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A SUBDETERMINANT INEQUALITY

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Let A be an n-square positive semi-definite hermitian matrix and let $D_m(A)$ denote the maximum of all order m principal subdeterminants of A. In this note we prove the inequality

$$(D_m(A))^{1/m} \geqq (D_{m+1}(A))^{1/(m+1)}\,, \qquad m = 1, \cdots, n-1\,,$$

and discuss in detail the case of equality. This result is closely related to Newton's and Szász's inequalities.

Let $A = (a_{ij})$ be an n-square positive semi-definite hermitian matrix with eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n \geqq 0$. We introduce some notation. For $1 \leqq m \leqq n$ let $Q_{m,n}$ denote the set of all $\binom{n}{m}$ sequences $\omega = (\omega_1, \cdots, \omega_m)$, $1 \leqq \omega_1 < \omega_2 < \cdots < \omega_m \leqq n$. Let $A[\omega \mid \omega]$ denote the m-square submatrix of A whose (i, j) entry is $a_{\omega_i \omega_j}$, $i, j = 1, \cdots, m$.

THEOREM. *If A is a positive semi-definite hermitian matrix then*

$$\begin{aligned} &\max_{\alpha \in Q_{m,n}} (\det (A[\alpha \mid \alpha]))^{1/m} \\ &\qquad \geqq \max_{\omega \in Q_{m+1,n}} (\det (A[\omega \mid \omega]))^{1/m+1}, \qquad m = 1, \cdots, n-1\,. \end{aligned} \tag{1}$$

Equality holds for a given m if and only if either A has rank less than m or $A[\omega^0 \mid \omega^0]$ is a multiple of the identity, where the sequence $\omega^0 \in Q_{m+1,n}$ is one that satisfies

$$\det (A[\omega^0 \mid \omega^0]) = \max_{\omega \in Q_{m+1,n}} \det A[\omega \mid \omega]\,. \tag{2}$$

There are two classical results that are closely related to the inequalities (1). These are Szász's inequalities and the Newton inequalities. Szász proved that [1, p. 119]

$$\begin{aligned} &\left(\prod_{\alpha \in Q_{m,n}} (\det (A[\alpha \mid \alpha]))^{1/\binom{n}{m}}\right)^{1/m} \\ &\qquad \geqq \left(\prod_{\omega \in Q_{m+1,n}} (\det (A[\omega \mid \omega]))^{1/\binom{n}{m+1}}\right)^{1/(m+1)}\,. \end{aligned} \tag{3}$$

Newton's inequalities [1, p. 106] state that if $E_m(A)$ is the mth elementary symmetric function of the nonnegative numbers $\lambda_1, \cdots, \lambda_n$ then

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$$(4) \qquad \left(E_m(A)\Big/\binom{n}{m}\right)^{1/m} \geqq \left(E_{m+1}(A)\Big/\binom{n}{m+1}\right)^{1/(m+1)} .$$

However,

$$(5) \qquad E_m(A) = \sum_{\alpha \in Q_{m,n}} \det(A[\alpha \mid \alpha])$$

and hence (4) can be written

$$(6) \qquad \begin{aligned} &\left(\sum_{\alpha \in Q_{m,n}} \det(A[\alpha \mid \alpha])\Big/\binom{n}{m}\right)^{1/m} \\ &\qquad \geqq \left(\sum_{\omega \in Q_{m+1,n}} \det(A[\omega \mid \omega])\Big/\binom{n}{m+1}\right)^{1/(m+1)} . \end{aligned}$$

Notice that (3) compares the geometric mean of the principal subdeterminants of order m with the geometric mean of the principal subdeterminants of order $m+1$. Also (6) makes the same kind of comparison for the arithmetic means of these quantities. The result (1) compares the maxima of the two sets of subdeterminants.

To prove the theorem we state and prove a preliminary lemma.

LEMMA. *If A is a positive semi-definite n-square hermitian matrix then*

$$(7) \qquad \max_{\alpha \in Q_{m,n}} \det(A[\alpha \mid \alpha]) \geqq (\det(A))^{m/n} , \qquad 1 \leqq m \leqq n .$$

Equality holds if and only if either the rank of A is less than m or A is a multiple of the identity matrix.

