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# TOPOLOGIES FOR LAPLACE TRANSFORM SPACES

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**In this paper four topologies are compared:**
**(i) an $L_2$-type topology on the space of functions having bilateral transforms,**
**(ii) an $L_1$ and (iii) an $L_2$-type topology on the space of transforms, and**
**(iv) finally that of one form of convergence of compact subsets for the space of analytic functions. It is shown that sequential convergence in (i) implies (iii) and (iv) and (ii) implies (i) and (iv) and hence (iii).**

In an earlier paper [2]; the author used equivalence classes of analytic functions to construct an imbedding space for Schwartz Distributions. The mechanism for constructing the mapping was the bilateral Laplace Transform, in this way the traditional approach to operational calculus was preserved. In that paper a topology was imposed on the imbedding space from the space of analytic functions. We now obtain some additional results about the possible topologies defined on the space of analytic functions.

THEOREM 1. *Let $F_j(t), j = 0, 1, 2, \cdots, -\infty < t < \infty$ be real valued functions such that for each $j$*

$$d_1(F_j) = \left[\int_0^\infty |e^{-\sigma_1 t}F_j(t)|^2 dt\right]^{1/2} < \infty$$

*and*

$$d_2(F_j) = \left[-\int_0^{-\infty} |e^{-\sigma_2 t}F_j(t)|^2 dt\right]^{1/2} < \infty$$

*where $-\infty < \sigma_1 < \sigma_2 < \infty$. If*

$$d(F_j - F_0) = d_1(F_j - F_0) + d_2(F_j - F_0) \to 0$$

*as $j \to \infty$ then*

(i) *$f_j(z) \to f_0(z)$ uniformly on compact subsets of $\sigma_1 < R(z) < \sigma_2$ where $f_j(z) = \int_{-\infty}^\infty e^{-zt}F_j(t)dt$*

(ii) $||f_j - f_0||_x \to 0$

*where*

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$$||f_j||_x = \left[\int_{-\infty}^{\infty} |f_j(x+iy)|^2 dy\right]^{1/2}$$

*for each*

$$\sigma_1 < x < \sigma_2 .$$

*Proof.* From [3], pp 245, $d_1 < \infty$, $d_2 < \infty$ implies the existence of $f_j(z)$ with the integral defining $f_j(z)$ converging absolutely for $\sigma_1 < R(z) < \sigma_2$. Rewrite this integral as

$$f_j(z) = \int_{-\infty}^{0} e^{-t(z-\sigma_2)} e^{-\sigma_2 t} F_j(t) dt$$
$$+ \int_{0}^{\infty} e^{-t(z-\sigma_1)} e^{-t\sigma_1} F_j(t) dt .$$

By the Cauchy-Schwarz Inequality

$$|f_j(z)| \leqq \left[\frac{1}{2[\sigma_2 - R(z)]} \int_{-\infty}^{0} |e^{-\sigma_2 t} F_j(t)|^2 dt\right]^{1/2}$$
$$+ \left[\frac{1}{2[R(z) - \sigma_1]} \int_{0}^{\infty} |e^{-\sigma_1 t} F_j(t)|^2 dt\right]^{1/2} .$$

Let

$$g_K(z) = \max_{z \in K} \left[\sqrt{\frac{1}{2[R(z) - \sigma_1]}} , \sqrt{\frac{1}{2[\sigma_2 - R(z)]}}\right]$$

for $K$ an arbitrary subset of $-\sigma_1 < R(z) < \sigma_2$. Now

$$\max_{z \in K} |f_j(z) - f_0(z)| \leqq g_K(z) d(F_j - F_0)$$

and (i) is established.

To prove (ii) we use an additional consequence of the theorem from [3], pp 254, namely for each $j$

$$\int_{-\infty}^{\infty} |e^{-xt} F_j(t)|^2 dt = \int_{-\infty}^{\infty} |f_j(x+iy)|^2 dt$$

for $\sigma_1 < x < \sigma_2$. Since

$$\int_{0}^{\infty} |e^{-xt} F_j(t)|^2 dt \leqq [d_1(F_j)]^2$$

$$\int_{-\infty}^{0} |e^{-xt} F_j(t)|^2 dt \leqq [d_2(F_j)]^2$$

it follows that

$$||f_j - f_0||_x^2 \leqq [d_1(F_j - F_0)]^2 + [d_2(F_j - F_0)]^2 .$$

By definition $d_1(F_j) \geqq 0$, $d_2(F_j) \geqq 0$, therefore $d(F_j - F_0) \to 0$ implies $d_1(F_j - F_0) \to 0$ and $d_2(F_j - F_0) \to 0$ and (ii) is proved.

