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L^2 EXPANSIONS IN TERMS OF GENERALIZED HEAT POLYNOMIALS AND OF THEIR APPELL TRANSFORMS

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The object of this paper is to characterize functions which have L^2 expansions in terms of polynomial solutions $P_{n,\nu}(x, t)$ of the generalized heat equation

$$(*) \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{2\nu}{x}\frac{\partial}{\partial x}\right]u(x, t) = \frac{\partial}{\partial t}u(x, t) .$$

and in terms of the Appell transforms $W_{n,\nu}(x, t)$ of the $P_{n,\nu}(x, t)$. H^* denotes the C^2 class of functions $u(x, t)$ which, for $a < t < b$, satisfy $(*)$ and for which

$$u(x, t) = \int_0^\infty G(x, y; t - t')u(y, t')d\mu(y),$$

$$d\mu(x) = 2^{(1/2)-\nu}\left[\Gamma\left(\nu + \frac{1}{2}\right)\right]^{-1}x^{2\nu}dx ,$$

for all $t, t', a < t' < t < b$, the integral converging absolutely, where $G(x, y; t)$ is the source solution of $(*)$. The principal results are the following:

THEOREM. **Let $u(x, t) \in H^*$, $-\sigma \leqq t < 0$, and**

$$u(x, t)[G(x; -t)]^{\frac{1}{2}} \in L^2$$

for each fixed t $-\sigma \leqq t < 0$, $0 \leqq x < \infty$. Then, for $-\sigma \leqq t < 0$,

$$\lim_{N\to\infty} \int_0^\infty G(x; -t) \left| u(x, t) - \sum_{n=0}^{N} a_n P_{n,\nu}(x, -t)\right|^2 d\mu(x) = 0 ,$$

and

$$\int_0^\infty G(x; -t) \mid u(x, t) \mid^2 d\mu(x) = \sum_{n=0}^{\infty} \mid a_n \mid^2 b_n^{-1} t^{2n} ,$$

where

$$b_n = [2^{4n} n!]^{-1} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} + n\right)} ,$$

and

$$a_n = b_n\int_0^\infty u(y, t) W_{n,\nu}(y, -t)d\mu(y) .$$

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THEOREM. If $u(x, t) \in H^*$, $0 < t \leqq \sigma$, and if

$$u(ix, t)[G(x; t)]^{(1/2)} \in L^2$$

for each fixed t, $0 < t \leqq \sigma$, $0 \leqq x < \infty$, then, for $0 < t \leqq \sigma$,

$$\lim_{N\to\infty} \int_0^\infty G(x; t) \left| u(ix, t) - \sum_{n=0}^{N} a_n P_{n,\nu}(x, -t) \right|^2 d\mu(x) = 0 ,$$

and

$$\int_0^\infty G(x; t) \mid u(ix, t) \mid^2 d\mu(x) = \sum_{n=0}^{\infty} \mid a_n \mid^2 b_n^{-1} t^{2n} ,$$

where b_n is given above and

$$a_n = b_n \int_0^\infty u(ix, t) W_{n,\nu}(x, t) d\mu(x) .$$

THEOREM. If $u(x, t) \in H^*$, $0 < \sigma \leqq t$, and if

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for each fixed t, $0 < \sigma \leqq t$, $0 \leqq x < \infty$, then, for $0 < \sigma \leqq t$,

$$\lim_{N\to\infty} \int_0^\infty G(ix; t) \left| u(x, t) - \sum_{n=0}^{N} a_n W_{n,\nu}(x, t) \right|^2 d\mu(x) = 0 ,$$

and

$$\int_0^\infty G(ix; t) \mid u(x, t) \mid^2 d\mu(x) = \sum_{n=0}^{\infty} t^{-2n} b_n^{-1} (2t)^{-2\nu-1} \mid a_n \mid^2 ,$$

where b_n is given above, and

$$a_n = b_n \int_0^\infty u(x, t) P_{n,\nu}(x, -t) d\mu(x) .$$

The theory is an extension, in part, of recent results of P. C. Rosenbloom and D. V. Widder.

