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MAXIMAL CONVEX FILTERS IN A LOCALLY CONVEX SPACE

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Let $E\,[\mathscr{T}]$ be a locally convex space, $\mathfrak{B}$ a saturated covering of E by bounded sets, and E' the topological dual of $E[\mathscr{T}]$. Let $\mathscr{T}_{\mathfrak{B}}$ be the topology on E' of uniform convergence on sets of $\mathfrak{B}$ and E'' the topological dual of $E'[\mathscr{T}_{\mathfrak{B}}]$. We assume E'' has the natural topology $\mathscr{T}_n$—that of uniform convergence on the equicontinuous sets of E'.

This article includes the following: (1) an intrinsic characterization for a bounded convex set B of E of the closure $\bar{B}$ of B in E''; (2) an intrinsic characterization of the closure $\bar{E}$ of E in E''; and (3) necessary and sufficient conditions that $\bar{E}$ be E''.

The spaces β. Let $\mathfrak{M}$ be the class of all closed convex neighborhoods[1] of 0 in $E[\mathscr{T}]$, and $B \in \mathfrak{B}$. A filter $\mathfrak{F}$ on B is called a *convex filter* if, for every $F \in \mathfrak{F}$, there exist $M, N \in \mathfrak{M}$ and $\chi \in E$ such that $\mathring{M} \supset N$, $F \supset (M + \chi) \cap B$, and $(N + \chi) \cap B \in \mathfrak{F}$. Clearly if $\mathfrak{F}$ and $\mathfrak{G}$ are two convex filters on B, such that every set of $\mathfrak{F}$ meets every set of $\mathfrak{G}$, then the least upper bound filter of $\mathfrak{F}$ and $\mathfrak{G}$ on B is also convex. Furthermore:

LEMMA 1. *For $M, N \in \mathfrak{M}$, if $\mathring{M} \supset N$, then there exists $K \in \mathfrak{M}$ such that $\mathring{M} \supset K \supset \mathring{K} \supset N$.*

Proof. If p and q are the distance functions of M and N, then $1/2(p + q)$ is the distance function of such a K.

THEOREM 1. *A convex filter $\mathfrak{F}$ on B is a maximal convex filter on B if and only if, for every two closed convex bodies K and L of E such that $\mathring{K} \supset L$, either $K \cap B \in \mathfrak{F}$ or $B \backslash L \in \mathfrak{F}$.*

Proof. Assume $\mathfrak{F}$ is maximal and let K and L be as above, and let $B \backslash L \notin \mathfrak{F}$. Let $x \in \mathring{L}$ and define a sequence $\{M_n\}$ in $\mathfrak{M}$ so that

$$\mathring{K} - x \supset M_1 \supset \mathring{M}_1 \supset L - x \quad \text{and} \quad \mathring{M}_n \supset M_{n+1} \supset \mathring{M}_{n+1} \supset L - x \quad (n \geqq 1).$$

Then the filter $\mathfrak{G}$ on B with base $\{(M_n + x) \cap B \mid n = 1, 2, 3, \cdots\}$ is

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[1] The notation and definitions are principally those of Gottfried Köthe, Topologische Lineare Räume I, Springer-Verlag, Berlin, 1960.

convex and $K \cap B \in \mathfrak{G} \subset \mathfrak{F}$.

Conversely let $\mathfrak{F}$ and $\mathfrak{G}$ be two convex filters on B such that $\mathfrak{F}$ is strictly weaker than $\mathfrak{G}$. Let $G \in \mathfrak{G}$, $M, N \in \mathfrak{M}$, and $x \in E$ such that $G \notin \mathfrak{F}$, $\mathring{M} \supset N$, $G \supset (M + x) \cap B$, and $(N + x) \cap B \in \mathfrak{G}$. Then neither $(M + x) \cap B$ nor $B \backslash (L + x) \in \mathfrak{F}$.

REMARKS 1. For every $x \in B, \mathfrak{V}_B(x) = \{V \cap B \subset B \mid V$ a neighborhood of x in $E\}$ is a maximal convex filter on B.

2. For a maximal convex filter $\mathfrak{F}$ on B, there is $x \in B$ such that $\mathfrak{F} = \mathfrak{V}_B(x)$ if and only if $\mathfrak{F}$ has nonempty intersection.

