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A GENERALISATION OF W^*-ALGEBRAS

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Using the theory of double centralisers due to B. E. Johnson, we define a QW^*-algebra as being a B^*-algebra, A, such that the algebra of double centralisers of each closed *-subalgebra B is contained in a suitable related closed *-subalgebra B_{00}.

After obtaining explicit descriptions of the algebras of double centralisers of commutative and noncommutative B^*-algebras, we prove that in the general noncommutative case a W^*-algebra is a QW^*-algebra, and a QW^*-algebra is an AW^*-algebra, while in the commutative case the QW^* and AW^* conditions are equivalent.

We prove that if A is QW^* then so are its centre, any maximal commutative *-subalgebra, and any subalgebra of the form eAe for e a projection in A.

We shall be concerned with centraliser theory, for the basic details of which reference may be made to Johnson [2], [3].

I should like to take this opportunity of expressing my sincere gratitude to Dr. J. H. Williamson, my research supervisor, for his advice and encouragement.

DEFINITION 1. A *left centraliser* $\mathscr{T}$ of the algebra A is a linear map $\mathscr{T}$ of A into itself such that $\mathscr{T}(xy) = (\mathscr{T}x)y$ for all $x, y \in A$.

A *right centraliser* $\mathscr{S}$ is a linear operator on A such that $\mathscr{S}(xy) = x(\mathscr{S}y)$ for all $x, y \in A$.

A *double centraliser* (the concept is due to Johnson [2]) is a pair of linear operators $(\mathscr{T}, \mathscr{S})$ such that $x\cdot(\mathscr{T}y) = (\mathscr{S}x)\cdot y$ for all $x, y \in A$.

The set of all double centralisers on A is denoted by $Q(A)$.

We will assume throughout that $xA = 0$ or $Ax = 0$ only holds for $x = 0$. We note that this holds for B^*-algebras since $xA = 0 \Rightarrow xx^* = 0 \Rightarrow x = 0$, and $Ax = 0 \Rightarrow x^*x = 0 \Rightarrow x = 0$.

It is not difficult to see that defining $(\mathscr{T}_x, \mathscr{S}_x) \in Q(A)$ for $x \in A$ by $\mathscr{T}_x(y) = xy$, $\mathscr{S}_x(y) = yx$, and algebraic operations in $Q(A)$ by

$$\lambda_1(\mathscr{T}_1, \mathscr{S}_1) + \lambda_2(\mathscr{T}_2, \mathscr{S}_2) = (\lambda_1\mathscr{T}_1 + \lambda_2\mathscr{T}_2, \lambda_1\mathscr{S}_1 + \lambda_2\mathscr{S}_2)$$
$$(\mathscr{T}_1, \mathscr{S}_1)\cdot(\mathscr{T}_2, \mathscr{S}_2) = (\mathscr{T}_1\mathscr{T}_2, \mathscr{S}_2\mathscr{S}_1)$$

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we have A embedded as a subalgebra of $Q(A)$, which is an algebra with identity. $A = Q(A)$ if and only if A has an identity. Also, for $(\mathscr{T}, \mathscr{S}) \in Q(A)$, $\mathscr{T}$ is a left centraliser and $\mathscr{S}$ is a right centraliser, and either of $\mathscr{T}, \mathscr{S}$ determines the other uniquely.

If A is commutative, the notions of right, left and double centraliser coincide, and for $(\mathscr{T}, \mathscr{S}) \in Q(A)$ we have $\mathscr{T} = \mathscr{S}$.

PROPOSITION 1. If A is a Banach algebra then all double centralisers are continuous.

Proof. Suppose $(\mathscr{T}, \mathscr{S}) \in Q(A)$ and say $x_n \to x$, $\mathscr{T}x_n \to y$. Then

$$\begin{aligned} z\cdot(\mathscr{T}x_n) &= (\mathscr{S}z)\cdot x_n \\ \to z\cdot y \qquad &\to (\mathscr{S}z)\cdot x = z\cdot(\mathscr{T}x) . \end{aligned}$$

So $z(y - \mathscr{T}x) = 0$ for all $z \in A$ i.e. $A(y - \mathscr{T}x) = 0$ and so $y = \mathscr{T}x$. Therefore $\mathscr{T}$ is a closed operator on the Banach space A, hence by the Closed Graph Theorem, $\mathscr{T}$ is continuous. Likewise so is $\mathscr{S}$.