1. Preliminary results. The generalized heat polynomial $P_{n,\nu}(x, t)$ is a polynomial defined by

$$P_{n,\nu}(x, t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} + n - k\right)} x^{2n-2k} t^k , \tag{1.1}$$

ν a fixed positive number. Note that when $\nu = 0$, $P_{n,0}(x, t) = v_{2n}(x, t)$, the ordinary heat polynomials defined in [8; p. 222]. For $t > 0$, $P_{n,\nu}(x, t)$ has the following integral representation.

$$P_{n,\nu}(x, t) = \int_0^\infty y^{2n} G(x, y; t) d\mu(y), \tag{1.2}$$

$$d\mu(y) = 2^{(1/2)-\nu} \left[\Gamma\left(\nu + \frac{1}{2}\right)\right]^{-1} x^{2\nu} dx .$$

As may readily be verified, for $-\infty < x, t < \infty$, $P_{n,\nu}(x, t)$ satisfies the generalized heat equation

$$(1.3)\qquad \Delta_x u(x, t) = \frac{\partial}{\partial t} u(x, t)\ ,$$

where $\Delta_x f(x) = f''(x) + (2\nu/x) f'(x)$. We denote by H the class of all C^2 functions which satisfy (1.3). The source solution of (1.3) is given by $G(x; t)$, where

$$(1.4)\qquad G(x, y; t) = \left(\frac{1}{2t}\right)^{\nu+\frac{1}{2}} \exp\left(-\frac{x^2+y^2}{4t}\right) \mathscr{I}\left(\frac{xy}{2t}\right)\ ,$$

with $\mathscr{I}(z) = C_\nu z^{(1/2)-\nu} I_{\nu-(1/2)}(z)$, $C_\nu = 2^{(1/2)-\nu}\Gamma(\nu + (1/2))$, $I_r(z)$ being the Bessel function of imaginary argument of order r, and where $G(x; t) = G(x, 0; t)$. For a detailed study of the properties of $G(x, y; t)$ see [1].

Corresponding to the generalized heat polynomial $P_{n,\nu}(x, t)$ is its Appell transform $W_{n,\nu}(x, t)$ defined by

$$(1.5)\qquad W_{n,\nu}(x,t) = G(x,t)P_{n,\nu}\left(\frac{x}{t}, -\frac{1}{t}\right),\ t > 0,\ n = 0, 1, 2, \cdots\ ,$$

which is also a solution of (1.3). It follows readily from the definition of $P_{n,\nu}(x, t)$ that

$$(1.6)\qquad W_{n,\nu}(x, t) = t^{-2n}G(x, t)P_{n,\nu}(x, -t),\ t > 0,\ n = 0, 1, 2, \cdots\ .$$

The importance of $P_{n,\nu}(x, t)$ and $W_{n,\nu}(x, t)$ in our theory is that they form a biorthogonal system on $0 \leqq x < \infty$. We have, for $t > 0$,

$$(1.7)\qquad \int_0^\infty W_{n,\nu}(x, t)P_{m,\nu}(x, -t)d\mu(x) = \frac{1}{b_n}\delta_{mn}\ ,$$

where

$$(1.8)\qquad b_n = \Gamma\left(\nu + \frac{1}{2}\right)\Big/\left[2^{4n}n!\,\Gamma\left(\nu + \frac{1}{2} + n\right)\right]\ .$$

A consequence of (1.7) is a fundamental generating function for the biorthogonal set $P_{n,\nu}(x, -t)$, $W_{n,\nu}(x, t)$. We have, for $0 \leqq x, y < \infty$, $-s < t < s$, $s > 0$,

$$(1.9)\qquad G(x, y; s + t) = \sum_{n=0}^{\infty} b_n W_{n,\nu}(y, s)P_{n,\nu}(x, t)\ .$$

