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RATIO LIMIT THEOREMS FOR MARKOV CHAINS

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In an irreducible, recurrent, Markov chain, with integer states, let $N_n(A)$ be the occupation time of A by time n, where A is a finite set of states. Our principal concern in this paper is to investigate various "ratio limit theorem" for $P_x(N_n(A) = k)$. Criteria are given for various ratio limits to exist. The limits (when they exist) are shown to be expressible in terms of an integral over the set of states E completed with its dual recurrent boundary $\hat{B}$. Applications are given to several specific Markov chains.

Throughout this paper $\{X_n\}$ will be an irreducible, recurrent Markov chain with states in a denumerable set E and with nth step transition probabilities P^n_{xy}. For convenience we may take E to be the integers. For discrete time Markov processes, ratio limits for the quantities $P_x(N_n(A) = k)$ were first investigated by Kac [4] for certain special cases of partial sums of independent random variables with a common distribution. Recently these quantities have been intensively studied by Kesten and Spitzer [9] for the irreducible chains formed by the successive partial sums of independent, identically distributed, integer-lattice-valued random vectors in r dimensions. They show the remarkable fact that in *all* such chains, for any two states x, y, any integer $k \geqq 0$, and any finite nonempty set A, the limits

$$\lim_{n \to \infty} \frac{P_x(N_n(A) = k)}{P_y(N_n(\{y\}) = 0)} \tag{1.1}$$

exist and they explicitly find their values.

Now, in general, limits (1.1) exist in very few recurrent chains and we shall have to be content with much weaker types of ratio limits if we want results of any generality. The weakest form of these ratio limits asserts that for any two states x, y, and any finite nonempty set A,

$$\lim_{t \to 1^-} \frac{\sum_{n=0}^{\infty} t^n P_x(N_n(A) = k)}{\sum_{n=0}^{\infty} t^n P_y(N_n(\{y\}) = 0)} \tag{1.2}$$

exists for all $k \geqq 0$. Although limits (1.2) exist in every positive-recurrent chain, there are null-recurrent chains in which these weak

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limits fail to hold even for sets A having just one point. In § 3 of this paper, we will show that a necessary and sufficient condition for the limits in (1.2) to exist for all finite sets A, arbitrary states x, y, and arbitrary nonnegative integers k, is that for some state, say 0, the limits

$$\lim_{t\to 1^-} \sum_{n=0}^{\infty} t^n (P_{00}^n - P_{x0}^n) \tag{1.3}$$

exist for all states x. We will also show (in § 3) that if the series

$$\sum_{n=0}^{\infty} (P_{00}^n - P_{x0}^n) \tag{1.4}$$

converges for all states x, then we may conclude that the Doeblin-type ratio limits,

$$\lim_{R\to\infty} \frac{\sum_{n=0}^{R} P_x(N_n(A) = k)}{\sum_{n=0}^{R} P_y(N_n(\{y\}) = 0)} \tag{1.5}$$

exist and have the same value as those in (1.2).

In § 4 we establish an interesting representation for these limits by using the boundary theory for recurrent chains. In § 5 we investigate several conditions under which the strong ratio limits (1.1) exist. We conclude the paper, in § 6, with the application of the results of the previous sections to several specific examples.

2. Notation.

In this section we shall introduce the notation to be used throughout the remainder of the paper.

Letters A, B etc., will denote nonempty subsets of E.

$$V_A = \inf\{k > 0 : X_k \in A\}$$

$$\delta_A(x) = \begin{cases} 1 \text{ if } x \in A\,, \\ 0 \text{ if } x \notin A\,, \end{cases}$$

$$N_n(A) = \begin{cases} \sum_{k=1}^{n} \delta_A(X_k) \text{ (the occupation time of } A \text{ by time } n), & n>0 \\ 0\,, \quad n = 0 \end{cases}$$

$$F_A^n(x, y) = P_x(V_A = n, X_n = y)\,.$$

As defined above, $F_A^n(x, y) = 0$ if $y \notin A$ and $F_A^0(x, y) = 0$.

$$Q_n(x; A) = P_x(V_A > n) = P_x(N_n(A) = 0)\,,$$

$$\Pi_A(x, y) = \sum_{n=1}^{\infty} F_A^n(x, y)\,,$$

$$P_{xy}^A = \begin{cases} \Pi_A(x, y)\,, & x \in A\,, \\ 0\,, & x \notin A\,; \end{cases}$$

P^A_{xy} is the transition matrix of the Markov chain, "restricted to A."

$$H_A(x, y) = \begin{cases} \Pi_A(x, y), & x \notin A, \\ \delta_{xy}, & x \in A; \end{cases}$$

$H_A(x, y)$ is the "harmonic measure" of A.

$$g_A(x, y) = \sum_{n=0}^{\infty} P_x(V_A > n, X_n = y);$$

$g_A(x, y)$ is the expected number of visits to y among $(X_0, X_1, \cdots)$ before the first entrance into A among $(X_1, X_2, \cdots)$. As defined above, $g_A(x, y) = \delta_{xy}$ if $y \in A$.

$$G_A(x, y) = \begin{cases} g_A(x, y), & x \notin A, \\ 0, & x \in A; \end{cases}$$

$G_A(x, y)$ is the "Green's function" of A.

$$\begin{aligned} {}_zP^n_{xy} &= \begin{cases} P_x(V_{\{z\}} \geqq n, X_n = y), & n > 0, \\ \delta_{xy} \text{ if } x \neq z;\ 0 \text{ if } x = z, n = 0, \end{cases} \\ {}_zP^*_{xy} &= \sum_{n=0}^{\infty} {}_zP^n_{xy}, \\ \pi_y &= {}_0P^*_{0y}. \end{aligned}$$

(Recall that π_y is the unique stationary measure of the chain with $\pi_0 = 1$. See [1] Theorem 7 p. 50)

$$\begin{aligned} K_A(x, y) &= g_A(x, y)\pi_y^{-1}, \\ \Pi^r_A(x, y) &= \begin{cases} \delta_{xy}, r = 0, \\ \sum_t \Pi_A(x, t)\Pi^{r-1}_A(t, y), r > 0, \end{cases} \\ C^n_{xy} &= P^n_{yy} - P^n_{xy}, \\ \tilde{C}_{xy} &= \lim_{t \to 1^-} \sum_{n=0}^{\infty} C^n_{xy} t^n, \end{aligned}$$

provided the limit on the right exists.

$$C^*_{xy} = \sum_{n=0}^{\infty} C^n_{xy},$$

provided the series converges.

The dual chain to P is the chain with transition matrix $\hat{P}_{xy} = (\pi_y/\pi_x)P_{yx}$. Quantities which refer to the dual chain will be denoted by $\hat{}$. For example $\hat{G}_A(x, y)$ is the Green's function for A for the dual chain. Finally, for any random variable Z,

$$E_x Z = E(Z \mid X_0 = x).$$

3. Weak ratio limits. We shall commence our investigation with the weak ratio limits for $P_x(V_A > n)$.

THEOREM 3.1. *In order that the limits*

$$\lim_{t\to 1^-} \frac{\sum_{n=0}^{\infty} t^n Q_n(x; A)}{\sum_{n=0}^{\infty} t^n Q_n(y; \{y\})} = M_y(x; A) \tag{3.1}$$

should exist for all states x, y and for all finite nonempty sets A, it is both necessary and sufficient that for some state, say 0, *and all states x, that the limit*

$$\lim_{t\to 1^-} \sum_{n=0}^{\infty} C_{x0}^n t^n = \tilde{C}_{x0} \tag{3.2}$$

exist. If we further know that the series

$$\sum_{n=0}^{\infty} C_{x0}^n = C_{x0}^* \tag{3.3}$$

converges for all states x, then we may conclude the stronger fact that the Doeblin-type limits

$$\lim_{R\to\infty} \frac{\sum_{n=0}^{R} Q_n(x; A)}{\sum_{n=0}^{R} Q_n(y; \{y\})} = M_y^*(x; A) \tag{3.4}$$

exist for all states x, y and for all finite nonempty sets A. In this latter case $M_y(x; A) = M_y^*(x; A)$. The $M_y(x; A)$ satisfy the following relations:

$$M_y(x; \{z\}) = (\pi_y/\pi_z)[\tilde{C}_{xz} + \delta_{xz}] \tag{3.5}$$

and if $B = A \cup \{z\}$, $z \notin A$,

$$M_y(x; B) = M_y(x; A) - \Pi_B(x, z) M_y(z; A) . \tag{3.6}$$

For ease we shall divide the proof into several lemmas.

LEMMA 3.1. *If the limits in* (3.2) *exist for all states x, then for any state y and all states x*

$$\lim_{t\to 1^-} \sum_{n=0}^{\infty} t^n C_{xy}^n = \tilde{C}_{xy} \tag{3.7}$$

exists. Similarly, if series (3.3) *converges for all states x, then for any state y and all states x*

$$(3.8) \qquad \sum_{n=0}^{\infty} C_{xy}^{n} = C_{xy}^{*} .$$

In this latter case $C_{xy}^{*} = \widetilde{C}_{xy}$.