We are particularly interested in C^*-algebras and in both the commutative and noncommutative cases explicit descriptions of their centraliser algebras may be given.

By the Gelfand Representation Theorem a commutative B^*-algebra is isometrically isomorphic to the space $C_0(Z)$ of all continuous functions vanishing at infinity on its carrier space, Z, a locally compact Hausdorff space.

PROPOSITION 2. For a locally compact Hausdorff space Z we have $QC_0(Z) = C(Z)$, the space of all bounded continuous functions on Z.

Proof. Certainly any $h \in C(Z)$ defines an element $\mathscr{T}_h$ of $QC_0(Z)$ by $\mathscr{T}_h f = h\cdot f$ for $f \in C_0(Z)$, for

$$f \in C_0(Z),\, h \in C(Z) \Rightarrow hf \in C_0(Z)$$

and

$$h(fg) = (hf)g .$$

We clearly have $\| \mathscr{T}_h \| \leqq \| h \|_\infty$. Suppose conversely we are given a centraliser $\mathscr{T}$ on $C_0(Z)$. Then for $f, g \in C_0(Z)$ we have

$$(\mathscr{T}f)g = \mathscr{T}(fg) = \mathscr{T}(gf) = (\mathscr{T}g)f$$

so for $z \in Z$ taking any $f \in C_0(Z)$ such that $f(z) \neq 0$ and defining $h(z) = \mathscr{T}f(z)/f(z)$ we have $h(z)$ well defined independently of f.

Being a quotient of continuous functions, h is continuous at z, for each $z \in Z$. And for any $g \in C_0(Z)$,

$$\mathscr{T} g(z) = \frac{\mathscr{T} f(z)}{f(z)} g(z) = h(z)g(z)$$

so

$$\mathscr{T} g = hg = \mathscr{T}_h g .$$

Now by Proposition 1, $\mathscr{T}$ is a bounded operator, so taking $f \in C_0(Z)$ such that $0 \leqq f \leqq 1$ and $f(z) = 1$ we have $h(z) = \mathscr{T} f(z)$ and $|\mathscr{T} f(z)| \leqq \|\mathscr{T} f\|_\infty \leqq \|\mathscr{T}\| \|f\|_\infty = \|\mathscr{T}\|$ so $\|h\|_\infty \leqq \|\mathscr{T}\|$ and we see $h \in C(Z)$.

Hence all $\mathscr{T}$ are of the form $\mathscr{T}_h$ and $\|\mathscr{T}\| = \|h\|_\infty$. So $QC_0(Z) = C(Z)$.

PROPOSITION 3. If A is a C^*-algebra over H, principal identity E, then $Q(A)$ is isometrically isomorphic to

$$\{T \in \mathscr{B}(H) \colon T = ETE,\ TA \cup AT \subset A\} .$$

Proof. Recall that the principal identity of a C^*-algebra A is defined to be the orthogonal projection of H onto $M = H \ominus N$ where $N = \{\xi \in H \colon A\xi = 0\}$. Equivalently M is the closure of

$$M_1 = \{T\xi : T \in A, \xi \in H\} .$$

Suppose given $(\mathscr{T}, \mathscr{S}) \in Q(A)$, then $\mathscr{T}$ is a bounded left centraliser.

Since A is a C^*-algebra it has an approximate identity (Segal [6]), $(Z_\lambda)_{\lambda \in \Lambda}$ say, so $\|Z_\lambda\| = 1$, and $SZ_\lambda \to S$, $Z_\lambda S \to S$ for each $S \in A$. So $\mathscr{T}(Z_\lambda S) \to \mathscr{T}(S)$. But $\mathscr{T}(Z_\lambda S) = \mathscr{T}(Z_\lambda)S = T_\lambda S$ where $T_\lambda = \mathscr{T}(Z_\lambda)$, so $\mathscr{T}(S) = \lim_\lambda T_\lambda S$ and $\|T_\lambda\| \leqq \|\mathscr{T}\| \|Z_\lambda\| = \|\mathscr{T}\|$. For $\xi \in M_1$, $\xi = S\eta$ some $S \in A, \eta \in H$ so $\mathscr{T}(S)\eta = \lim_\lambda T_\lambda S\eta = \lim_\lambda T_\lambda \xi$. Define $T\xi = \lim_\lambda T_\lambda \xi = \mathscr{T}(S)\eta$, then T maps M_1 into M and $\|T\xi\| \leqq \|\mathscr{T}\| \|\xi\|$ so $\|T\| \leqq \|\mathscr{T}\|$.