2. Inversion. For $t > s$, let us set

$$(2.1)\qquad \mathscr{K}(x, y; s, t) = \sum_{n=0}^{\infty} b_n\left(\frac{t}{s}\right)^{(\nu/2)+(1/4)} e^{(x^2/8t)-(y^2/8s)}\, W_{n,\nu}(x,t)P_{n,\nu}(y, -s)\ ,$$

where b_n is defined by (1.8). Then, as a consequence of the definitions and of (1.9), we have

$$\begin{aligned}(2.2)\quad &\mathscr{K}(x, y; s, t)\\ &= \left(\frac{t}{s}\right)^{(\nu/2)+(1/4)} e^{-(x^2(t-s))/(8t(t+s))}\, G(x\sqrt{2s/(t+s)}, y\sqrt{(t+s)/2s}; t-s)\,.\end{aligned}$$

From the well known properties of $G(x, y; t)$ — see [1; § 4] — the following results are immediate.

LEMMA 2.1.

(2.3) (a) $\mathscr{K}(x, y; s, t) \geqq 0,\ 0 \leqq x, y < \infty,\ s < t$,

(2.4) (b) $\lim_{y\to\infty} \mathscr{K}(x, y; s, t) = 0,\ 0 \leqq x < \infty, s < t$,

(2.5) (c) $\lim_{s\to t^-} \mathscr{K}(x, y; s, t) = 0$ *uniformly* $0 \leqq x, y < \infty$, $|y - x| \geqq \delta > 0$, δ *any fixed positive number.*

(d) *For* x *fixed*, $0 \leqq x < \infty$,

$$(2.6)\qquad \lim_{s\to t_-}\int_a^b \mathscr{K}(x, y; s, t)d\mu(y) = 1\,, \qquad 0 \leqq a < x < b \leqq \infty\,,$$
$$= 0\,, \qquad 0 \leqq a \leqq b < x < \infty\,,$$
$$= 0\,, \qquad 0 \leqq x < a < b \leqq \infty\,.$$

It is now easy to establish the following fundamental inversion theorem.

THEOREM 2.2. *If* φ *belongs to* $L^1(0, \infty)$ *and is continuous at* x, *then*

$$(2.7)\qquad \lim_{s\to t^-}\int_0^\infty \mathscr{K}(x, y; s, t)\varphi(y)d\mu(y) = \varphi(x)\,.$$

3. The Huygens property. A function $u(x, t)$ is said to have the Huygens property for $a < t < b$ if and only if $u(x, t) \in H$ there and for every $t, t', a < t' < t < b$,

$$(3.1)\qquad u(x, t) = \int_0^\infty G(x, y; t - t')u(y, t')d\mu(y)\,,$$

the integral converging absolutely. We denote the class of all functions with the Huygens property by H^*. Functions of class H^* have a complex integral representation as given in the following result.

LEMMA 3.1. *If* $u(x, t) \in H^*$, $a < t < b$, *then for* $a < t < t' < b$,

$$(3.2) \qquad u(x, t) = \int_0^\infty G(ix, y; t' - t)u(iy, t')d\mu(y) .$$

The fact that $P_{n,\nu}(x, t) \in H^*$ for $-\infty < t < \infty$, and $W_{n,\nu}(x, t) \in H^*$ for $0 < t < \infty$ enables us to conclude that certain integrals involving functions of H^* are constant. A general result was proved in [5], but we state here the specific forms required in this theory.

THEOREM 3.2. *If $u(x, -t) \in H^*$ for $0 < t < \infty$, then*

$$(3.3) \qquad \int_0^\infty u(x, -t) W_{n,\nu}(x, t)d\mu(x)$$

is a constant.

THEOREM 3.3. *If $u(x, t) \in H^*$ for $0 < t < \infty$, then*

$$(3.4) \qquad \int_0^\infty u(ix, t) W_{n,\nu}(x, t)d\mu(x)$$

is a constant.