Proof. It is readily seen that for $n > 0$ and any two states x, y,

$$(3.9) \qquad C_{xy}^{n} = \sum_{k=1}^{n} C_{x0}^{k}\,{}_0P_{0y}^{n-k} - \sum_{k=1}^{n} C_{y0}^{k}\,{}_0P_{0y}^{n-k} + {}_0P_{yy}^{n} - {}_0P_{xy}^{n} .$$

Since the chain is irreducible, the series

$$\sum_{n=0}^{\infty} {}_0P_{0y}^{n},\ \sum_{n=0}^{\infty} {}_0P_{xy}^{n},\ \sum_{n=0}^{\infty} {}_0P_{yy}^{n} ,$$

all converge (See [1], p. 45). Taking generating functions in (3.9) we see that (3.7) follows from (3.2) by Abel's theorem. Similarly, if series (3.3) converges, then the convergence of series (3.8) follows from (3.9) by Merten's theorem ([3], p. 228). Finally, the equality of C_{xy}^{*} and $\widetilde{C}_{xy}$, when the former exists, is a direct consequence of Abel's theorem.

Next we will show that the return times to fixed states are always "weakly" asymptotic.

LEMMA 3.2.[1] *In any irreducible, recurrent Markov chain we have, for any two states z and y,*

$$(3.10) \qquad \lim_{R\to\infty} \frac{\pi_z \sum_{n=0}^{R} Q_n(z; \{z\})}{\pi_y \sum_{n=0}^{R} Q_n(y; \{y\})} = 1 .$$

Proof. Assume $y \neq z$. Familiar generating-function relation for the tail of a power series show that for $|t| < 1$,

$$(3.11) \qquad \frac{\sum_{n=0}^{\infty} t^n Q_n(z; \{z\})}{\sum_{n=0}^{\infty} t^n Q_n(y; \{y\})} = \frac{\sum_{n=0}^{\infty} t^n P_{yy}^{n}}{\sum_{n=0}^{\infty} t^n P_{zz}^{n}} = R_{yz}(t) .$$

Now from a "last entrance decomposition," we see that

$$P_{yy}^{n} = {}_zP_{yy}^{n} + \sum_{k=0}^{n} P_{yz}^{k}\,{}_zP_{zy}^{n-k} ,$$

and thus

$$\sum_{n=0}^{\infty} t^n P_{yy}^{n} = \sum_{n=0}^{\infty} t^n\,{}_zP_{yy}^{n} + \sum_{n=0}^{\infty} t^n P_{yz}^{n} \sum_{n=0}^{\infty} t^n\,{}_zP_{zy}^{n} .$$

[1] Added in proof. Under the assumptions of the lemma, we also have that

$$\sum_{n=0}^{R} \sum_{x\in B} \pi(x) P_x(N_n(B) = k) \sim \sum_{n=0}^{R} \pi(y) P_y(N_n(\{y\}) = 0)$$

for any integer $k \geqq 0$ and any finite nonempty set B.

Using the above relation and well-known generating-function relations (see [1], p. 53 eqns. (1) and (2)), we see that $R_{yz}(t)$ may be written as

$$R_{yz}(t) = \sum_{n=0}^{\infty} t^n \, {}_zP_{yy}^n \left[1 - \sum_{n=1}^{\infty} F_{\{z\}}^n(z, z)t^n\right] + \sum_{n=1}^{\infty} F_{\{z\}}^n(y, z)t^n \sum_{n=0}^{\infty} t^n \, {}_zP_{zy}^n \; . \tag{3.12}$$

Now $R_{yz}(t)$ is a power series in t which converges at $t = 1$ to the value π_y/π_z. To see this, observe that the second term on the right in (3.12) is the product of two power series with positive coefficients which converge at $t = 1$ to the values 1 and ${}_zP_{zy}^*$, respectively. Thus the product series converges at $t = 1$ to ${}_zP_{zy}^*$. On the other hand, the first term on the right in (3.12) is the product of a power series with positive coefficients which converges at $t = 1$ to ${}_zP_{yy}^*$, and a power series which converges at $t = 1$ to 0. Thus Merten's theorem [*op. cit.*] implies that the product series converges at 1 to 0. We have thus shown that $R_{yz}(1) = {}_zP_{zy}^*$. But by [1] (Corollary 1, p. 49, and Theorem 7, p. 50) we have ${}_zP_{zy}^* = \pi_y/\pi_z$.

Denote the coefficient of t^n in $R_{yz}(t)$ by r_{yz}^n. Then from (3.11) we easily obtain

$$\sum_{n=0}^{R} Q_n(z; \{z\}) = \sum_{n=0}^{R} Q_n(y; \{y\} \sum_{j=0}^{R-n} r_{yz}^j \; . \tag{3.13}$$

Since $\sum_{j=0}^{\infty} r_{yz}^j$ converges, and since

$$\lim_{R\to\infty} \frac{Q_R(y, \{y\})}{\sum_{n=0}^{R} Q_n(y; \{y\})} = 0 \; ,$$

we have, by a well-known Abelian theorem on Norlünd summability (see [3; p. 64[),

$$\lim_{R\to\infty} \frac{\sum_{n=0}^{R} Q_n(z; \{z\})}{\sum_{n=0}^{R} Q_n(y; \{y\})} = \sum_{n=0}^{\infty} r_{yz}^n = \pi_y/\pi_z \; . \tag{3.14}$$

This completes the proof.

As a corollary of the proof we have the following:

COROLLARY 3.1. *In any irreducible, recurrent Markov chain, the series $R_{yz}(t)$ defined in* (3.11) *converges at $t = 1$ to π_y/π_z.*

We of course have that the weaker, Abelian version, is also universally valid.

COROLLARY 3.2. *In any irreducible, recurrent Markov chain,*

$$\lim_{t\to 1^-} \frac{\sum_{n=0}^{\infty} Q_n(z; \{z\})t^n}{\sum_{n=0}^{\infty} Q_n(y; \{y\})t^n} = \pi_y/\pi_z . \tag{3.15}$$

Proof. This follows directly from Corollary 3.1 (by Abel's theorem). We are now in a position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We shall proceed by induction on the number of points in A. Let us first assume that the series (3.3) converges for all states x and show that this leads to the conclusion that the limits in (3.4) exist for all states x, y and for all finite non-empty sets A. Suppose $A = \{y\}$. If $x = y$, then obviously the limit in (3.4) exists and has the value 1; so assume that $x \neq y$. Since for $x \neq y$ we have

$$\sum_{n=0}^{\infty} Q_n(x, y)t^n = (1-t)^{-1}\left[1 - \sum_{n=1}^{\infty} t^n F_{\{y\}}^n(x, y)\right] = (1-t)^{-1}\left[1 - \frac{\sum_{n=0}^{\infty} t^n P_{xy}^n}{\sum_{n=0}^{\infty} t^n P_{yy}^n}\right]$$

we see that for $x \neq y$,

$$\sum_{n=0}^{\infty} t^n Q_n(x; \{y\}) = \sum_{k=0}^{\infty} C_{xy}^k t^k \sum_{n=0}^{\infty} t^n Q_n(y; \{y\}) , \tag{3.16}$$

and thus

$$Q_n(x; \{y\}) = \sum_{k=0}^{n} C_{xy}^k \, Q_{n-k}(y; \{y\}) . \tag{3.17}$$

Consequently,

$$\frac{\sum_{n=0}^{R} Q_n(x; \{y\})}{\sum_{n=0}^{R} Q_n(y; \{y\})} = \frac{\sum_{n=0}^{R} Q_n(y; \{y\}) \sum_{j=0}^{R-n} C_{xy}^j}{\sum_{n=0}^{R} Q_n(y; \{y\})} . \tag{3.18}$$

By Lemma 3.1,

$$\lim_{R\to\infty} \sum_{j=0}^{R} C_{xy}^j = C_{xy}^* ,$$

and thus, by the same Abelian theorem as used in the proof of the previous lemma, we see that the limit as $R \to \infty$ in (3.18) exists and has the value C_{xy}^*.

From Lemma 3.2, we then have

$$\lim_{R\to\infty} \frac{\sum_{n=0}^{R} Q_n(x, \{z\})}{\sum_{n=0}^{R} Q_n(y; \{y\})} = (\pi_y/\pi_z)[C^*_{xz} + \delta_{xz}] ,$$

which establishes the existence of the limits in (3.4) for all sets A having exactly one point. Suppose we have established the existence of these limits for all sets A having exactly $r > 0$ points. Let $B = A \cup \{z\}$, where $z \notin A$ but is otherwise arbitrary. It is readily seen that

$$Q_n(x; B) = Q_n(x; A) - \sum_{k=1}^{n} F^k_B(x, z)Q_{n-k}(z; A) , \tag{3.19}$$

and thus

$$\frac{\sum_{n=0}^{R} Q_n(x; B)}{\sum_{n=0}^{R} Q_n(y; \{y\})} = \frac{\sum_{n=0}^{R} Q_n(x; A)}{\sum_{n=0}^{R} Q_n(y; \{y\})} - \frac{\sum_{n=0}^{R} Q_n(z; A) \sum_{j=1}^{R-n} F^j_B(x, z)}{\sum_{n=0}^{R} Q_n(y; \{y\})} . \tag{3.20}$$

By the induction assumption, the first term on the right converges to $M^*_y(x; A)$. If we multiply and divide the second term on the right by $\sum_{n=0}^{R} Q_n(z; A)$, and apply the Abelian theorem mentioned above, then we may conclude that the second term on the right in (3.20) converges to $\Pi_B(x, z)M^*_y(z; A)$. Thus the limit in (3.20) exists and has value $M^*_y(x; A) - \Pi_B(x, z)M^*_y(z; A)$. By induction, we then have that the limit in (3.4) exists for all finite sets A, and moreover, that relation (3.6) holds.