LEMMA 2. *Every maximal convex filter on B is a weak Cauchy filter.*

Proof. Let $\mathfrak{F}$ be a maximal convex filter on B,

$$u \in E', \ M = \{x \in E \mid |ux| \leqq 1/2\} \quad \text{and} \quad N = \{x \in E \mid |ux| \leqq 1/4\}.$$

Then $M, N \in \mathfrak{M}$ and $\mathring{M} \supset N$. Since B is weakly precompact, there exist $x_1, x_2, \cdots, x_n \in E$ such that $\bigcup_{i=1}^{n} (N + x_i) \supset B$, and so $(M + x_i) \cap B \in \mathfrak{F}$ for some $1 \leqq i \leqq n$. For $x, y \in (M + x_i) \cap B$, we have $|ux - uy| \leqq 1$.

For a maximal convex filter $\mathfrak{F}$ on B and $u \in E'$, let $\mathfrak{F}(u)$ denote the limit of the restriction of u to B according to the filter $\mathfrak{F}$.

LEMMA 3. *For every maximal convex filter $\mathfrak{F}$ on B, the mapping $u \rightarrow \mathfrak{F}(u)$ on E' is linear and $\mathscr{T}_{\mathfrak{B}}$ continuous.*

Proof. Linearity is easily proved. Also let V be the polar set of the absolutely convex hull of $2B$, $u \in V$, and $F \in \mathfrak{F}$ such that $|ux - \mathfrak{F}(u)| \leqq 1/2$ for every $x \in F$. Then, for such an x, we have $|\mathfrak{F}(u)| \leqq |\mathfrak{F}(u) - ux| + |ux| \leqq 1$.

We shall denote by $\beta = \beta_B$ the set of all maximal convex filters on B. By Lemma 3 there is a mapping π_B from β_B into E'' such that $\pi_B(\mathfrak{F})(u) = \mathfrak{F}(u)$ for every $\mathfrak{F} \in \beta_B$ and $u \in E'$.

THEOREM 2. *If either $\mathscr{T}$ is the weak topology or B is convex, then π_B is a one-to-one mapping of β_B onto the $\mathscr{T}_n$-closure $\bar{B}$ of B in E''.*

Proof. For $\mathfrak{F} \in \beta_B$, $\pi_B(\mathfrak{F})$ is in the weak closure of B in E''. For given $u_1, \cdots, u_n \in E'$ and $\varepsilon > 0$, let $F_1, \cdots, F_n \in \mathfrak{F}$ such that $|u_i x - \mathfrak{F}(u_i)| \leqq \varepsilon$ $(1 \leqq i \leqq n)$ and $x \in \bigcap_{i=1}^n F_i$. Then $|\mathfrak{F}(u_i) - u_i x| \leqq \varepsilon$, $(1 \leqq i \leqq n)$.

Also, if B is convex, $\pi_B(\mathfrak{F})$ is in the $\mathscr{T}_n$-closure $\bar{B}$ of B in E''. Suppose the contrary. Then there is a continuous real linear functional w on E'' and a real number r such that $w(\pi_B(\mathfrak{F})) < r$ and $wz > r$ for every $z \in \bar{B}$.

Assume first that E is a real vector space. Let u be the restriction of w to E', so $u \in E$. Let $F \in \mathfrak{F}$ such that $|ux - \mathfrak{F}(u)| < r - w(\pi_B(\mathfrak{F}))$ for every $x \in F$. Then, for such an x, we have $wx = ux - \mathfrak{F}(u) + \mathfrak{F}(u) < r$. But $x \in B$.

Now let E be a complex vector space. Then there is a complex linear functional v on E'' such that $w = \Re v$. Let u be the restriction of v to E and $F \in \mathfrak{F}$ such that $|ux - \mathfrak{F}(u)| \leqq r - w(\pi_B(\mathfrak{F}))$ for every $x \in F$. Then for such an x we have $wx = \Re(vx) = \Re(ux - \mathfrak{F}(u)) + \Re(\mathfrak{F}(u)) < r$. Again, we have a contradiction.

Thus $\pi_B(\beta_B) \subset \bar{B}$ if $\mathscr{T}$ is the weak topology or B is convex.
On the other hand, if $z \in \bar{B}$, then:

$$\mathfrak{V}_B(z) = \{V \cap B \subset B \mid V \text{ a neighborhood of } z \text{ in } E''[\mathscr{T}_n]\} \in \beta_B$$

and $\pi_B(\mathfrak{V}_B(z)) = z$. Let V be a neighborhood of z in $E''[\mathscr{T}_n]$, and let U and W be closed convex neighborhoods of 0 in $E''[\mathscr{T}_n]$ such that $\mathring{U} \supset W$ and $U + U \subset V - z$. Let $\chi \in (U + z) \cap (-\mathring{W} + z) \cap B$, $M = U \cap E$, and $N = V \cap E$. Then $M, N \in \mathfrak{M}$ and $\mathring{M} \supset N$, $V \supset (M + \chi) \cap B$, and $(N + \chi) \cap B = (W + \chi) \cap B \in \mathfrak{V}_B(z)$. Thus $\mathfrak{V}_B(z)$ is convex.