THEOREM 3.4. *If $u(x, t) \in H^*$ for $0 < t < \infty$, then*

$$(3.5) \qquad \int_0^\infty u(x, t)P_{n,\nu}(x, -t)d\mu(x)$$

is a constant.

4. **L^2 expansions.** We establish criteria for a function $u(x, t)$ so that the series $\sum_{n=0}^\infty a_n P_{n,\nu}(x, -t)$ converges in mean, with weight functions $G(x, -t)$, to $u(x, t)$.

THEOREM 4.1. *Let $u(x, t) \in H^*$ for $-\sigma \leqq t < 0$, and*

$$u(x, t)[G(x, -t)]^{1/2} \in L^2$$

for $-\sigma \leqq t < 0$, $0 \leqq x < \infty$. Then, for $-\sigma \leqq t < 0$,

$$(4.1) \qquad \lim_{N\to\infty} \int_0^\infty G(x, -t) \left| u(x, t) - \sum_{n=0}^N a_n P_{n,\nu}(x, -t) \right|^2 d\mu(x) = 0$$

and

$$(4.2) \qquad \int_0^\infty G(x, -t) \mid u(x, t) \mid^2 d\mu(x) = \sum_{n=0}^\infty \frac{\mid a_n \mid^2}{b_n} t^{2n} ,$$

where b_n is given by (1.8) and

$$(4.3) \qquad a_n = b_n \int_0^\infty u(y, t) W_{n,\nu}(y, -t)d\mu(y) .$$

Proof. For t fixed, let $\phi(x, t)$ be a continuous function vanishing outside a finite interval and such that, for $\varepsilon > 0$,

$$(4.4) \qquad \int_0^\infty | u(x, -t)[G(x, t)]^{1/2} - \phi(x, t)|^2 \, d\mu(x) < \varepsilon, \; 0 < t \leqq \sigma .$$

Now set

$$(4.5) \qquad \psi_n(x, t) = P_{n,\nu}(x, -t)[G(x, t)]^{1/2} , \qquad 0 < t \leqq \sigma .$$

Then, by (2.1), we have

$$(4.6) \qquad \mathscr{K}(x, y; s, t) = \sum_{n=0}^\infty b_n t^{-2n} \psi_n(x, t)\psi_n(y, s) ,$$

where b_n is defined by (1.8). Hence

$$\begin{aligned}\int_0^\infty \mathscr{K}(x, y; s, t)\phi(y, t) d\mu(y) &= \int_0^\infty \phi(y, t) d\mu(y) \sum_{n=0}^\infty b_n t^{-2n} \psi_n(x, t)\psi_n(y, s) \\ &= \sum_{n=0}^\infty b_n t^{-2n} \psi_n(x, t) \int_0^\infty \psi_n(y, s)\phi(y, t) d\mu(y) .\end{aligned}$$

If we set

$$(4.7) \qquad A_n(t) = b_n t^{-2n} \int_0^\infty \psi_n(y, t)\phi(y, t) d\mu(y) ,$$

and apply Theorem 2.2, we find that

$$(4.8) \qquad \sum_{n=0}^\infty A_n(t)\psi_n(x, t) = \lim_{s \to t^-} \int_0^\infty \mathscr{K}(x, y; s, t)\phi(y, t) d\mu(y) = \phi(x, t) .$$

If we multiply both sides of (4.8) by $\phi(x, t) d\mu(x)$ and integrate between 0 and ∞, we obtain

$$\sum_{n=0}^\infty A_n(t) \int_0^\infty \psi_n(x, t)\phi(x, t) d\mu(x) = \int_0^\infty \phi^2(x, t) d\mu(x) ,$$

or, by (4.7),

$$(4.9) \qquad \sum_{n=0}^\infty \frac{t^{2n}}{b_n} A_n^2(t) = \int_0^\infty \phi^2(x, t) d\mu(x) .$$

Now, let

$$(4.10) \qquad c_n(t) = b_n t^{-2n} \int_0^\infty u(y, -t)[G(y, t)]^{1/2} \psi_n(y, t) d\mu(y) .$$