Now let us assume that the limits in (3.2) exist for all states x, and show this leads to the conclusion that the limits (3.1) exist as required.

From (3.16), Lemma 3.1, and Corollary 3.2 we have at once that the limits in (3.1) exist for all sets A having a single point, and moreover, that $M_y(x, \{z\}) = (\pi_y/\pi_z)[\tilde{C}_{xz} + \delta_{xz}]$. Suppose we have established the existence of the limits in (3.1) for all sets A having exactly $r > 0$ points. Again let $B = A \cup \{z\}, z \notin A$. From (3.19) we have

$$\frac{\sum_{n=0}^{\infty} Q_n(x; B)t^n}{\sum_{n=0}^{\infty} Q_n(y; \{y\})t^n} = \frac{\sum_{n=0}^{\infty} Q_n(x; A)t^n}{\sum_{n=0}^{\infty} Q_n(y; \{y\})t^n} - \frac{\sum_{k=1}^{\infty} F^k_B(x, z)t^n \sum_{n=0}^{\infty} Q_n(z; A)t^n}{\sum_{n=0}^{\infty} Q_n(y; \{y\})t^n}$$

and thus, by the induction assumption and Abel's theorem, we have that the limit on the right, as $t \to 1^-$, exists, and has the value $M_y(x; A) - \Pi_B(x, z)M_y(z; A)$. Hence, by induction, we have that the limits in (3.1) exist for all finite sets A. This establishes the sufficiency portion of Theorem 3.1.

Now suppose we know that the limits in (3.1) exist for all finite sets A and for all states x, y; then, in particular, they exist for $A = \{0\}$. From (3.16) we see at once that this implies the existence of the limits in (3.2) for all states x, and thus condition (3.2) is necessary.

Finally, a simple Abelian-type argument shows that $M_y(x; A) = M_y^*(x; A)$ whenever the latter exists. This completes the proof of Theorem 3.1.

Before proceeding further let us make some comments on the preceding results.

Whenever series (3.3) converges for all states x, then as noted in the proof of the above theorem, Abel's theorem gives us that the limits in (3.2) exist for all states x. On the other hand, if the limits in (3.2) exist for all states x, then series (3.3) may fail to converge but we do have that it is at least $(C; 1)$ summable.[2] To see this, observe that if

$$S_{xy}^n = \sum_{k=0}^{n-1} C_{xy}^k , \tag{3.21}$$

then, for $x \neq y$,

$$S_{xy}^n = \sum_{r=1}^{n} F_{\{y\}}^r(x, y)[E_y N_n(\{y\}) - E_y N_{n-r}(\{y\})] + Q_n(x; \{y\}) E_y N_n(\{y\}) ,$$

and thus $S_{xy}^n \geqq 0$. Consequently, whenever the limits

$$\lim_{t \to 1^-} \sum_{n=0}^{\infty} C_{xy}^n t^n = \tilde{C}_{xy}$$

exist, we have by a well-known Tauberian theorem (see [3], p. 154),

$$\lim_{n \to \infty} n^{-1} \sum_{r=0}^{n-1} S_{xy}^r = \tilde{C}_{xy} .$$

If the chain is positive recurrent, then $E_x V_A < \infty$ for all nonempty sets A. But $\sum_{n=0}^{\infty} Q_n(x; A) = E_x V_A$, and thus the limits (3.4) exist in *all* positive recurrent chains. Moreover, in this case, $M_y(x; A) = (E_y V_{\{y\}})^{-1} E_x V_A$. Since series (3.3) need not converge in a periodic positive recurrent chain, we see that this condition is not necessary for the existence of the limits in (3.4) (at least in the positive recurrent case). In this regard let us point out that the series (3.3) does converge for all states x in every *aperiodic* positive recurrent chain (see [6])

Under the same conditions as Theorem 3.1, the following extension also holds.

THEOREM 3.2. *If the limits* (3.2) *exist for all states x, then for any nonempty finite set A, any two states x, y, and any nonnegative*

[2] This fact will be needed in §4.

integer k,

$$(3.22)\qquad \lim_{t\to 1^-}\frac{\sum_{n=0}^{\infty}P_x(N_n(A)=k)t^n}{\sum_{n=0}^{\infty}P_y(N_n(\{y\})=0)t^n}=\sum_z \Pi_A^k(x,z)M_y(z;A)\ .$$

Moreover, if series (3.3) *converges for all states* x, *then*

$$(3.23)\qquad \lim_{R\to\infty}\frac{\sum_{n=0}^{R}P_x(N_n(A)=k)}{\sum_{n=0}^{\infty}P_y(N_n(\{y\})=0)}=\sum_z \Pi_A^k(x,z)M_y(z;A)\ .$$

Proof. For brevity we shall only prove the assertion in (3.23). The proof of (3.22) is very similar. We proceed by induction on k. For $k=0$ the assertion in (3.23) is just that of Theorem 3.1, and thus (3.23) holds for $k=0$.

Suppose we have established the result for all $k\leqq k_0$. Now,

$$(3.24)\qquad P_x(N_n(A)=k_0+1)=\sum_z\sum_{j=1}^{n}F_A^j(x,z)P_z(N_{n-j}(A)=k_0)\ ,$$

and thus

$$(3.25)\qquad \frac{\sum_{n=0}^{R}P_x(N_n(A)=k_0+1)}{\sum_{n=0}^{R}P_y(N_n(\{y\})=0)}=\sum_{z\in A}\frac{\sum_{n=0}^{R}P_z(N_n(A)=k_0)\sum_{j=1}^{R-n}F_A^j(x,z)}{\sum_{n=0}^{R}P_y(N_n(\{y\})=0)}\ .$$

Since we may write each term on the right in the above expression as

$$\frac{\sum_{n=0}^{R}P_z(N_n(A)=k_0)\sum_{j=1}^{R-n}F_A^j(x,z)}{\sum_{n=0}^{R}P_z(N_n(A)=k_0)}\cdot\frac{\sum_{n=0}^{R}P_z(N_n(A)=k_0)}{\sum_{n=0}^{R}P_y(N_n(\{y\})=0)}\ ,$$

and since $\sum_{j=1}^{\infty}F_A^j(x,z)=\Pi_A(x,z)$,

$$\lim_{R\to\infty}\frac{P_z(N_R(A)=k_0)}{\sum_{n=0}^{R}P_z(N_n(A)=k_0)}=0\ ,$$

we see (by the induction assumption and the same Abelian theorem as used in the previous proofs) that the limit, as $R\to\infty$, in (3.25) exists and has the value

$$\sum_{z\in A}\Pi_A(x,z)\sum_{u\in A}\Pi_A^{k_0}(z,u)M_y(u;A)=\sum_{u\in A}\Pi_A^{k_0+1}(x,u)M_y(u;A)\ .$$

Thus, the assertion of the theorem is true for $k=k_0+1$, and thus, by induction, the theorem holds for all values of k. This completes the proof.

Observe that in the special case in which $A = \{y\}$ and $k > 0$, the limits in (3.23) have the value 1. For this special case we can show, by the same methods as used to establish the result in general, that this result holds in *every* recurrent chain.

The same arguments we used to establish the above results enable us to show that limits of the more general expressions

$$\frac{\sum_{n=0}^{R} P_x(N_n(\{y_i\}) = k_i,\ 1 \leqq i \leqq r)}{\sum_{n=0}^{R} P_y(N_n(\{y\}) = 0)}$$

exist, and to compute their values. Since these results are quite complicated to write down, however, we shall not pursue these generalizations.

4. Representation. By using the boundary theory for recurrent chains, as developed by Kemeny and Snell in [8], we may establish interesting representations for the limits found in the last section. For convenience, we shall summarize below that portion of this theory which we shall need. For details we refer the reader to [8].

We shall be interested in the dual, i.e., exit boundary, of the chain. If we choose a state y as a "taboo" state, then a boundary point ξ corresponds to a sequence $\{t_n\}$ of states such that $|t_n| \to \infty$ and the limits

$$\lim_{n\to\infty} K_{\{y\}}(x, t_n)$$

exist for all states x. The set of boundary points, $\hat{B}$, so obtained does not depend on which state y is chosen as the taboo state, and $E^* = E \cup \hat{B}$ is a compact metric space which gives a discrete topology to E. For each x and A, the functions $K_A(x, \cdot)$, $\hat{G}_A(\cdot, x)$, and $\hat{H}_A(\cdot, x)$ can be extended to E^* as continuous functions. For a fixed y, let ${}_yQ_{xt} = P_{xt}$ if $t \neq y$, and let ${}_yQ_{xy} = 0$. Then ${}_yQ_{xt}$ is the substochastic transition matrix of a transient chain (the yth associated transient chain). A function f is called superregular for ${}_yQ$ if ${}_yQf(x) \leqq f(x)$. If ${}_yQf = f$, then f is called regular for ${}_yQ$. A nonnegative regular function is minimal if every nonnegative regular function $g \leqq f$ is a multiple of f. A point $\xi \in \hat{B}$ is called minimal if $K_{\{y\}}(\cdot, \xi)$ is a minimal regular function for ${}_yQ$. (Again, the minimal points of $\hat{B}$ can be shown to be independent of y.) If h is a nonnegative superregular function for ${}_yQ$, then the transient chain with transition matrix, ${}_yQ_{xt}h(t)h(x)^{-1}$ is called the h-process for ${}_yQ$. The following representation theorem follows by applying the results in § 5 of [8] to the measure $\nu(x) = f(x)\pi_x\pi_y^{-1}$ on the dual chain.