Let K and L be closed convex bodies of E such that $\mathring{K} \supset L$. Let $x \in \mathring{L}$, $M = K - x$, and $N = L - x$. Either $z \in$ interior $M^{\circ\circ} + x$——in which case $K \cap B = (M + x) \cap B = (M^{\circ\circ} + x) \cap B \in \mathfrak{V}_B(z)$——or $z \notin N^{\circ\circ} + x$——in which case $E'' \backslash (N^{\circ\circ} + x)$ is a neighborhood of z in E'' and so $B \backslash L = [E'' \backslash (N^{\circ\circ} + x)] \cap B \in \mathfrak{V}_B(z)$. Thus $\mathfrak{V}_B(z) \in \beta_B$.

Finally, let $u \in E'$, $\varepsilon > 0$, and $F \in \mathfrak{V}_B(z)$ such that $|ux - \mathfrak{V}_B(z)(u)| \leqq \varepsilon/2$ for every $x \in F$. Let $V = \{w \in E'' \mid |wu - zu| \leqq \varepsilon/2\}$. Then, for $x \in F \cap V$, we have $|\mathfrak{V}_B(z)(u) - zu| \leqq |\mathfrak{V}_B(z)(u) - ux| + |ux - zu| \leqq \varepsilon$. Therefore, $\pi_B(\mathfrak{V}_B(z))(u) = zu$ for $u \in E''$, and so $\pi_B(\mathfrak{V}_B(z)) = z$.

REMARK. Thus $\pi_B(\mathfrak{V}_B(z)) = z$ for $z \in \bar{B}$ and $\mathfrak{F} = \mathfrak{V}_B(\pi_B(\mathfrak{F}))$ for $\mathfrak{F} \in \beta_B$.

COROLLARY 1. *If either $\mathscr{T}$ is the weak topology or B is convex, then every maximal convex filter on B is a $\mathscr{T}$-Cauchy filter.*

COROLLARY 2. *If either $\mathscr{T}$ is the weak topology or B is convex,* *then for every* $\mathfrak{F} \in \beta_B$ *and* $M \in \mathfrak{M}$, *there exist* $x \in B$ *such that* $(M + x) \cap B \in \mathfrak{F}$.

Proof. Let $F \in \mathfrak{F}$ such that $F - F \subset M$ and $x \in F$.
For $M \in \mathfrak{M}$ and $x \in B$ we define:

$$\nu_B(M, x) = \{\mathfrak{F} \in \beta_B \mid (\mathring{M} + x) \cap B \in \mathfrak{F}\}$$
$$\mu_B(M, x) = \{\mathfrak{F} \in \beta_B \mid \pi_B(\mathfrak{F}) \in \text{interior } M^{\circ\circ} + x\} .$$

For $M, N \in \mathfrak{M}$ and $x, y \in B$, if $z \in (\mathring{M} + x) \cap (\mathring{N} + y) \cap B$ and $K = (M + x - z) \cap (N + y - z)$, then $\nu_B(M, x) \cap \nu_B(N, y) = \nu_B(K, z)$ and $\mu_B(M, x) \cap \mu_B(N, y) = \mu_B(K, z)$. Hence the class of all sets of the form $\nu_B(M, x)$ and the class of all sets of the form $\mu_B(M, x)$ (for $M \in \mathfrak{M}$ and $x \in B$) form bases of topologies, called the ν- and μ-topologies respectively, on β_B.

THEOREM 3. *If* $\pi_B(\beta_B) \subset \bar{B}$ *(in particular if either* $\mathscr{T}$ *is the weak topology or* B *is convex), then* ν- *and* μ-*topologies coincide and* π_B *is a homeomorphism of* β_B *onto* $\bar{B}$.

Proof. If $\pi_B(\beta_B) \subset \bar{B}$, then, for $M \in \mathfrak{M}$ and $x \in B$, we have $\mu_B(M, x) \subset \nu_B(M, x)$, and so the identity mapping of β_B with the μ-topology onto β_B with the ν-topology is continuous.