Consider

$$(4.11) \qquad I = \int_0^\infty \left\{ u(x, -t)[G(x, t)]^{1/2} - \sum_{k=0}^n c_k(t)\psi_k(x, t) \right\}^2 d\mu(x) .$$

Since, by 1.7, we have

$$(4.12)\qquad \int_0^\infty \psi_n(x, t)\psi_m(x, t)d\mu(x) = \frac{t^{2n}}{b_n}\,\delta_{mn}\ ,$$

with b_n given in (1.8), it follows that

$$\begin{aligned}
I &= \int_0^\infty [u(x, -t)]^2 G(x, t)d\mu(x) - \sum_{k=0}^{n} c_k^2(t)\frac{t^{2k}}{b_k} \\
&\leqq \int_0^\infty [u(x, -t)]^2 G(x, t)d\mu(x) + \sum_{k=0}^{n}\frac{t^{2k}}{b_k}[A_k(t) - c_k(t)]^2 - \sum_{k=0}^{n} c_k^2(t)\frac{t^{2k}}{b_k} \\
&= \int_0^\infty [u(x, -t)]^2 G(x, t)d\mu(x) + \sum_{k=0}^{n}\frac{t^{2k}}{b_k}A_k^2(t) - 2\sum_{k=0}^{n}\frac{t^{2k}}{b_k}A_k(t)c_k(t) \\
&= \int_0^\infty \left\{u(x, -t)[G(x, t)]^{1/2} - \sum_{k=0}^{n} A_k(t)\psi_k(x, t)\right\}^2 d\mu(x) \\
&\leqq 2\int_0^\infty \{u(x, -t)[G(x, t)]^{1/2} - \phi(x, t)\}^2 d\mu(x) \\
&\qquad + 2\int_0^\infty \left\{\phi(x, t) - \sum_{k=0}^{n} A_k(t)\psi_k(x, t)\right\}^2 d\mu(x)\ .
\end{aligned}$$

By (4.4), we have

$$\begin{aligned}
I &< 2\varepsilon + 2\int_0^\infty \phi^2(x, t)d\mu(x) + 2\int_0^\infty \sum_{k=0}^{n} A_k^2(t)\psi_k^2(x, t)d\mu(x) \\
&\qquad - 4\int_0^\infty \phi(x, t)d\mu(x)\sum_{k=0}^{n} A_k(t)\psi_k(x, t) \\
&< 2\varepsilon + 2\int_0^\infty \phi^2(x, t)d\mu(x) + 2\sum_{k=0}^{n} A_k^2(t)\frac{t^{2n}}{b_n} \\
&\qquad - 4\sum_{k=0}^{n} A_k(t)\int_0^\infty \phi(x, t)\psi_k(x, t)d\mu(x) \\
&< 2\varepsilon + 2\left\{\int_0^\infty \phi^2(x, t)d\mu(x) - \sum_{k=0}^{n} A_k^2(t)\frac{t^{2n}}{b_n}\right\}\ .
\end{aligned}$$

It follows, therefore, by (4.9), that if n is sufficiently large, $I < 4\varepsilon$.

Hence

$$(4.13)\qquad \lim_{N\to\infty}\int_0^\infty \left| u(x, -t)[G(x, t)]^{1/2} - \sum_{k=0}^{N} c_k(t)\psi_k(x, t)\right|^2 d\mu(x) = 0\ ,$$

or, by (4.5), we have (4.1) with $c_k(t) = a_k$. Theorem 3.4 establishes the fact that a_k is independent of t.

Parseval's equation (4.2) follows since

$$\begin{aligned}
\int_0^\infty G(x, t)\,|\,u(x, -t)\,|^2\, d\mu(x) &= \int_0^\infty \left|\sum_{n=0}^{\infty} c_n(t)\psi_n(x, t)\right|^2 d\mu(x) \\
&= \sum_{n=0}^{\infty} |\,a_n\,|^2\,\frac{t^{2n}}{b_n}\ ,
\end{aligned}$$

with the last equality a result of (4.12).