THEOREM 4.1. *Suppose* (i) $f(x) \geqq 0$, (ii)$f(y) = 0$, *and* (iii) $Pf(x) \leqq f(x) + \delta_{xy}$. *Then, there is a unique probability measure* Γ_f, *vanishing on the nonminimal points of* $\hat{B}$, *such that*

$$f(x) + \delta_{xy} = \pi_y \int_{E^*} K_{\{y\}}(x, \xi)\Gamma_f(d\xi) .$$

Finally, if $h(x) = f(x) + \delta_{xy}$, then h is a superregular function for ${}_yQ$. Let $\theta^A(x)$ be the probability that, starting from y, this h process last visits the set A at the point x, ($\theta^A(x)$ depends on y and f). Then for $x \in A$, by results in [8], we have

$$\sum_{t \in A} (I - P^A)_{xt} f(t) = [\theta^A(x) - \delta_{xy}]\pi_y/\pi_x . \tag{4.1}$$

Moreover, if $\{A_n\}$ is a fundamental sequence of sets, i.e., the A_n are monotone increasing and $\cup_n A_n = E$, then for any continuous function g on E^*,

$$\lim_{n \to \infty} \sum_x \theta^{A_n}(x) g(x) = \int_{E^*} g(\xi)\Gamma_f(d\xi) . \tag{4.2}$$

That is, the θ^{A_n} converge weakly to Γ_f.

We are now in a position to establish the following result.

THEOREM 4.2. *Suppose the limits in* (3.2) *exist for all states* x. *Then, for any finite nonempty set* A, *and any two states* x, y *we have, for* $x \notin A$,

$$M_y(x; A) = (\pi_y/\pi_x) \int_{E^*} \hat{G}_A(\xi, x)\beta(d\xi) , \tag{4.3}$$

while for $x \in A$,

$$M_y(x; A) = (\pi_y/\pi_x) \int_{E^*} \hat{H}_A(\xi, x)\beta(d\xi) , \tag{4.4}$$

where β *is a unique probability measure on* E^* *which vanishes on the nonminimal boundary points.*

We shall first establish the desired representation for all sets A having exactly one point. This will be an immediate consequence of (3.5) and the following lemma.

LEMMA 4.1. *For each* y *and all states* x,

$$\tilde{C}_{xy} = (\pi_y/\pi_x) \int_{E^*} \hat{G}_{\{y\}}(\xi, x)\beta(d\xi) , \tag{4.5}$$

where β is a unique probability measure on E^* which vanishes on the nonminimal boundary points.

Note. In [8] a corresponding representation was shown to hold for C^*_{xy}. In the proof given below to establish (4.5), we shall use techniques similar to those used in [8] to establish the result for C^*_{xy}.

Proof. In the remarks following the proof of Theorem 3.1, we showed that

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} S^k_{xy} = \tilde{C}_{xy} . \tag{4.6}$$

From Fatou's lemma, we then have

$$\sum_t P_{xt}\tilde{C}_{ty} \leqq \liminf_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \sum_t P_{xt}S^k_{ty} \leqq \tilde{C}_{xy} + \delta_{xy} .$$

Consequently, for each fixed y, we see that $\tilde{C}_{xy}$ satisfies conditions (i)–(iii) of Theorem 4.1, and thus there is a unique probability measure $\beta_{\{y\}}$, vanishing on the nonminimal boundary points, such that

$$\tilde{C}_{xy} + \delta_{xy} = \pi_y \int_{E^*} K_{\{y\}}(x, \xi)\beta_{\{y\}}(d\xi) .$$

But since for each $\xi \in E^*$,

$$K_{\{y\}}(x, \xi) = [\hat{G}_{\{y\}}(\xi, x) + \delta_{xy}]\pi_x^{-1} , \tag{4.7}$$

we have

$$\tilde{C}_{xy} = (\pi_y/\pi_x)\int_{E^*} \hat{G}_{\{y\}}(\xi, x)\beta_{\{y\}}(d\xi) .$$

To complete the proof we must now show that the measure $\beta_{\{y\}}$ is independent of y. In order to do this we may proceed as follows. Let A be any finite set containing x; then,

$$\sum_t \hat{P}_{yt}\hat{H}_A(t, x) = \hat{H}_A(y, x) + \hat{P}^A_{yx} - \delta_{yx}, x \in A .$$

If we iterate the above relation n times, we obtain the identity

$$\sum_t \hat{P}^{n+1}_{yt}\hat{H}_A(t, x) = \hat{H}_A(y, x) + \sum_{t\in A} \hat{N}^n_{yt}[\hat{P}^A - I]_{tx} ,$$

where $\hat{N}^n_{yt} = (I + \hat{P} + \cdots + \hat{P}^n)_{yt}$. Now, $\sum_t \pi_t \hat{P}^A_{tx} = \pi_x$, and thus we may rewrite the above identity as

$$\sum_t \hat{P}^{n+1}_{yt}\hat{H}_A(t, x) = \hat{H}_A(y, x) + \sum_{t\in A} \left[\left(\frac{\pi_t}{\pi_y}\right)\hat{N}^n_{yy} - \hat{N}^n_{yt}\right][I - \hat{P}^A]_{tx} . \tag{4.8}$$

But

$$(\pi_t/\pi_y)\hat{N}^n_{yy} - \hat{N}^n_{yt} = (\pi_t/\pi_y)S^n_{ty} .$$

Using this identity we obtain from (4.8) the identity

$$\sum_t \frac{(\hat{N}^{n+1} - I)_{yt}H_A(t, x)}{n+1} = \hat{H}_A(y, x) + (\pi_x/\pi_y)\sum_{t\in A}(I - P^A)_{xt}\Big[\frac{1}{n+1}\sum_{k=0}^{n} S^k_{ty}\Big] .$$

Consequently, from (4.6) we have that the limit, as $n \to \infty$, on the right-hand side of the above expression exists and has the value

$$\hat{H}_A(y, x) + (\pi_x/\pi_y)\sum_{t\in A}(I - P^A)_{xt}\tilde{C}_{ty} .$$

For a fixed x and A, let

$$\varphi_n(y) = \sum_t \frac{\hat{N}^{n+1}_{yt}\hat{H}_A(t, x)}{n+1} .$$

From the above, we then have that $\lim_{n\to\infty}\varphi_n(y) = \varphi(y)$ exists. But

$$\hat{N}^{n+1} - I = \hat{P}\hat{N}^n ,$$

and thus

$$\varphi_n(y) = \sum_t \hat{P}_{yt}\varphi_{n-1}(t)n/n + 1 + \hat{H}_A(y, x)(n+1)^{-1} .$$

By dominated convergence, we then have that $\varphi(y)$ satisfies the relation $\hat{P}\varphi(x) = \varphi(x)$, and as the $\hat{P}$ chain is recurrent, we must have that $\varphi(x)$ is a constant (independent of y).[3] Denote this constant by $\tilde{\lambda}_A(x)$. We have thus established the following identity:

$$[\tilde{\lambda}_A(x) - \hat{H}_A(y, x)]\pi_y/\pi_x = \sum_{t\in A}(I - P^A)_{xt}\tilde{C}_{ty},\ x \in A ,$$

and, in particular, for $y \in A$ we have

$$[\tilde{\lambda}_A(x) - \delta_{yx}]\pi_y/\pi_x = \sum_{t\in A}(I - P^A)_{xt}\tilde{C}_{ty},\ x, y \in A . \tag{4.9}$$

However, from (4.1), for any finite set A containing x,

$$\sum_{t\in A}(I - P^A)_{xt}\tilde{C}_{ty} = (\pi_y/\pi_x)[\theta^A(x) - \delta_{xy}] , \tag{4.10}$$

where $\theta_A(x)$ is the probability that, starting from y, x is the last state to be visited in the set A by the h process for ${}_yQ$ determined by the function $\tilde{C}_{\cdot y} + \delta_{\cdot y} = h(\cdot)$. From (4.9) and (4.10) we see that if $y \in A$,

[3] See Theorem 3 on p. 226 of [**6**].

then $\theta^A(x) = \tilde{\lambda}_A(x)$. Consequently, (by (4.2)) for any fundamental sequence of sets $\{A_n\}$, we have that $\{\tilde{\lambda}_{A_n}\}$ converges weakly to $\beta_{\{y\}}$, and as the $\tilde{\lambda}_{A_n}$ are independent of y, we must have that $\beta_{\{y\}}$ is too. This completes the proof.

