

ON A COINCIDENCE THEOREM OF F. B. FULLER

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Our objective is the following theorem.

THEOREM (1.1). **Let X denote a space dominated by a finite n -polyhedron K such that $H^n(X; Z)$ is torsion free. Let M be a compact topological n -manifold which is 1-connected and let $f, g: X \rightarrow M$ be two given maps. Then, there is a map $g' \sim g: X \rightarrow M$ such that f and g' are coincidence free if, and only if, the (rational) Lefschetz coincidence class $L(f, g) = 0$.**

If X is a finite n -polyhedron and M is a triangulated, 1-connected compact manifold, the theorem is implicit in the work of F. B. Fuller ([5], [6]). In this case, one shows that the obstruction to deforming $f \times g: K \rightarrow M \times M$ to a map $f' \times g': K \rightarrow M \times M - \Delta$, Δ the diagonal of $M \times M$, is zero. Thus, one obtains maps $f' \sim f, g' \sim g$, such that f' and g' are coincidence free. Then, as shown in Fuller's thesis [6], one observes that since M is a manifold (local homogeneity is all that is necessary), deforming both f and g to obtain coincidence free maps is equivalent to deforming just one of them to achieve coincidence freeness. We will use a direct approach, employing general obstruction theory for deforming cross sections into fiber subspaces, simple techniques in fiber spaces to relax the conditions on X and the fact that the classical computations relating $H^n(M \times M, M \times M - \Delta)$ and $H^n(M \times M)$ where M is a triangulated manifold remain valid for topological manifolds by employing the techniques of tangent fiber spaces [4] or the recent result of Kister [7] that microbundles are bundles. An immediate application of this theorem is an obstruction theory proof of the following converse of the Lefschetz Fixed Point Theorem for compact 1-connected topological manifolds.

COROLLARY (1.2). *Let M denote a compact 1-connected topological manifold, $f: M \rightarrow M$ a given map and L_f the Lefschetz number for f . If $L_f = 0$, there is a map $g, g \sim f$, such that g is fixed point free.*

The corollary for triangulated manifolds, because of our previous remarks, is implicit in Fuller [6]. The triangulated case (which is doubtless classical but hard to find explicitly stated anywhere) may also be shown using the theory of Nielson-Reidemeister fixed point classes and results of Wecken [11]. An alternative proof of the corollary in

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the topological case, without obstruction theory, may be given using Nielsen-Reidemeister fixed point classes, results of Weier [12] and the existence of a local index theory for fixed points in the category of ANR's ([2] or [3]). Finally, we remark that (1.1) was discovered by the author before he became aware of Fuller [6] in order to answer a question, raised to him by E. E. Floyd, which is answered by (1.2). (1.2) is easily seen to be false for polyhedra in general.

Singular homology is employed throughout.

2. Preliminaries. We recall first the necessary obstruction theory [6].

Let $\xi = (E, p, B, F)$, $\xi_0 = (E_0, p_0, B, F_0)$ denote locally trivial fiber spaces over B such that $E_0 \subset E$, $p_0 = p|_{E_0}$ and for each $x_0 \in B$ there is a neighborhood U of x_0 and a homeomorphism of pairs

$$(p^{-1}(U), p_0^{-1}(U)) \xleftarrow{\varphi_U} (U \times F, U \times F_0)$$

such that $p\varphi_U(x, y) = x$, $(x, y) \in U \times F$.

DEFINITION (2.1). (ξ, ξ_0) is called a *locally trivial pair*, with fiber (F, F_0) .

Suppose now that (ξ, ξ_0) is a locally trivial pair, as above, such that the fibre (F, F_0) is n -simple. Then, $\pi_n(F, F_0)$ is a local system of groups on B . Suppose further that B is a locally finite simplicial complex, with k -skeleton denoted by B^k , and $f: B \rightarrow E$ is a given cross section such that $f(B^{n-1}) \subset E_0$. We paraphrase Theorem 3.2 of [6] as follows.