Also π_B from β_B with the ν-topology onto $\bar{B}$ is continuous. Let $\mathfrak{F} \in \beta_B$ and V a neighborhood of $\pi_B(\mathfrak{F})$ in $E''[\mathscr{T}_n]$. Let U be a closed convex neighborhood of 0 in E'' such that $U + U \subset V - \pi_B(\mathfrak{F})$, $M = U \cap E$, and $x \in (\mathring{U} + \pi_B(\mathfrak{F})) \cap B$. Then $(\mathring{M} + x) \cap B \in \mathfrak{V}_B(\pi_B(\mathfrak{F})) = \mathfrak{F}$, and so $\mathfrak{F} \in \nu_B(M, x)$. Also if $\mathfrak{G} \in \nu_B(M, x)$, there is a neighborhood W of $\pi_B(\mathfrak{G})$ such that $W \cap B = (M + x) \cap B = (U + x) \cap B$ so

$$\pi_B(\mathfrak{G}) \in \overline{W \cap B} \subset U + x \subset U + U + \pi_B(\mathfrak{F}) \subset V .$$

Finally π_B^{-1} from $\bar{B}$ onto β_B with the μ-topology is continuous by the definition of the sets μ.

COROLLARY 1. *If either* $\mathscr{T}$ *is the weak topology or* B *is convex, then* B *is closed in* $E''[\mathscr{T}_n]$ *if and only if every maximal convex filter on* B *has nonempty intersection.*

COROLLARY 2. *B is weakly compact if and only if every maximal weakly convex filter on B has nonempty intersection.*

2. The space η. Let $\mathfrak{A}$ denote the class of all convex sets of $\mathfrak{B}$ and $\alpha = \bigcup_{B \in \mathfrak{A}} \beta_B$ the topological union of the spaces β_B. Let π be

the continuous function from α into $E''[\mathscr{T}_n]$ defined by $\pi(\mathfrak{F}) = \pi_B(\mathfrak{F})$ if $\mathfrak{F} \in \beta_B$. For $A, B \in \mathfrak{A}$ such that $A \subset B$, define a mapping g_{BA} from β_A into β_B by $g_{BA}(\mathfrak{F}) = \mathfrak{V}_B(\pi_A(\mathfrak{F}))$ (for $\mathfrak{F} \in \beta_A$). Then $g_{BA} = \pi_B^{-1}\pi_A$ and consequently is a homeomorphism of β_A into β_B. Also, if $A \subset B \subset C$, then $g_{CA} = g_{CB}\, g_{BA}$.

THEOREM 4. *Let $A, B \in \mathfrak{A}$ such that $A \subset B$, and let $\mathfrak{F} \in \beta_A$ and $\mathfrak{G} \in \beta_B$. The following three conditions are equivalent;*

(a) $\mathfrak{G} = g_{BA}(\mathfrak{F})$;
(b) $\pi(\mathfrak{F}) = \pi(\mathfrak{G})$;
(c) *Every set of $\mathfrak{G}$ contains a set of $\mathfrak{F}$.*

Proof. $\mathfrak{F} = \mathfrak{V}_A(\pi_B(\mathfrak{F}))$, $\mathfrak{G} = \mathfrak{V}_B(\pi_B(\mathfrak{G}))$, and $g_{BA}(\mathfrak{F}) = \mathfrak{V}_B(\pi_B(\mathfrak{F}))$. Hence (a) and (b) are equivalent. Also (b) implies (c): Given $G \in \mathfrak{G}$ there is a neighborhood V of $\pi(\mathfrak{G}) = \pi(\mathfrak{F})$ such that $G = V \cap B \supset V \cap A \in \mathfrak{F}$. Also (c) implies (b): If $\pi(\mathfrak{F}) \neq \pi(\mathfrak{G})$, then $\pi(\mathfrak{F})$ and $\pi(\mathfrak{G})$ have disjoint neighborhoods V and W in E'', and so $W \cap A$ is a set of $\mathfrak{G}$ containing no set of $\mathfrak{F}$.