We may now complete the proof of Theorem 4.2. First, observe that from (4.7), we see that (4.3) and (4.4) for the set $A = \{z\}$ are equivalent to the single relation,

$$M_y(x; \{z\}) = \pi_y \int_{E^*} K_{\{z\}}(x, \xi)\beta(d\xi) \;. \tag{4.11}$$

Now suppose we have established the relation

$$M_y(x; A) = \pi_y \int_{E^*} K_A(x, \xi)\beta(d\xi) \;, \tag{4.12}$$

for all sets A having exactly $r > 0$ points. Let $B = A \cup \{z\}$, where $z \notin A$ but is otherwise arbitrary. From the relation

$$\begin{aligned} P_x(V_B > n, X_n = t) &= P_x(V_A > n, X_n = t) \\ &\quad - \sum_{k=1}^{n} F_B^k(x, z) P_z(V_A > n - k, X_{n-k} = t) \;, \end{aligned}$$

we obtain the identity

$$g_B(x, y) = g_A(x, y) - \Pi_B(x, z) g_A(z, y) \;,$$

and thus for each $\xi \in E^*$,

$$K_B(x, \xi) = K_A(x, \xi) - \Pi_B(x, z) K_A(z, \xi) \;.$$

Consequently, by the induction assumption and equation (3.6),

$$M_y(x; B) = \pi_y \int_{E^*} K_B(x, \xi)\beta(d\xi) \;,$$

which establishes (4.12) for the set B. By induction, we then have that (4.12) holds for all finite nonempty sets.

If $x \notin A$ we have that $g_A(x, t) = G_A(x, t)$, and thus for $x \notin A$, (4.12) becomes (4.3). On the other hand, for any x, t we have

$$g_A(x, t) = \sum_z P_{xz} G_A(z, t) + \delta_{xt} \;,$$

and thus

$$K_A(x, t) = \left[\sum_z \hat{G}_A(t, z)\hat{P}_{zx} + \delta_{xt}\right]\pi_x^{-1} \;.$$

But if $x \in A$, the expression in braces is $\hat{H}_A(t, x)$. Consequently, from (4.12) and the above we obtain (4.4). This completes the proof of

Theorem 4.2.

From the proof of the above theorem we have the following:

COROLLARY 4.1. *For any two states x, y and any finite nonempty set A,*

$$M_y(x; A) = \pi_y \int_{E^*} K_A(x, \xi)\beta(d\xi) , \tag{4.13}$$

provided the limits in (3.2) *exist for all states x.*

COROLLARY 4.2. *Under the same conditions as Theorem* 4.2 *we have $M_y(x; A)\pi_y^{-1}$ is independent of y. Moreover*

$$\sum_{x \in A} \pi_x M_y(x; A) = \pi_y . \tag{4.14}$$

Let us see what the representation in Theorem 4.2 becomes in a positive recurrent chain. If the chain is positive recurrent, then it is readily seen that the representation in Corollary 4.1 is equivalent to the identity

$$(\pi_y E_y V_{\{y\}})^{-1} E_x V_A = \sum_t K_A(x, t)\beta(\{t\}) + \int_{\hat{B}} K_A(x, \xi)\beta(d\xi) .$$

However, it is a well-known and easily verifiable fact that

$$E_x V_A = \sum_t g_A(x, t) = m^{-1} \sum_t K_A(x, t)(m\pi_t) , \tag{4.15}$$

where $m = (\sum_t \pi_t)^{-1}$. By the uniqueness of β we must then have

$$\beta(\hat{B}) = 0, \text{ and } \beta(\{t\}) = m\pi_t .$$

From the above results on the positive recurrent case we see that we may view the resuls of Theorem 4.2 as an extension of the identity (4.15) to those null chains for which the limits in (3.2) exist for all states x.

5. Strong ratio limits. Let us now turn our attention to the problem of the strong ratio limits. It is quite easy to give examples of aperiodic recurrent chains (both positive and null) in which series (3.3) converges for all states x (and thus the weak ratio limits (3.4) exist), but in which the strong ratio limits fail to exist. The following is a sufficient condition for null recurrent chains which is frequently satisfied.

THEOREM 5.1. *If the limits* (3.2) *exist for all states x and if for some state, say z_0, we have*

$$\sum_{n=0}^{\infty} t^n Q_n(z_0; \{z_0\}) \sim (1-t)^{-\alpha} L\Big(\frac{1}{1-t}\Big),\ t \to 1^- , \tag{5.1}$$

where $\alpha > 0$ *and* $L(u)$ *is a slowly varying function, then for any state* y,

$$Q_n(y; \{y\}) \sim (\pi_{z_0}/\pi_y)\Gamma(\alpha)^{-1} n^{\alpha-1} L(n) , \tag{5.2}$$

and for any finite nonempty set A, *arbitrary states* x, y, *and arbitrary nonnegative integer* k,

$$\lim_{n\to\infty} \frac{P_x(N_n(A) = k)}{P_y(N_n(\{y\}) = 0)} = \sum_z \Pi_A^k(x, z) M_y(z; A) . \tag{5.3}$$

Proof. We know (see (3.15)) that

$$\lim_{t\to 1^-} \frac{\sum_{n=0}^{\infty} t^n Q_n(y; \{y\})}{\sum_{n=0}^{\infty} t^n Q_n(z_0; \{z_0\})} = \pi_{z_0}/\pi_y ,$$

and thus, if (5.1) holds, we have

$$\sum_{n=0}^{\infty} t^n Q_n(y; \{y\}) \sim (1-t)^{-\alpha} L\Big(\frac{1}{1-t}\Big)\pi_{z_0}/\pi_y,\ t \to 1^- .$$

Since $Q_n(y; \{y\})$ is nonnegative, by Karamata's Tauberian theorem ([3], p. 154) we have

$$\sum_{k=0}^{n} Q_k(y; \{y\}) \sim (\pi_{z_0}/\pi_y)\frac{n^\alpha L(n)}{\Gamma(\alpha+1)},\ n \to \infty , \tag{5.4}$$

and since $Q_n(y; \{y\})$ is monotone, another Tauberian theorem ([2], p. 517) asserts that from (5.4) we may conclude that (5.2) holds. Now if the limits in (3.2) exist for all x, then by Theorem 3.2 and (5.1) we have

$$\sum_{n=0}^{\infty} t^n P_x(N_n(A) = k) \sim \pi_y^{-1}(1-t)^{-\alpha} L\Big(\frac{1}{1-t}\Big)\pi_{z_0} \sum_z \Pi_A^k(x, z) M_y(z; A) ,$$

and thus,

$$\sum_{n=0}^{\infty} t^n P_x(N_n(A) \leqq k) \sim \pi_y^{-1}(1-t)^{-\alpha} L\Big(\frac{1}{1-t}\Big)\pi_{z_0} \sum_z \sum_{j=0}^{k} \Pi_A^j(x, z) M_y(z; A) .$$

Since $N_n(A)$ is nondecreasing, $P_x(N_n(A) \leqq k)$ is nonincreasing in n for each fixed k. Consequently, by applications of the Tauberian theorems mentioned above, we may conclude

$$P_x(N_n(A) \leqq k) \sim \pi_y^{-1}\Gamma(\alpha)^{-1} n^{\alpha-1} L(n)\pi_{z_0} \sum_z \sum_{j=0}^{k} \Pi_A^j(x, z) M_y(z; A) .$$

Thus we must have

$$P_x(N_n(A) = k) \sim \pi_y^{-1}\Gamma(\alpha)^{-1}n^{\alpha-1}L(n)\pi_{z_0}\sum_z \Pi_A^k(x, z)M_y(z; A) .$$

Now (5.3) follows from (5.2) and the above expression.

REMARK. Again observe that in the special case when $A = \{y\}$, the limit value is 1 for all $k > 0$. If we trace through the above proof for this special case, we will see that we never use the assumption that the limits in (3.2) exist. Thus we have

COROLLARY 5.1. *If* (5.1) *holds for some state* z, *then* (5.2) *holds for any state* y, *and moreover, for any integer* $k > 0$,

$$\lim_{n\to\infty} \frac{P_x(N_n(\{y\}) = k)}{P_y(N_n(\{y\}) = 0)} = 1 . \tag{5.5}$$

Observe that whenever (5.5) holds, we have the curious result that for $0 \leqq j \leqq k$.

$$\lim_{n\to\infty} P_x(N_n(\{y\}) = j \mid N_n(\{y\}) \leqq k) = (k + 1)^{-1} . \tag{5.6}$$

In the case when the dual boundary has only one point we may establish the existence of the strong ratio limits for one point sets with less assumptions. In fact we have the following.