An example illustrating the theorem is given by $u(x, t) = e^{a^2t}\mathscr{I}(ax)$. This function satisfies the hypotheses for $-\infty < t < 0$ and we find that

$$\int_0^\infty G(x, t)\mathscr{I}^2(ax)e^{-2a^2t}d\mu(x) = \mathscr{I}(2a^2t) , \qquad 0 < t < \infty , \tag{4.14}$$

whereas

$$\begin{aligned}\sum_{n=0}^\infty | a_n |^2 \frac{t^{2n}}{b_n} &= \sum_{n=0}^\infty b_n(a^2t)^{2n}2^{4n} \\ &= \mathscr{I}(2a^2t) , \qquad 0 < t < \infty ,\end{aligned} \tag{4.15}$$

since

$$\begin{aligned}a_n &= b_n\int_0^\infty e^{-a^2t}\mathscr{I}(ay)W_{n,\nu}(y, t)d\mu(y) , \qquad 0 < t < \infty \\ &= (2a)^{2n}b_n .\end{aligned} \tag{4.16}$$

Although, in this example, $u(x, t) \in H^*$ for $-\infty < t < \infty$, the expansion (4.1) does not hold in the extended strip. Note that, in this case, the requirement that $u(x, t)[G(x, -t)]^{1/2}$ be in L^2 fails for $0 < t < \infty$. A modification of Theorem 4.1 when $u(x, t) \in H^*$ for $0 < t \leqq \sigma$ is given by the following result.

THEOREM 4.2. *If $u(x, t) \in H^*$ for $0 < t \leqq \sigma$, and if*

$$u(ix, t)[G(x, t)]^{1/2} \in L^2$$

for each fixed t, $0 < t \leqq \infty$, $0 \leqq x < \infty$, then for $0 < t \leqq \sigma$,

$$\lim_{N\to\infty}\int_0^\infty G(x, t)\left|u(ix, t) - \sum_{n=0}^N a_nP_{n,\nu}(x, -t)\right|^2 d\mu(x) = 0 , \tag{4.17}$$

and

$$\int_0^\infty G(x, t) \,| u(ix, t) |^2 \, d\mu(x) = \sum_{n=0}^\infty \frac{t^{2n}}{b_n} | a_n |^2 , \tag{4.18}$$

where b_n is given by (1.8) *and*

$$a_n = b_n\int_0^\infty u(ix, t)W_{n,\nu}(x, t)d\mu(x) , \qquad 0 < t \leqq \sigma . \tag{4.19}$$

Proof. As in the preceding proof, we have

$$\lim_{N\to\infty}\int_0^\infty \left|u(ix, t)[G(x, t)]^{1/2} - \sum_{n=0}^N c_n(t)\psi_n(x, t)\right|^2 d\mu(x) = 0 ,$$

with

$$c_n(t) = b_n t^{-2n}\int_0^\infty u(iy, t)[G(y, t)]^{1/2}\psi_n(y, t)d\mu(y) .$$

Hence (4.17) holds with $c_n(t) = a_n$, which, by Theorem 3.5, is independent of t. Further,

$$\begin{aligned}\int_0^\infty G(x, t)\,|\,u(ix, t)\,|^2\, d\mu(x) &= \int_0^\infty \left|\sum_{n=0}^\infty c_n(t)\psi_n(x, t)\right|^2 d\mu(x)\\ &= \sum_{n=0}^\infty \frac{t^{2n}}{b_n}\,|\,a_n\,|^2\end{aligned}$$

which is the Parseval equation (4.18).