THEOREM (2.2). *There exists an obstruction class*

$$\bar{d}^n(f) \in H^n(B; \pi_n(F, F_0))$$

such that if $\bar{d}^n(f) = 0$, then f is homotopic (rel B^{n-2}) to a cross section $f': B \rightarrow E$ such that $f'(B^n) \subset E_0$.

REMARK (2.3). It is understood that when cross sections are homotopic we mean homotopic in the family of cross sections.

Suppose now we add the assumption that $\pi_k(F, F_0) = 0$ for $k < n$ and $\pi_1(B)$ acts trivially on $\pi_n(F, F_0)$. If $g: B \rightarrow E$ is a given cross section, then by (2.2) there is a cross section $f: B \rightarrow E$, homotopic to g , such that $f(B^{n-1}) \subset E_0$. Lundell [6, p. 167] has computed $\bar{d}^n(f)$ as follows. Designate $\pi_n(F, F_0)$ by π . The Hurewicz homomorphism $h: \pi \rightarrow H_n(F, F_0)$ is an isomorphism. Let $\theta^n \in H^n(F, F_0; \pi)$ denote the fundamental class of (F, F_0) , i.e., the class corresponding to h^{-1} under the universal coefficient isomorphism $H^n(F, F_0; \pi) \approx \text{Hom}(H_n(F, F_0); \pi)$. Consider the diagram

$$\begin{CD} H^n(E, E_0; \pi) @>k^*>> H^n(E; \pi) \\ @V{i^*}VV @VV{f^*}V \\ H^n(F, F_0; \pi) @>>> H^n(B; \pi) \end{CD}$$

where i and k are inclusion maps.

PROPOSITION (2.4) ([6]). $\bar{d}^n(f) = f^*k^*i^{*-1}(\theta^n)$.

Since $f^* = g^*$, $\bar{d}^n(f)$ depends only on g and we may employ the notation $\bar{d}^n(f) = \bar{d}^n(g)$ and state the following.

PROPOSITION (2.5). Given a locally trivial pair (ξ, ξ_0) satisfying all the assumptions imposed above, then a given cross section $g: B \rightarrow E$ is homotopic to a cross section $g': B \rightarrow E$ such that $g'(B^n) \subset E_0$ if, and only if, $\bar{d}^n(g) = g^*k^*i^{*-1}(\theta^n) = 0$.

Let $\xi = (E, p, B, F)$ and $\xi_0 = (E_0, p_0, B, F_0)$, where (ξ, ξ_0) is a locally trivial pair. Let $\varphi: K \rightarrow B$ denote a map. Then, φ induces a locally trivial pair $(\xi', \xi'_0) = \varphi^{-1}(\xi, \xi_0)$ over K in the usual manner as follows. $\xi' = (E', p', K, F')$, where

$$\begin{aligned} E' &= \{(x, y) \in K \times E: \varphi(x) = p(y)\}, \\ p'(x, y) &= x. \end{aligned}$$

$\xi'_0 = (E'_0, p'_0, K, F'_0)$, where

$$\begin{aligned} E'_0 &= \{(x, y) \in K \times E_0: \varphi(x) = p_0(y)\} \\ p'_0(x, y) &= x. \end{aligned}$$

Suppose $g: B \rightarrow E$ is a given cross section in ξ . Then, $g'(x) = (x, g\varphi(x))$, $x \in K$, defines a corresponding cross section in g' . Letting $\bar{\varphi}(x, y) = y$ we have the following commutative diagram

$$\begin{CD} E' @>\bar{\varphi}>> E \\ @V{p'}VV @VV{p}V \\ K @>\varphi>> B \end{CD}$$

and $\bar{\varphi}(E'_0) \subset E_0$.

LEMMA (2.6). Suppose $\gamma_0: B \rightarrow E_0$ is a map (not necessarily a cross section) such that $g \sim \gamma_0$ (as maps). If (ξ, ξ_0) is a fibered pair in the sense of [1] (e.g. if B is paracompact), g is homotopic to a cross section $g_0: B \rightarrow E_0$.