THEOREM 5.2. *Suppose that the dual boundary of the* P *chain has only one point and* $Q_n(y; \{y\}) > 0$ *for all* n. *Then for each fixed* m,

$$\lim_{n\to\infty} \frac{Q_{n+m}(x, \{y\})}{Q_n(y; \{y\})} = \begin{cases} (\pi_y/\pi_x)\mu_{xy}, & x \neq y , \\ 1, & x = y , \end{cases} \tag{5.7}$$

if and only if

$$\lim_{n\to\infty} \frac{Q_{n+1}(y; \{y\})}{Q_n(y; \{y\})} = 1 . \tag{5.8}$$ [4]

Proof. Clearly if (5.7) holds, then setting $y = x$ and $m = 1$, we obtain (5.8). To establish the converse, we need only consider the case $m = 0$. If $x = y$, there is nothing to prove; so from now on assume $x \neq y$. Now we have

$$Q_n(x, \{y\}) = \sum_{r=0}^{\infty} F_{\{y\}}^{r+1+n}(x, \{y\}) = \sum_t G_{\{y\}}(x, t)\, {}_yP_{ty}^{n+1} ,$$

[4] If the chain is null recurrent, then $Q_n(y, \{y\}) > 0$ for all n. (See the proof of Theorem 5.3). The author has constructed an example of a recurrent chain in which the dual boundary has exactly 2 points and such that (5.8) holds, but in which the limits in (5.7) *fail* to hold. However, in this example it is not known if condition (3.2) holds, so that the weak limits may also not exist.

and thus,

$$(5.9) \qquad \frac{Q_n(x, \{y\})}{Q_n(y; \{y\})} = \frac{Q_{n+1}(y; \{y\})}{Q_n(y, \{y\})} \sum_t \hat{G}_{\{y\}}(t, x) R_n(t) \, \pi_y / \pi_x \; ,$$

where

$$(5.10) \qquad R_n(t) = \frac{{}_y\hat{P}_{yt}^{n+1}}{\hat{Q}_{n+1}(y; \{y\})} \; .$$

Now for $t \neq y$,

$$R_n(t) = P_y(\hat{X}_{n+1} = t \mid \hat{V}_{\{y\}} > n + 1) \; ,$$

and thus

$$(5.11) \qquad R_n(t) \geqq 0 \text{ and } \sum_{t \neq y} R_n(t) \doteq 1 \; .$$

Since the $\hat{P}$ chain is irreducible, we have, for some m_0, that $\hat{P}_{ty}^{m_0} = \alpha > 0$, and as

$$P_y(n < \hat{V}_{\{y\}} \leqq n + m_0 + 1) \geqq P_y(\hat{V}_{\{y\}} > n + 1, \hat{X}_{n+1} = t) \hat{P}_{ty}^{m_0} \; ,$$

we have,

$$\frac{P_y(\hat{V}_{\{y\}} > n + 1, \hat{X}_{n+1} = t)}{\hat{Q}_{n+1}(y; \{y\})} \leqq (1/\alpha) \left[\frac{Q_n(y; \{y\})}{Q_{n+1}(y; \{y\})} - \frac{Q_{n+m_0+1}(y; \{y\})}{Q_{n+1}(y; \{y\})} \right] .$$

From (5.8) and the above inequality, we then have, for each fixed t,

$$(5.12) \qquad \lim_{n \to \infty} R_n(t) = 0 \; .$$

The assumption that the dual boundary has only one point is equivalent to the fact that the limit

$$(5.13) \qquad \lim_{|t| \to \infty} \hat{G}_{\{y\}}(t, x) = \mu_{xy}$$

exists. From (5.8) and (5.9), we see that to establish (5.7) it is only necessary to show

$$\lim_{n \to \infty} \sum_t \hat{G}_{\{y\}}(t, x) R_n(t) = \mu_{xy} \; .$$

However, this is an easy consequence of (5.11), (5.12), and (5.13).

Although the conditition that the dual boundary have only one point is very restrictive, we shall see in § 6 that it applies to some of the important chains of practical interest. In particular, it was shown by Spitzer [13] to hold for all partial sums of independent, identically distributed, integer-valued random variables except those with mean 0 and finite variance.[5] The above theorem for this class of chains was established by Kesten and Spitzer in [9], and the proof given above was patterned after their proof.

Theorem 5.1 includes certain cases not covered by the present theorem. For example, it was shown in [9] that, for the partial sum of independent, integer-valued random variables with a common distribution having zero mean and finite variance, the conditions of Theorem (5.1) hold; but in this case it was shown that the dual boundary has two points. On the other hand, the present theorem clearly includes cases not covered by Theorem (5.1)—notably, some positive recurrent chains are included in the present result.

If the chain is null recurrent and has only a single point in its dual boundary, then, as was noted in [8], we must have that C^*_{xy} exists for all states x, y. Moreover, in this case we must have that $C^*_{xy} = (\pi_y/\pi_x)\mu_{xy}$. On the other hand, if the chain is positive recurrent, the weak and strong limits need no longer be the same.

Usually it is not easy to verify when condition (5.8) holds. In the null-recurrent case, a simple sufficient condition is the following.

THEOREM 5.3. *If, in a null-recurrent chain, P^n_{yy} is a monotone function of n, then* (5.8) *holds.*

Proof. We must have $Q_n(y; \{y\}) > 0$ for all n. For if $Q_{n_0}(y; \{y\}) = 0$, then $Q_{n+n_0}(y; \{y\}) = 0$ for all $n \geqq 0$, and thus $\sum_{n=0}^{\infty} Q_n(y; \{y\}) < \infty$. But this cannot be true in a null-recurrent chain. Thus we have

$$\frac{Q_n(y; \{y\})}{Q_{n-1}(y; \{y\})} \leqq 1 .$$

Now, it is well-known that

$$1 = \sum_{k=0}^{n} P^{n-k}_{yy} Q_k(y; \{y\}) ,$$

and thus

$$Q_n(y; \{y\}) + \sum_{k=0}^{n-1} (P^{n-k}_{yy} - P^{n-1-k}_{yy}) Q_k(y; \{y\}) = 0 .$$

Since P^n_{yy} is nonincreasing, we have

$$Q_n(y; \{y\}) \geqq Q_{n-1}(y; \{y\})(1 - P^n_{yy}) .$$

The fundamental limit theorem for Markov chains (see [1]) then gives us

[5] Of course, the partial sums must also be such that they constitute an irreducible, recurrent chain.

$$\liminf_{n\to\infty} \frac{Q_n(y; \{y\})}{Q_{n-1}(y; \{y\})} \geqq \lim_{n\to\infty} (1 - P^n_{yy}) = 1 .$$

Let us now extend the result of Corollary 5.1.

THEOREM 5.4. *If for a state y, $Q_n(y; \{y\}) > 0$ for all n and there is a positive $\alpha \leqq 1/2$ such that*

$$\sup_n \frac{Q_{[\alpha n]}(y; \{y\})}{Q_n(y; \{y\})} < \infty \tag{5.14}$$

and (5.8) *holds, then* (5.5) *holds for any $k > 0$ and any initial point x.*

Proof. Consider first the case $x = y$. Let V_k denote the time of the kth visit to y.[6] Then clearly,

$$P_y(N_n(\{y\}) = k) = \sum_{r=1}^{n} P_y(V_k = r)Q_{n-r}(y; \{y\}) .$$

By (5.8) we have

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{\sum_{r=1}^{m} P_y(V_k = r)Q_{n-r}(y; \{y\})}{Q_n(y; \{y\})} = \sum_{r=1}^{\infty} P_y(V_k = r) = 1 .$$

To complete the proof, we must therefore show that for all $k > 0$,

$$\lim_{m\to\infty}\overline{\lim_{n\to\infty}} \sum_{r=m+1}^{n} P_y(V_k = r)Q_{n-r}(y; \{y\})Q_n(y; \{y\})^{-1} = 0 . \tag{5.15}$$

If $x = y$, then clearly (5.5) holds for $k = 0$. Suppose we have already established that (5.5) holds for all $k < k_0$ when $x = y$. Then,

$$\lim_{n\to\infty} \frac{P_y(N_n(\{y\}) < k_0)}{P_y(N_n(\{y\}) = 0)} = \lim_{n\to\infty} \frac{P_y(V_{k_0} > n)}{Q_n(y; \{y\})} = k_0 . \tag{5.16}$$

Now, decompose the sum in (5.15) as follows:

$$\sum_{r=m+1}^{n} = \sum_{r=m+1}^{[\alpha n]} + \sum_{[\alpha n]+1}^{n-m} + \sum_{n-m+1}^{n} .$$

Since

$$\sum_{m+1}^{[\alpha n]} \leqq \frac{Q_{n-[\alpha n]}(y; \{y\})}{Q_n(y, \{y\})} \sum_{r=m+1}^{[\alpha n]} P_y(V_{k_0} = r) ,$$

we obtain from (5.14)

$$\lim_{m\to\infty}\overline{\lim_{n\to\infty}} \sum_{r=m+1}^{[\alpha n]} = 0 .$$

[6] That is, $V_1 = V_{\{y\}}$, and for $k > 1$, $V_k = \inf \{r > V_{k-1}: X_r = y\}$.