The example of the preceding theorem satisfies these hypotheses for $0 < t < \infty$, and we have, for $0 < t < \infty$,

$$\int_0^\infty G(x, t)e^{2a^2t}\mathscr{F}^2(iax)d\mu(x) = \mathscr{F}(2a^2t) ,$$

whereas

$$a_n = b_n\int_0^\infty e^{a^2t}\mathscr{F}(iax)W_{n,\nu}(x, t)d\mu(x) ,$$

so that

$$\sum_{n=0}^\infty \frac{t^{2n}}{b_n}\,|\,a_n\,|^2 = \mathscr{F}(2a^2t) .$$

Criteria for expansions in terms of $W_{n,\nu}(x, t)$ are given in the following result.

THEOREM 4.3. *If* $u(x, t) \in H^*$ *for* $0 < \sigma \leqq t$, *and if*

$$u(x, t)[G(ix, t)]^{1/2} \in L^2$$

for each fixed t, $0 \leqq \sigma < t$, $0 \leqq x < \infty$, *then for* $0 < \sigma \leqq t$,

$$\lim_{N\to\infty}\int_0^\infty G(ix, t)\left|u(x, t) - \sum_{n=0}^N a_n W_{n,\nu}(x, t)\right|^2 d\mu(x) = 0 , \tag{4.20}$$

and

$$\int_0^\infty G(ix, t)\,|\,u(x, t)\,|^2\, d\mu(x) = \sum_{n=0}^\infty \frac{t^{-2n}}{b_n}(2t)^{-2\nu-1}\,|\,a_n\,|^2 , \tag{4.21}$$

where b_n *is given by* (1.8) *and*

$$a_n = b_n\int_0^\infty u(x, t)P_{n,\nu}(x, -t)d\mu(x) \qquad \sigma \leqq t < \infty , \tag{4.22}$$

Proof. Again, as in Theorem 4.1, since $u(x, t)[G(ix, t)]^{1/2} \in L^2$, we have

$$(4.23) \qquad \lim_{N\to\infty}\int_0^\infty \left| u(x, t)[G(ix, t)]^{1/2} - \sum_{n=0}^{N} c_n(t)\psi_n(x, t)\right|^2 d\mu(x) = 0 ,$$

with

$$(4.24) \qquad c_n(t) = b_n t^{-2n}\int_0^\infty u(x, t)[G(ix, t)]^{1/2}\psi_n(x, t)d\mu(x) .$$

Now, (4.23) can be written in the form

$$\lim_{N\to\infty}\int_0^\infty G(ix, t)\left| u(x, t) - \sum_{n=0}^{N} c_n(t)(2t)^{\nu+(1/2)}t^{2n}W_{n,\nu}(x, t)\right|^2 d\mu(x) = 0 ,$$

with (4.24) becoming

$$c_n(t) = b_n t^{-2n}(2t)^{-\nu-(1/2)}\int_0^\infty u(x, t)P_{n,\nu}(x, -t)d\mu(x) .$$

Hence, if we set $a_n = c_n(t)t^{2n}(2t)^{\nu+(1/2)}$, a_n is independent of t, by Theorem 3.6, and (4.20) is established. Moreover, Parseval's formula is

$$\int_0^G u(ix, t) \mid u(x, t) \mid^2 d\mu(x) = \sum_{n=0}^{\infty} \mid c_n^2(t) \mid^2 \frac{t^{2n}}{b_n}$$
$$= \sum_{n=0}^{\infty} t^{-2n}(2t)^{-2\nu-1} \frac{\mid a_n \mid^2}{b_n} .$$

Note that the function $u(x, t) = G(x, k; t)$ satisfies the conditions of the theorem for $0 < t < \infty$. In this case, we have

$$a_n = b_n k^{2n} ,$$

and hence

$$\sum_{n=0}^{\infty} t^{-2n}(2t)^{-2\nu-1}\frac{\mid a_n \mid^2}{b_n} = \left(\frac{1}{2t}\right)^{2\nu+1}\mathscr{I}\left(\frac{k^2}{2t}\right) ,$$

whereas

$$\int_0^\infty G(ix; t) \mid G(x, k; t) \mid^2 d\mu(x) = \left(\frac{1}{2t}\right)^{2\nu+1}\mathscr{I}\left(\frac{k^2}{2t}\right) .$$

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