Proof. This lemma is a special case of Lemma (5.1) given in § 5.

LEMMA (2.7). Suppose $\varphi: K \rightarrow B$ is a dominating map, i.e., there is a map $\psi: B \rightarrow K$ such that $\varphi\psi \sim 1$, and (ξ, ξ_0) is a fibered pair. Then, there is a cross section $g_0: B \rightarrow E_0, g_0 \sim g$, if, and only if, there is a cross section $g'_0: K \rightarrow E'_0, g'_0 \sim g'$.

Proof. The necessity is simple and hence we confine ourselves to the sufficiency argument. Suppose, therefore, that there is a cross section $g'_0: K \rightarrow E'_0$ such that $g'_0 \sim g'$. Let $\gamma = \bar{\varphi}g'\psi$ and $\gamma_0 = \bar{\varphi}g'_0\psi$. Obviously $\gamma \sim \gamma_0$. On the other hand, $\gamma = g\varphi\psi \sim g$. Therefore, $g \sim \gamma_0, \gamma_0: B \rightarrow E_0$. Lemma (2.6) applies to give the cross section $g_0: B \rightarrow E_0, g_0 \sim g$.

Continuing with the above notation where (ξ, ξ_0) is locally trivial fibered pair and $g: B \rightarrow E$ is a given cross section let us assume further

- (i) $\pi_i(F, F_0) = 0 \ i < n, \pi_n(F, F_0)$ is simple ,
- (ii) $\pi_i(B)$ acts trivially on $\pi_n(F, F_0)$,
- (A) (iii) there is a dominating map $\varphi: K \rightarrow B$ where K is a finite polyhedron of dimension $\leq n$.

We have the following commutative diagram where (ξ', ξ'_0) is the locally trivial fibered pair induced over K by φ, g' is the cross section in ξ' associated with g as above, and $\pi = \pi_n(F, F_0)$.

$$(B) \quad \begin{array}{ccccc} & & H^n(E', E'_0; \pi) & \xrightarrow{k'^*} & H^n(E'; \pi) & \xrightarrow{g'^*} & H^n(K; \pi) \\ & \swarrow i'^* & \uparrow \bar{\varphi}^* & & \uparrow \bar{\varphi}^* & & \uparrow \varphi^* \\ H^n(F, F_0; \pi) & \xleftarrow{\approx} & & & & & \\ & \nwarrow i^* & H^n(E, E_0; \pi) & \xrightarrow{k^*} & H^n(E; \pi) & \xrightarrow{g^*} & H^n(B; \pi) \end{array}$$

i, i', k, k' are inclusion maps.

PROPOSITION (2.8). The cross section $g: B \rightarrow E$ is homotopic to a cross section $g_0: B \rightarrow E_0$ if, and only if, $g^*k^*i^{*-1}(\theta^n) = 0$, where θ^n is the fundamental class of (F, F_0) .

Proof. The assumptions (A) tell us that, via (2.5), the associated cross section $g': K \rightarrow E'$ is homotopic to a cross section $g'_0: K \rightarrow E'_0$ if, and only if, $\bar{d}^n(g') = g'^*k'^*i'^{*-1}(\theta^n) = 0$. Thus (2.7) implies that there is a cross section $g_0: B \rightarrow E_0, g \sim g_0$ if, and only if, $g^*k^*i^{*-1}(\theta^n) = 0$. But diagram (B) gives

$$\varphi^*g^*k^*i^{*-1}(\theta^n) = g'^*k'^*i'^{*-1}(\theta^n) .$$

Since φ^* is injective the proposition follows.

The preceding material will be applied to the following situation.