Next,

$$\sum_{r=[\alpha n]+1}^{n-m} \leqq Q_m(y; \{y\})\left[\frac{P_y(V_{k_0} > [\alpha n])}{Q_{[\alpha n]}(y, \{y\})} \frac{Q_{[\alpha n]}(y, \{y\})}{Q_n(y, \{y\})}\right]$$

and thus, by (5.14) and (5.16),

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \sum_{r=[\alpha n]+1}^{n-m} = 0 .$$

Finally, since

$$\sum_{n-m+1}^{n} \leqq \frac{P_y(V_{k_0} > n-m)}{Q_{n-m}(y, \{y\})} \frac{Q_{n-m}(y, \{y\})}{Q_n(y, \{y\})} - \frac{P_y(V_{k_0} > n)}{Q_n(y, \{y\})} ,$$

from (5.8) and (5.16) we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{n-m+1}^{n} = 0 .$$

Thus, when $x = y$, (5.15) holds for $k = k_0$, and consequently (5.5) holds for all $k \geqq 0$. Now for an arbitrary x we have for $k > 0$,

$$P_x(N_n(\{y\}) = k) = \sum_{r=1}^{n} F^r_{\{y\}}(x, y)P_y(N_{n-r}(\{y\}) = k-1) ,$$

and thus

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{\sum_{r=1}^{m} F^r_{\{y\}}(x, y)P_y(N_{n-r}(\{y\}) = k-1)}{Q_n(y, \{y\})} = 1 .$$

To complete the proof, we must therefore show that

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \sum_{r=m+1}^{n} = 0 . \tag{5.17}$$

If $x = y$, then, by what was just shown above, we must have that (5.17) holds. If $x \neq y$, we have

$$F^{n+m_0}_{\{y\}}(y, \{y\}) \geqq {}_yP^{m_0}_{yx} F^n_{\{y\}}(x, y) ,$$

and since $\sum_{n=0}^{\infty} {}_yP^n_{yx} = \pi_x/\pi_y$, we must have ${}_yP^{m_0}_{yx} > 0$ for some m_0. Thus, if we set $\beta = {}_yP^{m_0}_{yx}$, we have

$$\sum_{n=m+1}^{n} \leqq \sum_{r=n+1+m_0}^{n+m_0} \frac{F^r_y(y, \{y\})P_y(N_{n+m_0-r}(\{y\}) = k-1)}{\beta Q_{n+m_0}(y, \{y\})} \frac{Q_{n+m_0}(y, \{y\})}{Q_n(y, \{y\})} ,$$

and accordingly (5.17) holds in general. This completes the proof.

6. Examples. In this section we shall apply the results of the last section to various specific Markov chains. We note again that for the important case of partial sums of independent, identically distributed, integer-lattice-valued random vectors in r dimensions, Kesten and Spitzer [9] have shown that the strong ratio limits (1.1) exist for all nonempty finite sets A.

EXAMPLE 1. *Random Walk on the Nonnegative Integers.* The state space of this chain is the nonnegative integers, and its transition matrix is given for $x \geqq 0$ by

$$(6.1) \qquad P_{xy} = \begin{cases} p_x, & y = x + 1 \,, \\ r_x, & y = x \,, \\ q_x, & y = x - 1 \,, \\ 0, & \text{elsewhere,} \end{cases}$$

where $p_x + q_x + r_x = 1$. For $x \geqq 0$ we have $q_{x+1}, p_x > 0$, while for $x = 0$ we have $q_0 = 0$. Let

$$(6.2) \qquad \pi_x = \frac{p_0 p_1 \cdots p_{x-1}}{q_1 \cdots q_x} \,,$$

and observe that

$$(6.3) \qquad p_x \pi_x = q_{x+1} \pi_{x+1} \,.$$

We gather together below some essential facts about random walks which we shall need. For details we refer the reader to [5].

Associated with each random walk is a sequence of polynomials $Q_x(t)$ on $[-1, 1]$, and a probability distribution $\Psi(t)$ with support on $[-1, 1]$. Each polynomial $Q_x(t)$ is of exact degree x, and the polynomials satisfy the reccurrence relation

$$(6.4) \qquad \begin{aligned} &tQ_x(t) = p_x Q_{x+1}(t) + r_x Q_x(t) + q_x Q_{x-1}(t), \; x \geqq 0 \,, \\ &Q_0(t) = 1 \,, \qquad Q_{-1}(t) = 0 \,. \end{aligned}$$

Moreover, we have

$$(6.5) \qquad P^n_{xy} = \pi_y \int_{-1}^{1} t^n Q_x(t) Q_y(t) d\Psi(t)$$

and, in particular,

$$(6.6) \qquad \delta_{xy} = P^0_{xy} = \pi_y \int_{-1}^{1} Q_x(t) Q_y(t) d\Psi(t) \,.$$

The integrals in the above expressions include the mass m_1 at 1 and m_{-1} at -1. If the walk is *symmetric* (i.e., $r_x = 0$ for all x), then $m_{-1} = m_1$. If the walk is asymmetric, then $m_{-1} = 0$. If the walk is positive recurrent, then $m_1 = (\Sigma_x \pi_x)^{-1}$, while for a walk that is not positive recurrent we have $m_1 = 0$. From (6.3) and (6.4) we obtain

$$(1-t)\pi_x Q_x(t) = -\pi_x p_x(Q_{x+1}(t) - Q_x(t)) + \pi_{x-1}p_{x-1}(Q_x(t) - Q_{x-1}(t))\ ,$$

and thus

$$-(1-t)\sum_{z=0}^{y} \pi_z Q_z(t) = \pi_y p_y(Q_{y+1}(t) - Q_y(t))\ .$$

Another summation yields

$$Q_{x+1}(t) = 1 - (1-t)\sum_{y=0}^{x} (p_y\pi_y)^{-1}\sum_{z=0}^{y} \pi_z Q_z(t)\ . \tag{6.7}$$

Setting $t = 1$ in the above, we see that

$$Q_x(1) = 1 \text{ all } x \geqq 0\ . \tag{6.8}$$

We are now in a position to establish the following important property of random walks.

THEOREM 6.1. *In every null-recurrent random walk and in every positive-recurrent asymmetric random walk, the series*

$$C^*_{xy} = \sum_{n=0}^{\infty} (P^n_{yy} - P^n_{xy})$$

converges. Moreover,

$$C^*_{xy} = \begin{cases} m_1\pi_y \sum_{n=x}^{y-1} (p_n\pi_n)^{-1} \sum_{k=0}^{n} \pi_k\ , & x < y\ , \\ \pi_y \sum_{k=y}^{x-1} (p_k\pi_k)^{-1}\left\{1 - m_1 \sum_{j=0}^{k} \pi_j\right\}\ , & x > y\ , \\ 0\ , & x = y\ . \end{cases} \tag{6.9}$$

Proof. From (6.5) we see that

$$\begin{aligned} &\sum_{n=0}^{N} (P^n_{yy} - P^n_{xy}) \\ &\quad = \pi_y\int_{-1}^{1}(1-t)^{-1}(1-t^{N+1})Q_y(t)[Q_y(t) - Q_x(t)]d\Psi(t)\ . \end{aligned} \tag{6.10}$$

Suppose $y > x$. Then from (6.7) and (6.10) we have

$$\begin{aligned} &\sum_{n=0}^{N} (P^n_{yy} - P^n_{xy}) \\ &\quad = -\pi_y\int_{-1}^{1}(1-t^{N+1})Q_y(t)\sum_{k=x}^{y-1} (p_k\pi_k)^{-1}\sum_{r=0}^{k} \pi_r Q_r(t)d\Psi(t)\ . \end{aligned}$$

By (6.6), we see that

$$\pi_y\int_{-1}^{1} Q_y(t)\sum_{k=x}^{y-1}(p_k\pi_k)^{-1}\sum_{r=0}^{k}\pi_r Q_r(t)d\Psi(t)=0\ ,$$

and thus

$$\sum_{n=0}^{N}(P_{yy}^n-P_{xy}^n) \tag{6.11}$$
$$=\pi_y\int_{-1}^{1} t^{N+1}Q_y(t)\sum_{k=x}^{y-1}(p_k\pi_k)^{-1}\sum_{r=0}^{k}\pi_r Q_r(t)d\Psi(t)\ .$$

From the above, we obtain

$$C_{xy}^*=m_1\pi_y\sum_{n=x}^{y-1}(p_n\pi_n)^{-1}\sum_{k=0}^{n}\pi_k$$
$$+\lim_{N\to\infty}\pi_y m_{-1}(-1)^{N+1}Q_y(-1)\sum_{n=x}^{y-1}(p_n\pi_n)^{-1}\sum_{k=0}^{n}\pi_k Q_k(-1)\ ,$$

and thus C_{xy}^* exists if and only if $m_{-1}=0$. In that case,

$$C_{xy}^*=m_1\pi_y\sum_{n=x}^{y-1}(p_n\pi_n)^{-1}\sum_{k=0}^{n}\pi_k\ .$$

Now suppose $y<x$. A similar computation to that used above then shows that

$$\sum_{n=0}^{N}(P_{yy}^n-P_{xy}^n)$$
$$=\pi_y\int_{-1}^{1}(1-t^{N+1})Q_y(t)\sum_{k=y}^{x-1}(p_k\pi_k)^{-1}\sum_{j=0}^{k}\pi_j Q_j(t)d\Psi(t)\ .$$

From (6.6), we obtain

$$\pi_y\int_{-1}^{1} Q_y(t)\sum_{k=y}^{x-1}(p_k\pi_k)^{-1}\sum_{j=0}^{k}\pi_j Q_j(t)d\Psi(t)=\pi_y\sum_{k=y}^{x-1}(p_k\pi_k)^{-1}\ ,$$

and thus

$$\sum_{n=0}^{N}(P_{yy}^n-P_{xy}^n)$$
$$=\pi_y\sum_{k=y}^{x-1}(p_k\pi_k)^{-1}-\pi_y\int_{-1}^{1} t^{N+1}Q_y(t)\sum_{k=y}^{x-1}(p_k\pi_k)^{-1}\sum_{j=0}^{k}\pi_j Q_j d\Psi$$

Consequently, taking the limit as $N\to\infty$, for $m_{-1}=0$ we obtain

$$C_{xy}^*=\pi_y\sum_{k=y}^{x-1}(p_k\pi_k)^{-1}-\pi_y m_1\sum_{n=y}^{x-1}(p_n\pi_n)^{-1}\sum_{k=0}^{n}\pi_k$$
$$=\pi_y\sum_{n=y}^{x-1}(p_n\pi_n)^{-1}\left\{1-m_1\sum_{k=0}^{n}\pi_k\right\}\ .$$

This establishes the theorem.