So extend T to a map of M into M and define $T = 0$ on $H \ominus M$, so we have $T = ETE$ and $\mathscr{T}(S)\eta = \lim_\lambda T_\lambda S\eta = TS\eta$. Therefore $\mathscr{T}(S) = TS$ and $\|\mathscr{T}\| \leqq \|T\|$. So $\|\mathscr{T}\| = \|T\|$.

We have

$$\begin{aligned} (\mathscr{S} S)Z_\lambda &= S(\mathscr{T} Z_\lambda) = STZ_\lambda \\ \to \mathscr{S} S \quad & \qquad\qquad \to ST . \end{aligned}$$

So $\mathscr{S}(S) = ST$ for all $S \in A$, and as for $\mathscr{T}$, $\|\mathscr{S}\| = \|T\|$. Since $TS, ST \in A$ for all $S \in A$ we have $TA \cup AT \subset A$. Conversely given any

T such that $T = ETE$ and $TA \cup AT \subset A$, the maps $S \to TS$, $S \to ST$ both map A into itself and define a double centraliser of A. Hence result.

Denote the set $\{T \in \mathscr{B}(H)\colon T = ETE,\ TA \cup AT \subset A\}$ by $I(A)$, the idealiser of A in $E \cdot \mathscr{B}(H) \cdot E$.

Now let us suppose that B is a closed *-subalgebra of the B^*-algebra A. We define $B_0 = \{x \in A\colon Bx = xB = 0\}$ and $B_{00} = (B_0)_0$. Then B_{00} is a closed *-subalgebra of A containing B. Should it be necessary to make explicit mention of the algebra A we will write $B_0(A)$, etc.

Suppose two elements x_1, x_2 of B_{00} give the same double centraliser on B, so $x_1 y = x_2 y$ and $y x_1 = y x_2$ for all $y \in B$. Then $(x_1 - x_2)B = B(x_1 - x_2) = 0$ so $x_1 - x_2 \in B_0$. But $(x_1 - x_2)^* \in B_{00}$ so we have

$$(x_1 - x_2)^*(x_1 - x_2) = 0$$

and hence $x_1 - x_2 = 0$. So $x_1 = x_2$.

DEFINITION 2. A B^*-algebra A is said to be a *QW^*-algebra* if for each closed *-subalgebra B of A all double centralisers of B are given by elements of B_{00}. We see that for each double centraliser the corresponding element of B_{00} is unique, and so we may briefly say that A is QW^* if and only if $Q(B) \subset B_{00}$ for all closed *-subalgebras B.

We recall the definition of an AW^*-algebra (Kaplansky [4]).

DEFINITION 3. A B^*-algebra A is said to be an *AW^*-algebra* if

(i) every set of orthogonal projections in A has a least upper bound in A.

(ii) every maximal commutative *-subalgebra B of A is generated by its projections.

We also recall that a W^*-algebra is a C^*-algebra, over H say, which is closed in the weak operator topology defined by seminorms $\| T \|_{\xi,\eta} = |\langle T\xi, \eta \rangle|$ for $\xi, \eta \in H$. Denote weak closure by ${}^{-w}$.

PROPOSITION 4. For A a C^*-algebra, $I(A) \subset A^{-w}$.

Proof. By von Neumann's Double Commutant Theorem, $A^{-w} = \{T \in \mathscr{B}(H)\colon T = ETE,\ T \in A''\}$ where as usual A'' denotes the double commutant of A.