Proof. We use some properties of the compound matrix of A, denoted by $C_m(A)$. The essential facts concerning $C_m(A)$ are [1, pp. 17, 24, 70]:

(i) $\det(C_m(A)) = (\det(A))^{\binom{n-1}{m-1}}$ (Sylvester-Franke theorem);

(ii) if A is positive semi-definite hermitian, so is $C_m(A)$;

(iii) the characteristic roots of $C_m(A)$ are the $\binom{n}{m}$ products

$$\prod_{i=1}^{m} \lambda_{\omega_i}, \qquad \omega \in Q_{m,n} .$$

We want to prove that

$$(8) \qquad \max_{\alpha \in Q_{m,n}} \det(A[\alpha \mid \alpha]) \geqq (\det(A))^{m/n} .$$

If we apply the Hadamard determinant theorem [1, p. 114] to $C_m(A)$ then we conclude from (i)

$$(9) \qquad \prod_{\alpha \in Q_{m,n}} \det(A[\alpha \mid \alpha]) \geqq \det(C_m(A)) = (\det(A))^{\binom{n-1}{m-1}} .$$

If for every $\alpha \in Q_{m,n}$, $\det(A[\alpha \mid \alpha])$ were strictly less than $(\det(A))^{m/n}$ then from (9) we would conclude that

$$(\det(A))^{\binom{n-1}{m-1}} < ((\det(A))^{m/n})^{\binom{n}{m}} = (\det(A))^{\binom{n-1}{m-1}}, \tag{10}$$

a contradiction. Thus (8) holds. If (8) were equality suppose first that not all $\det(A[\alpha \mid \alpha])$, $\alpha \in Q_{m,n}$ are equal. Then from (9) we would obtain the same contradiction (10). Thus for equality to hold in (8)

$$\det(A[\alpha \mid \alpha]) = (\det(A))^{m/n}$$

for all $\alpha \in Q_{m,n}$. This means that all the main diagonal elements of $C_m(A)$ are equal. If this common value is 0 then A has rank at most $m-1$. If the common value is nonzero then (9) is equality throughout and as we know from the case of equality in the Hadamard determinant theorem $C_m(A)$ is a multiple of the identity. Thus from (iii) we know that the characteristic roots

$$\prod_{i=1}^{m} \lambda_{\alpha_i}, \quad \alpha \in Q_{m,n}, \quad m < n,$$

are equal. But then it follows that $\lambda_1 = \cdots = \lambda_n$ and hence A is a multiple of the identity, completing the proof of the lemma.

To prove the inequality (1) we apply the lemma to submatrices. Let ω^0 be a sequence in $Q_{m+1,n}$ for which

$$\det(A[\omega^0 \mid \omega^0]) = \max_{\omega \in Q_{m+1,n}} \det(A[\omega \mid \omega]). \tag{11}$$

For $\alpha \in Q_{m,n}$ and α a subsequence of ω^0, i.e., $\alpha \subset \omega^0$, we know that $A[\alpha \mid \alpha]$ is an m-square submatrix of $A[\omega^0 \mid \omega^0]$. Hence, by the lemma,

$$\max_{\alpha \in Q_{m,n}, \alpha \subset \omega^0} \det(A[\alpha \mid \alpha]) \geqq (\det(A[\omega^0 \mid \omega^0]))^{m/(m+1)}. \tag{12}$$

Thus a fortiori

$$\max_{\alpha \in Q_{m,n}} \det(A[\alpha \mid \alpha]) \geqq (\det(A[\omega^0 \mid \omega^0]))^{m/(m+1)}. \tag{13}$$

Applying (11) we obtain the inequality (1) from (13).

If equality holds in (1) then (12) must be equality as well. Therefore either the rank of $A[\omega^0 \mid \omega^0]$ is less than m or $A[\omega^0 \mid \omega^0]$ is a multiple of the $(m+1)$-square identity matrix. If the former is the case then $\det(A[\omega^0 \mid \omega^0]) = 0$ and hence, since (13) is equality, every mth order principal subdeterminant of A is 0. Thus the rank of A is less than m.

REFERENCE

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