COROLLARY. *Let A and $B \in \mathfrak{A}$, $\mathfrak{F} \in \beta_A$, and $\mathfrak{G} \in \beta_B$. The following three conditions are equivalent:*

(a) $\pi(\mathfrak{F}) = \pi(\mathfrak{G})$.
(b) *There exists $C \in \mathfrak{A}$ such that $C \supset A \cup B$ and $g_{CA}(\mathfrak{F}) = g_{CB}(\mathfrak{G})$.*
(c) *There exists $C \in \mathfrak{A}$ and $\mathfrak{H} \in \beta_C$ such that $C \supset A \cup B$ and every set of $\mathfrak{H}$ contains a set of $\mathfrak{F}$ and a set of $\mathfrak{G}$.*

Now let R be the equivalence relation $\pi(\mathfrak{F}) = \pi(\mathfrak{G})$ on α, η the quotient space α/R, ρ the canonical mapping of α onto η, and σ the mapping from η into E'' such that $\pi = \sigma\rho$.

THEOREM 5. *σ is a homeomorphism of η onto the $\mathscr{T}_n$-closure $\bar{E}$ of E in E''.*

Proof. We need only prove $\sigma(\eta) = \pi(\alpha) \supset \bar{E}$. Consider the dual system $\langle E', \bar{E}\rangle$. Since every $u \in E'$ is uniformly continuous on E, the topology induced on $\bar{E}$ by $\mathscr{T}_n$ is admissible for this dual system. For $z \in \bar{E}$, there is a closed absolutely convex set $B \in \mathfrak{B}$ such that $|zu| \leqq 1$ for every $u \in B^\circ$. Hence, $z \in B^{\circ\circ}$ = the closure of B in any admissible topology = the $\mathscr{T}_n$-closure $\bar{B}$ of B.

For $B \in \mathfrak{A}$, the weakest topology on β_B for which every function of the form $\mathfrak{F} \to \mathfrak{F}(u)$ (for $u \in E'$) is continuous will be called the *weak topology* of β_B. Clearly β_B in the weak topology is homeomorphic with $\bar{B}$ in the topology induced on $\bar{B}$ by the weak-star topology of E''.

THEOREM 6. *The following three conditions are equivalent:*

(a) $\bar{E} = E''$;

(b) $\bar{B}$ *is weak-star compact for every* $B \in \mathfrak{A}$;

(c) β_B *is weakly compact for every* $B \in \mathfrak{A}$.

Proof. Clearly (b) and (c) are equivalent. Also (a) implies (b); by the Alaoglu—Bourbaki theorem, for $B \in \mathfrak{A}$, the weak-star closure of B in $E'' = \bar{E}$ is weak-star compact; but since $\mathscr{T}_n$ is an admissible topology for the dual systm $\langle E', \bar{E}\rangle$, this weak-star closure is $\bar{B}$. Finally (b) implies (a): regarding $\mathfrak{B}$ as a total class of bounded subsets of $\bar{E}$, by the Mackey-Arens theorem $\mathscr{T}_{\mathfrak{B}}$ is an admissible topology for the dual system $\langle E', \bar{E}\rangle$, and so $E'' = \bar{E}$.

THEOREM 7. *For* $B \in \mathfrak{A}$, β_B *is weakly compact if and only if for every maximal weakly-convex filter* $\mathfrak{F}$ *on* B, *there is a maximal* $\mathscr{T}$-*convex filter on* B *which is stronger than* $\mathfrak{F}$.

Proof. Let β_B^w be the space of all maximal weakly convex filters on B and π_B^w the homeomorphism of β_B^w into E'' with the weak-star topology. In general $B \subset \pi_B(\beta_B) = \bar{B} \subset$ weak-star closure of $B = \pi_B^w(\beta_B^w)$.

If β_B is weakly compact, then $\pi_B^w(\beta_B^w) = \pi_B(\beta_B) = \bar{B}$. So, for $\mathfrak{F} \in \beta_B^w$, $\pi_B^w(\mathfrak{F}) \in \bar{B}$ and hence $\mathfrak{B}_B(\pi_B^w(\mathfrak{F})) \in \beta_B$ is stronger than $\mathfrak{F}$.

Conversely, let $\mathfrak{F} \in \beta_B^w$ and $\mathfrak{G} \in \beta_B$ stronger than $\mathfrak{F}$. Then $\pi_B^w(\mathfrak{F}) = \pi_B(\mathfrak{G})$, and so $\pi_B^w(\beta_B^w) \subset \pi_B(\beta_B)$.

COROLLARY. $\bar{E} = E''$ *if and only if, for every* $B \in \mathfrak{A}$ *and every maximal weakly-convex filter* $\mathfrak{F}$ *on* B, *there is a* $\mathscr{T}$-*convex filter on* B *stronger than* $\mathfrak{F}$.

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