Let M denote a topological n -manifold (connected, separable metric). Let $\xi = (M \times M, p, M, M)$, where $p(b, y) = b$ and

$$\xi_0 = (M \times M - \Delta, p_0, M, M - b)$$

where Δ is the diagonal in $M \times M$ and p_0 is the restriction of p . Then, (ξ, ξ_0) is a locally trivial fibered pair with fiber $(M, M - b)$, $b \in M$. Let R denote a principal ideal domain and suppose M is R -orientable, i.e., $\pi_1(M)$ acts trivially on $H_n(M, M - b; R)$ or, alternatively, (ξ, ξ_0) is R -orientable. Then, the inclusion map $i: (M, M - b) \rightarrow (M \times M, M \times M - \Delta)$ induces an isomorphism (see [4])

$$i^*: H^n(M \times M, M \times M - \Delta; R) \xrightarrow{\approx} H^n(M, M - b; R) \approx R.$$

Suppose now that $f: X \rightarrow M$ is any given map where X is an arbitrary space. Let $E = X \times M$, $E_0 = X \times M - \Delta_f$, where Δ_f is the graph of f . Let $q: E \rightarrow X$ be projection on the first factor and $q_0 = q|_{E_0}$. For fixed $x \in X$, $q^{-1}(x) = x \times M = M$ and $q_0^{-1}(x) = x \times (M - f(x)) = M - f(x)$. Let $\xi(f) = (E, q, X, M)$, $\xi_0(f) = (E_0, q_0, X, M - f(x))$.

PROPOSITION (2.9). $(\xi(f), \xi_0(f))$ is a locally trivial fibered pair with fiber $(M, M - f(x))$. If (ξ, ξ_0) is R -orientable, so is $(\xi(f), \xi_0(f))$.

Proof. $(\xi(f), \xi_0(f))$ may be identified with $f^{-1}(\xi, \xi_0)$.

REMARK (2.10). There is a natural map $\bar{f}: E = X \times M \rightarrow M \times M$ given by $\bar{f}(x, y) = (f(x), y)$ which gives rise to a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & M \times M \\ q \downarrow & & \downarrow p \\ X & \xrightarrow{f} & M \end{array}$$

and $\bar{f}(E_0) = \bar{f}(X \times M - \Delta_f) \subset M \times M - \Delta$.

If M is 1-connected and $\dim M \geq 3$, then $M - b$ is 1-connected and $\pi_k(M, M - b) = 0$ for $k < n$ and $\pi_n(M, M - b)$ is a simple system. If furthermore, X is dominated by a finite polyhedron of dimension $\leq n$, then $(\xi(f), \xi_0(f))$ satisfies the conditions in (A).

REMARK (2.11). If M is a compact, 1-connected 2-manifold, then $M = S^2$ (the 2-sphere) and the preceding remarks remain valid.

3. Some computations. We discuss first Lefschetz coincidence theory in this setting.

Let M denote an orientable (= Z -orientable) compact topological

n -manifold, R any principal ideal domain. Let $\mu \in H^n(M; R)$, $\bar{\mu} \in H_n(M; R)$ denote fundamental classes corresponding to a given orientation of M . Then for $b \in M$, the inclusion map $j: M \rightarrow (M, M - b)$ induces an isomorphism [8]

$$H^n(M; R) \xleftarrow[\cong]{j^*} H^n(M, M - b; R)$$

and we define U_b by $j^*(U_b) = \mu$. Now consider the fibered pair (ξ, ξ_0) where $\xi = (M \times M, p, M, M)$, $\xi_0 = (M \times M - \Delta, p_0, M, M - b)$ (see § 2) which gives rise to the diagram

$$H^n(M, M - b; R) \xleftarrow[\cong]{i^*} H^n(M \times M, M \times M - \Delta; R) \xrightarrow{k^*} H^n(M \times M; R),$$

where i and k are inclusions. Define U by $i^*(U) = U_b$ and set $\underline{U} = k^*(U)$. This class \underline{U} is studied by Milnor [7] in case R is a field and M is an R -orientable differentiable manifold. The techniques carry over to the topological situation by using the techniques of tangent fiber spaces [4] or a recent result of J. Kister [7] and we will make use of this fact. For convenience we refer to \underline{U} as the *basic R -class* for M . It is well-defined up to orientation

DEFINITION (3.1). Let M denote an R -orientable compact topological manifold with basic R -class \underline{U} . Let $f, g: X \rightarrow M$ denote two maps from a space X to M . These maps induce $f \times g: X \times X \rightarrow M \times M$. Furthermore, let $\delta: X \rightarrow X \times X$ denote the diagonal map. The class

$$\delta^*(f \times g)^*(\underline{U}) = L(f, g; R)$$

is called the *R -Lefschetz coincidence class* for f and g .