Suppose $T \in I(A)$, $S \in A'$, $R \in A$, then certainly $T = ETE$ and $(ST - TS)R = S(TR) - T(SR) = TRS - TRS = 0$. So $(ST - TS)E = 0$

and therefore $ST = TSE$. Since $T^* \in I(A)$, $S^* \in A'$ we have $S^*T^* = T^*S^*E$ so $TS = EST$. Thus $TS = EST = ETSE = TSE = ST$ and so $T \in A''$. Hence $I(A) \subset A''$.

THEOREM 1. *For a B^*-algebra A, $W^* \Rightarrow QW^* \Rightarrow AW^*$.*

If A is commutative, carrier space Z, then A is $QW^* \Leftrightarrow A$ is $AW^* \Leftrightarrow Z$ is extremally disconnected.

Proof. If A is a W^*-algebra and B is a closed *-subalgebra of A with principal identity E, then since A is W^* we note $E \in A$, and by Proposition 4, $I(B) \subset B^{-w} \subset A^{-w} = A$. Also we easily see that $B_0 = (I-E)A(I-E)$ so $B_{00} = EAE$. Thus $Q(B) \subset B_{00}$ by Proposition 3 and hence A is QW^*.

Suppose now that A is a commutative B^*-algebra, carrier space Z, so by the Gelfand Representation Theorem A is isometrically isomorphic to $C_0(Z)$.

It is well known that A is AW^* if and only if Z is an extremally disconnected compact Hausdorff space.

Suppose A is QW^*, then taking $B = A$ we see that A has an identity, so Z is compact Hausdorff.

Let U be any open dense subset of Z.

Then taking $B = \{f \in C(Z): f = 0 \text{ on } Z\backslash U\} = C_0(U)$, B is a closed *-ideal in A so $Q(B) = C(U) \subset A$.

So any continuous function f on U is extendible to Z. Therefore Z is extremally disconnected (see Gillman and Jerison [1], p. 96).

Now suppose that Z is an extremally disconnected compact Hausdorff space, and suppose B is a closed *-subalgebra of $A = C(Z)$.

Let $(Z_\lambda)_{\lambda \in \Lambda}$ be the sets of constancy of B (see Rickart [5], Ch. 3, § 2), then these form an upper semicontinuous decomposition of Z, so the space of these sets, Z' say, is a compact Hausdorff space and B may be considered as a space of continuous functions on Z'.

B is self-adjoint and separates points of Z', so by the Stone-Weierstrass Theorem, *either* B consists of all continuous functions on Z', in which case B has an identity so $Q(B) = B$, *or* B consists of all continuous functions on Z' vanishing at some point Z_0 of Z'. So $Q(B) =$ all continuous functions on $Z'\backslash\{Z_0\}$.

Given any function on $Z'\backslash\{Z_0\}$ it corresponds to a function f on $Z\backslash Z_0 = Y$ say.

Y is open, so $\bar{Y}$ is a compact open subset of Z, and therefore $\bar{Y}$ is extremally disconnected (Gillman and Jerison [1], p. 23). So there exists an extension of f to $\bar{Y}$, and defining $f = 0$ on $Z\backslash\bar{Y}$ we extend f to a continuons function on Z.

Now since

$$B_0 = \{g \in C(Z) : g = 0 \text{ on } Y\}$$
$$= \{g \in C(Z) : g = 0 \text{ on } \bar{Y}\}$$

and

$$B_{00} = \{g \in C(Z) : g = 0 \text{ on } Z \backslash \bar{Y}\}$$

we therefore have $Q(B) \subset B_{00}$.

So A is QW^* and we have proved our theorem for A commutative.

Now let us return to the general case and suppose A to be QW^*.

(i) Suppose (e_α) is a set of orthogonal projections in A (so $\alpha \neq \beta \Rightarrow e_\alpha e_\beta = 0$).

Let $B =$ closed $*$-subalgebra of A generated by the e_α's.
$=$ closed linear hull of the e_α's.

Now there exists a unique $e \in B_{00}$ such that $ex = xe = x$ for all $x \in B$ and e^*, $e^2 \in B_{00}$ with

$$e^*x = xe^* = x$$
$$e^2x = xe^2 = x \quad \text{for all } x \in B .$$

So $e^2 = e^* = e$ and thus e is a projection.

