $F \subset D$ and define A_F to be the subspace of $C(X)$ spanned by $\{E^\beta\}_{\beta \in F}$. Then A_F is a closed translation-invariant subspace of $C(X)$. Condition (iia) says that A_F contains the constants, condition (iib) says that A_F is self adjoint, and condition (iic) says that A_F is an algebra. Thus $A \sim \sum_{\beta \in F} E^\beta$ gives a one-to-one correspondence between the classes (i) and (ii).

Let $F \subset D$ satisfy conditions (iia, b, c), and define Γ_F to be the intersection of the kernels of the elements of F. Then Γ_F is a closed normal subgroup of X. Let $p: X \to X/\Gamma_F$ be the projection and define D_F to be the set of irreducible representations of X/Γ_F. Then $D_F \subset D$ under $\sigma \to \sigma \cdot p$. Now $A_F \subset C(X/\Gamma_F)$ under $f(x_F) = f(x)$, and $A_F = C(X/\Gamma_F)$ by the Stone-Weierstrass Theorem. It follows that $D_F = F$. Thus $\Gamma \leftrightarrow \{\beta \in D : \Gamma \subset \text{Kernel. } \beta\}$ is a one-to-one correspondence between the classes (iii) and (ii), and the composition of this with our other correspondence is given by $\Gamma \leftrightarrow C(X/\Gamma)$.

4.4. *Application to compact simply connected Lie groups.* Let X be a compact simply connected Lie group. Then $X = X_1 \times \cdots \times X_r$ where the X_i are compact simply connected simple Lie groups. By Corollary 3.3 and Theorem 4.1, the closed translation-invariant subalgebras of $C(X)$ containing the constants are enumerated by the enumeration of the closed normal subgroups $\Gamma \subset X$. X has center $Z = Z_1 \times \cdots \times Z_r$ where Z_i is the center of X_i, and the closed normal subgroups of X are just the groups $\Gamma = X' \cdot Z'$ where

(i) X' is a product of zero or more of the groups X_i and

(ii) Z' is a subgroup of Z.

Thus our enumeration problem is reduced to a knowledge of the centers of the groups X_i. This information is known; it is summarized in the following table for the convenience of the reader. Here $Z_{(m)}$ denotes the cyclic group of order m and the groups listed are all the compact simple simply connected Lie groups.

For example, the number of closed translation-invariant subalgebras

TABLE 1

group	description	center
$SU(n)$ $(n > 1)$	special unitary group in n complex variables	$Z_{(n)}$
Spin (n) $(2 < n \neq 4)$	two sheeted covering group of the rotation group in n real variables	$Z_{(2)}$ (n odd), $Z_{(2)} \times Z_{(2)}$ $(n = 4k)$, $Z_{(4)}$ $(n = 4k + 2)$
$Sp(n)$	unitary group in n quaternion variables	$Z_{(2)}$
G_2	automorphism group of the Cayley algebra	$\{1\}$
F_4	elliptic group of the Cayley projective plane	$\{1\}$
E_6	. . . . .	$Z_{(3)}$
E_7	. . . . .	$Z_{(2)}$
E_8	. . . . .	$\{1\}$

of $C(X)$ properly containing the constants is one (just $C(X)$) for $X = G_2$, F_4 or E_8; it is two for $X = \text{Spin}\,(2k+1)$, $Sp(n)$, E_6 or E_7; it is three for $X = \text{Spin}\,(4k+2)$ and five for $X = \text{Spin}\,(4k)$; for $X = SU(n)$ it is the number of divisors of n, counting both 1 and n.

Reference

1. J. A. Wolf, *Self adjoint function spaces on Riemannian symmetric manifolds*, Trans. Amer. Math. Soc. **113** (1964), 299-315.

University of California, Berkeley