REMARK (3.2). When $R = Q$, the field of rationals, $L(f, g; Q)$ is the Lefschetz class usually considered and we distinguish it by the notation $L(f, g)$.

The following theorem in which we employ the notation in (3.1) is immediate.

THEOREM (Lefschetz). *If f and g are coincidence free, then $L(f, g; R) = 0$.*

Let us assume now that R is a field and M is R -orientable as above. Following Milnor [7] arrange a basis

$$(1) \quad 1 = \alpha_1, \alpha_2, \dots, \alpha_N = \mu.$$

for $H^*(M; R)$ in increasing order of dimension. We find it also convenient to introduce the notation

(2) $K_p =$ integer subscripts in (1) of p -dimensional elements.

Then, the basic R -class \underline{U} takes on the form

$$(3) \quad \underline{U} = \sum c_{ij} \alpha_i \times \alpha_j = \sum_p \left(\sum_{i \in K_p} \sum_{j \in K_{n-p}} c_{ij} \alpha_i^p \times \alpha_j^{n-p} \right),$$

where the matrix

$$(4) \quad C_{p,n-p} = \| c_{ij} \|, \quad (i, j) \in K_p \times K_{n-p}$$

is square and if

$$(5) \quad Y_{n-p,p} = \| y_{ij} \|, \quad y_{ij} = \langle \alpha_i \cup \alpha_j, \bar{\mu} \rangle, \quad (i, j) \in K_{n-p} \times K_p,$$

we have

$$(6) \quad (-1)^{np} C_{p,n-p} Y_{n-p,p} = I \text{ (an identity matrix).}$$

Define

$$(7) \quad \beta_j^{n-p} = (-1)^p \sum_{k \in K_{n-p}} c_{jk} \alpha_k^{n-p}, \quad j \in K_p.$$

Then it is easily verified that

$$(8) \quad \langle \alpha_i^p \cup \beta_j^{n-p}, \bar{\mu} \rangle = \delta_{ij}, \quad i, j \in K_p,$$

so that the β 's form a dual basis. A simple computation now gives the classical formula

$$(9) \quad L(f, g; R) = \sum_p \left((-1)^p \sum_{i \in K_p} f^*(\alpha_i^p) \cup g^*(\beta_i^{n-p}) \right)$$

where f, g are as in (3.1). If $X = M, f = 1$, and R is the field of rationals, another straightforward computation shows

$$(10) \quad L(1, g) = (-1)^n \left(\sum_p (-1)^p \text{tr}(g_p^*) \right) \mu = (-1)^n L_g \mu$$

where $\text{tr} g_p^*$ is the trace of g^* in dimension p and $L_g = \sum_p (-1)^p \text{tr}(g_p^*)$ is the usual Lefschetz number for g .

REMARK (3.2). It is clear that the preceding material is applicable to compact topological manifolds which are not necessarily orientable if Z_2 coefficients are employed.

4. Proof of Theorem (1.1). The necessity is immediate and we confine ourselves to the "if part."

Let (ξ, ξ_0) denote the fibered pair in § 3, i.e., $\xi = (M \times M, p, M, M)$, $\xi_0 = (M \times M - \mathcal{A}, p_0, M, M - b)$. The given map $f: X \rightarrow M$ induces the fiber space $(\xi(f), \xi_0(f))$ over X (§ 2) which satisfies conditions (A)