Although it is clear that $d(F)$ is a norm on the set of functions with $d < \infty$ and $||f||_x$ is a norm on a subset of the functions analytic for $\sigma_1 < R(z) < \sigma_2$, the topology of uniform convergence on compact sets gives only a countably normed space or a Fréchet matric. (See [1], pg. 139.)

THEOREM 2. *Let* $f_j(z)$, $j = 0, 1, 2, \cdots$ *be functions analytic for* $\sigma_1 < R(z) < \sigma_2$ *such that for each* $j$

$$\int_{-\infty}^{\infty} |f_j(x + iy)|\, dy < \infty, \quad \sigma_1 < x < \sigma_2$$

*and* $\lim_{|y|\to\infty} f_j(x + iy) = 0$ *uniformly on closed subintervals of* $\sigma_1 < x < \sigma_2$. *If* $||f_j - f||_x \to 0$ *as* $j \to \infty$ *for each* $\sigma_1 < x < \sigma_2$. *then*

(i) $d'(F_j - F_0) \to 0$ *as* $j \to \infty$ *where*

$$d'(F_j - F_0) = d_1'(F_j - F_0) + d_2'(F_j - F_0)$$

*and*

$$F_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{xt}e^{ity}f_j(x + iy)dy\ , \tag{1}$$

$\sigma_1 < x < \sigma_2$. (T*he prime denotes that the metric is the same as in Theorem* 1, *except that* $\sigma_1$ *is replaced by* $x_1$ *and* $\sigma_2$ *by* $x_2$ *where* $\sigma_1 < x_1 < x_2 < \sigma_2$.)

(ii) $f_j(z) \to f_0(z)$ *uniformly on compact subsets of* $x_1 < R(z) < x_2$.

*Proof*. From [3], pg. 265 the hypotheses of the theorem are sufficient to insure that each $f_i(z)$ is a bilateral transform and also the validity of the inversion formula (1).

Since $|f_j(x + iy)| \to 0$ as $|y| \to \infty$ each $|f_j(x + iy)|$ is bounded for $\sigma_1 < x < \sigma_2$ if $|f_j(x + iy)| \leqq M_j(x)$ then

$$\int_{-\infty}^{\infty} |f_j(x + iy)|^2\, dy \leqq M_j(x)\int_{-\infty}^{\infty} |f_j(x + iy)|\, dy < \infty\ .$$

Consider, for $\sigma_1 < x < x_1$

$$\begin{aligned}\int_0^{\infty} |e^{x_1 t}F_j(t)|^2\, dt &= \frac{1}{(2\pi)^2}\int_0^{\infty}\left|e^{-t(x_1-x)}\int_{-\infty}^{\infty} e^{ity}f_j(x + iy)dy\right|^2 dt \\ &\leqq \frac{1}{(2\pi)^2}||f_j||_x^2\int_0^{\infty} e^{-2t(x_1-x)}dt\end{aligned}$$

but $0 < x_1 - x$ so that

$$C_1 = \frac{1}{2\pi}\int_0^{\infty} e^{-2t(x_1-x)}dt < \infty$$

and $d_1'(F_j) \leqq C_1 \| f_j \|_x$. Likewise

$$\int_{-\infty}^{0} | e^{-x_2 t} F_j(t) |^2 \, dt \leqq \frac{1}{(2\pi)^2} \| f_j \|_{x'}^2 \int_{-\infty}^{0} e^{t(x'-x_2)} dt$$

for $x_2 < x' < \sigma_2$, or

$$d_2'(F_j) \leqq C_2 \| f_j \|_{x'}, \quad \text{where} \quad C_2 = \frac{1}{2\pi} \int_{-\infty}^{0} e^{t(x'-x_2)} dt \ .$$

If $\| f_j - f_0 \|_x \to 0$ as $j \to \infty$ for all $\sigma_1 < x < \sigma_2$ then

$$\begin{aligned} d'(F_j - F_0) &= d_1'(F_j - F_0) + d_2'(F_j - F_0) \\ &\leqq C_1 \| f_j - f_0 \|_x + C_2 \| f_j - f_0 \|_{x'} \end{aligned}$$

or $d'(F_j - F_0) \to 0$.

Part (ii) follows by applying Theorem 1 to part (i).

In a subsequent paper the author expects to extend these to multidimensional Laplace Transforms and to Laplace Transforms on Locally Compact Abelian Groups.

## References

1. W. Thron, *Introduction to the Theory of Functions of a Complex Variable*, John Wiley, New York, 1953.
2. D. E. Myers, *An imbedding space for Schwartz distributions*, Pacific J. Math. **11** (1961), 1467-1477.
3. D. V. Widder, *The Laplace Transform*, Princeton, 1946.

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