As a corollary we have the following result.

COROLLARY 6.1. *Under the same condition as for Theorem* 6.1 *we have, for* $z \neq x$,

$$\lim_{y\to\infty} \pi_y^{-1} G_{\{z\}}(x, y) = \begin{cases} \sum_{n=z}^{x-1} (p_n \pi_n)^{-1} , & z < x , \\ 0 , & z > x . \end{cases} \tag{6.12}$$

Proof. By Theorem 22 of [6], for $z \neq x, y$ we have

$$G_{\{z\}}(x, y) = C^*_{zy} - C^*_{xy} + (\pi_y/\pi_z) C^*_{xz} . \tag{6.13}$$

From (6.9), we see that when $y > \max(x, z)$,

$$\begin{aligned} \frac{C^*_{zy} - C^*_{xy}}{\pi_y} &= m_1 \left\{ \sum_{n=z}^{y-1} - \sum_{n=x}^{y-1} \right\} (p_n \pi_n)^{-1} \sum_{k=0}^{n} \pi_k \\ &= \begin{cases} m_1 \sum_{n=z}^{x-1} (p_n \pi_n)^{-1} \sum_{k=0}^{n} \pi_k , & z < x , \\ -m_1 \sum_{n=x}^{z-1} (p_n \pi_n)^{-1} \sum_{k=0}^{n} \pi_k , & z > x . \end{cases} \end{aligned}$$

Thus if $y > \max(x, z)$, we have

$$\pi_y^{-1} G_{\{z\}}(x, y) = \begin{cases} \sum_{n=z}^{x-1} (p_n \pi_n)^{-1} , & x > z , \\ 0 , & x < z , \end{cases}$$

which establishes (6.12).

REMARK. Observe that in the null-recurrent case the matrix C^*_{xy} is triangular with 0's on and above the main diagonal. Also observe that in the null-recurrent case the limit in (6.12) is C^*_{xz}/π_z, while in the positive-recurrent case this limit is different from the above expression.

Theorem 6.1 shows that the stronger Doeblin form of weak ratio limits always holds for null recurrent random walks. From Corollary 6.1 we see that Theorem 5.2 is applicable to all random walks except symmetric positive-recurrent walks.

EXAMPLE 2. *The Aperiodic Renewal Chain.* This chain also has the nonnegative integers as its state space. Its transition matrix is given by

$$P_{xy} = \begin{cases} p_{x+1} , & y = x + 1 , \\ q_{x+1} = 1 - p_{x+1} , & y = 0 , \\ 0 , & \text{elsewhere,} \end{cases} \tag{6.14}$$

where $0 < p_x < 1$, $x > 0$.

Let

$$\pi_x = p_1 \cdots p_x \,, \qquad (\pi_0 = 1) \,,$$
$$\sigma_x = \sum_{k=0}^{x-1} \pi_k \,, \qquad \lim_{x \to \infty} \sigma_x = \sigma_\infty \,.$$

It was shown in [6] that

$$\text{(6.15)} \qquad C^*_{xt} = \begin{cases} \dfrac{\sigma_t}{\sigma_\infty} - \dfrac{\pi_t}{\pi_x}\dfrac{\sigma_x}{\sigma_\infty} \,, & t \geqq x \,, \\ \dfrac{\pi_t}{\pi_x}\Big(1 - \dfrac{\sigma_x}{\sigma_\infty}\Big) + \dfrac{\sigma_t}{\sigma_\infty} \,, & t < x \,, \end{cases}$$

where $1/\sigma_\infty = 0$ if $\sigma_\infty = \infty$.

If $t > \max(x, y)$, we obtain

$$\frac{C^*_{yt} - C^*_{xt}}{\pi_t} = \frac{1}{\sigma_\infty}\Big(\frac{\sigma_x}{\pi_x} - \frac{\sigma_y}{\pi_y}\Big)$$

and thus, by (6.13),

$$\text{(6.16)} \qquad \begin{aligned} \lim_{t \to \infty} \pi_t^{-1} G_{\{y\}}(x, t) &= \frac{1}{\sigma_\infty}\Big(\frac{\sigma_x}{\pi_x} - \frac{\sigma_y}{\pi_y}\Big) + \frac{C^*_{xy}}{\pi_y} \\ &= \begin{cases} 0 \,, & y \geqq x \,, \\ \dfrac{1}{\pi_x} \,, & y < x \,. \end{cases} \end{aligned}$$

Again observe that this limit is C^*_{xy}/π_y only in the null case. This establishes the following:

THEOREM 6.2. *In any aperiodic, recurrent renewal chain,* C^*_{xy} *exists for all states* x, y. *Moreover if* $y \neq x$,

$$\text{(6.17)} \qquad \lim_{t \to \infty} \pi_t^{-1} G_{\{y\}}(x, t) = \begin{cases} 0 \,, & y > x \,, \\ \pi_x^{-1} \,, & y < x \,. \end{cases}$$

The above theorem shows then that all the stronger forms of the weak ratio limits exist in every renewal chain, and that Theorem 5.2 is applicable. It is quite simple, however, to exhibit renewal chains for which some or none of the strong ratio limit results hold. To see this, all we need do is recall the familiar fact ([1], Sec. 8) that given any sequence $\{W_k\}$ of independent, identically distributed positive integer-valued random variables, we may construct a renewal chain such that the successive returns to 0 have waiting times W_k. Since all the conditions for the various strong ratio limits to hold involve conditions on the tails of the distribution of W_1, and since we may readily construct such distributions which do or do not satisfy these conditions, we see that there are renewal chains for which these strong ratio limits hold, and renewal chains for which they do not hold.

REMARK. The dual of the renewal chain presents another example of a Markov chain with a single boundary point for which series (3.3) always converges.

As our next example, we consider the reflecting-barrier process.

EXAMPLE 3. *Reflecting Barrier Process.* Let U_i be independent, identically distributed, integer-valued random variables assuming positive and negative values and such that $E(e^{i\theta U_1}) = 1$ if and only if $\theta = 2n\pi$, and let

$$T_n = (T_{n-1} + U_n)^+ = \max(0, T_{n-1} + U_n) .$$

It is a familiar fact (see, e.g., [11]) that

$$P(T_n = 0 \mid T_0 = x) = P(M_{n-1} = 0, S_n \leqq -x) , \tag{6.18}$$

where $S_n = U_1 + \cdots + U_n$, and $M_n = \max(0, S_1, \cdots, S_n)$. From (6.18) we see at once that

$$(P_{00}^n - P_{x0}^n) \geqq 0 ,$$

and thus

$$\sum_{n=0}^{\infty} (P_{00}^n - P_{x0}^n)$$

converges. As the T_n process is irreducible, by Lemma 3.1 C_{xy}^* exists for all states x, y. Thus the Doeblin-type ratio limits are always valid for recurrent T_n chains. By results in [12] (see eqn. 4.21) condition (5.1) of Theorem 5.1 is satisfied if and only if

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} P(S_k \leqq 0) = \alpha , \qquad 0 \leqq \alpha < 1 , \tag{6.19}$$

and thus, when the above condition is satisfied, the strong results of Theorem 5.1 are applicable. In particular, if $P(S_k \leqq 0)$ converges to a value <1 (e.g., if U_1 has a symmetric distribution), we see that (6.19) holds.

Since

$$G_{\{0\}}(x, y) = \sum_{n=0}^{\infty} P(S_n = y - x, S_i > -x, 0 \leqq i \leqq n)$$

we see that $G_{\{0\}}(x, y)$ is the same as the Green's functisn for the half line $(-\infty, 0]$ for the S_n process. From results in [13] (see E3 P. 332) it easily follows that the dual boundary of the T_n process has exactly one point whenever the S_n are recurrent. But if the S_n are recurrent, then the T_n must be null recurrent (see [11]), and since $P_{00}^n = P(M_n = 0)$ is clearly a monotone function of n, we see from Theorems 5.2 and 5.3 that

$$\lim_{n\to\infty} \frac{P_x(V_{\{0\}} > n)}{P_0(V_{\{0\}} > n)} = C_{x0}^* + \delta_{x0} \tag{6.20}$$

whenever the S_n are recurrent. In particular, (6.20) holds whenever $EU_1 = 0$.

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