Also $ee_\alpha = e_\alpha e = e_\alpha$ all α, so $e \geqq e_\alpha$ all α.

Now suppose f is a projection in A such that $f \geqq$ all e_α. Then $fe_\alpha = e_\alpha f = e_\alpha$ all α, so since all $x \in B$ are limits of linear combinations of the e_α's, we have $fx = xf = x$ for all $x \in B$.

Now

$$y \in B_0 \Rightarrow yfx = yx = 0$$
$$xyf = 0 \qquad \text{all } x \in B \Rightarrow yf \in B_0$$

so for all $y \in B_0$,

$$fey = f0 = 0$$
$$yfe = 0 \qquad \text{thus } fe \in B_{00} .$$

But

$$fex = fx = x$$
$$xfe = xe = x$$

all $x \in B$, so since e is unique, $e = fe$.

So $ef = fe = e$ and $e \leqq f$.

Hence e is a least upper bound in A for the e_α's.

(ii) Suppose B is a maximal commutative $*$-subalgebra of A. Then by Proposition 5 below, B is QW^*, thus since B is commutative it follows from the above result that B is AW^*, and is a maximal commutative $*$-subalgebra of itself and therefore generated by its projections.

Thus we have both conditions for A to be AW^*.

The obvious question of interest arising from this theorem is whether or not the QW^* and the AW^* conditions are equivalent in the noncommutative case, but so far we have not been able to settle this problem.

We now prove some results for QW^*-algebras similar to those holding for W^*- and AW^*-algebras. We are indebted to the referee for pointing out case (iv) of Proposition 5 as generalising cases (i) and (ii).

PROPOSITION 5. If A is a QW^*-algebra then so also are the following closed $*$-subalgebras of A:

(i) the centre Z of A,

(ii) any maximal commutative $*$-subalgebra of A,

(iii) the subalgebra eAe for any projection e in A,

(iv) S'' for any subset S of A such that $S^* = S$, where S'' is the double commutant of S in A.

Proof. We first prove (iv) from which (i) and (ii) follow immediately.

(iv) Suppose B is a closed $*$-subalgebra of S''.

Since A is QW^* any double centraliser on B is given by some $x \in B_{00}(A)$.

To prove $x \in B_{00}(S'')$, since $B_0(S'') \subset B_0(A)$, we need only show $x \in S''$.

Let $y \in S'$, $z \in B \subset S''$, then

$$(xy - yx)z = x(yz) - y(xz) = xzy - xzy = 0$$
$$z(xy - yx) = (zx)y - (zy)x = yzx - yzx = 0$$

so $xy - yx \in B_0(A)$.

Now

$$u \in B_0(A) \Rightarrow yuz = 0$$
$$zyu = yzu = 0 \quad \text{all } z \in B \Rightarrow yu \in B_0(A)\ ,$$

and likewise $u \in B_0(A) \Rightarrow uy \in B_0(A)$.

Therefore since $x \in B_{00}(A)$, $xyu = 0$ and $uxy = 0$ for all $u \in B_0(A)$, so $xy \in B_{00}(A)$, and likewise $yx \in B_{00}(A)$. So $(xy - yx)^* \in B_{00}(A)$ and hence $xy - yx = 0$ for all $y \in S'$. Thus $x \in S''$ and the result follows.

(i) We have $Z = A'$, $Z' = A$ so $Z = Z''$, and clearly $Z = Z^*$, so the result follows from (iv).

(ii) Suppose C is a maximal commutative $*$-subalgebra of A, then by maximality C is closed and $C' = C$, so $C = C''$ and the result follows from (iv).

(iii) Let B be a closed $*$-subalgebra of eAe, then since A is QW^*

any double centraliser on B is given by some $x \in B_{00}(A)$. Since $B \subset eAe$ we have $y \in B_0(A) \Rightarrow ey, ye \in B_0(A)$ and $x \in B_{00}(A) \Rightarrow exe \in B_{00}(A)$.

But for $z \in A$ we have

$$zexe = (zx)e = zx$$
$$exez = e(xz) = xz$$

so by the uniqueness of x in $B_{00}(A)$ we have $x = exe$.

Thus $x \in eAe$ and so $x \in B_{00}(eAe)$. Hence eAe is QW^*.

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