(§ 2). The fiber of $(\xi(f), \xi_0(f))$ over $x \in X$ is $(M, M - b)$, $b = f(x)$. Letting $\xi(f) = (E, q, X, M)$, $\xi_0(f) = (E_0, q_0, X, M - b)$, we have the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & M \times M \\ q \downarrow & & \downarrow p \\ X & \xrightarrow{f} & M \end{array}$$

where $\bar{f}(x, y) = (f(x), y)$. The other given map $g: X \rightarrow M$ induces a cross section $\tilde{g}: X \rightarrow E$ by $\tilde{g}(x) = (x, g(x))$, $x \in X$. We note at this point that $\bar{f}\tilde{g} = (f \times g)\delta$, where $\delta: X \rightarrow X \times X$ is the diagonal map and $f \times g: X \times X \rightarrow M \times M$ is the product map. Our objective is to deform the cross section \tilde{g} into $E_0 = X \times M - \Delta_f$. Consider the commutative diagram

$$\begin{array}{ccccc} & & H^n(E, E_0; \pi) & & \xrightarrow{\tilde{k}^*} H^n(E; \pi) \\ & \swarrow \tilde{i}^* & \uparrow \bar{f}^* & & \searrow \tilde{g}^* \\ H^n(M, M - b; \pi) & & & & H^n(X; \pi) \\ & \swarrow i^* & & & \\ & & H^n(M \times M, M \times M - \Delta; \pi) & \xrightarrow{k^*} & H^n(M \times M; \pi) \end{array}$$

where $i, \tilde{i}, k, \tilde{k}$ are inclusions and $\pi = \pi_n(M, M - b) \approx Z$. If θ^n is the fundamental class of $(M, M - b)$, we need to show, by (2.8), that $\tilde{g}^*\tilde{k}^*\tilde{i}^{*-1}(\theta^n) = 0$. We may choose an orientation $\mu \in H^n(M; \pi)$ of M so that $j^*(\theta^n) = \mu$ where $j: M \rightarrow (M, M - b)$ is the inclusion map. This orientation determines

$$\underline{U} \in H^n(M \times M, M \times M - \Delta; \pi) \quad \text{and} \quad \underline{U} \in H^n(M \times M; \pi)$$

such that $\underline{U} = k^*i^{*-1}(\theta^n)$. The corresponding π -Lefschetz class is given by $\delta^*(f \times g)^*(\underline{U}) = L(f, g; \pi)$. On the other hand

$$\begin{aligned} \tilde{g}^*\tilde{k}^*\tilde{i}^{*-1}(\theta^n) &= \tilde{g}^*\tilde{k}^*\tilde{i}^{*-1}i^*(U) = \tilde{g}^*\tilde{k}^*\bar{f}^*(U) \\ &= \tilde{g}^*\bar{f}^*k^*(U) = \tilde{g}\bar{f}^*(\underline{U}) = \delta^*(f \times g)^*(\underline{U}) . \end{aligned}$$

Therefore, $L(f, g; \pi) = \tilde{g}^*\tilde{k}^*i^{*-1}(\theta^n)$ and hence the integral Lefschetz class is precisely the obstruction class in question. Now, if Q is the field of rationals, the inclusion homomorphism $l: \pi \rightarrow Q$ induces $l^*: H^n(X; \pi) \rightarrow H^n(X; Q)$. It is clear that $l^*(L(f, g; \pi)) = L(f, g; Q) = L(f, g)$, where the Lefschetz class $L(f, g)$ is chosen in terms of the orientation μ . But the assumption that $H^n(X; \pi)$ is torsion free together with the fact that X is dominated by a finite polyhedron implies that l^* is injective. Therefore, $L(f, g) = 0$ implies $L(f, g; \pi) = 0$, which in turn implies $\tilde{g}\tilde{k}^*i^{*-1}(\theta^n) = 0$ and the theorem follows.

Since a compact topological n -manifold is always dominated by a finite polyhedron of dimension n , we have the following corollary, which is a converse to the Lefschetz Fixed Point Theorem.

COROLLARY (4.1). *Let M denote a compact, 1-connected topological manifold, $f: M \rightarrow M$ a given map and L_f the Lefschetz number for f . If $L_f = 0$, there is a map $g, g \sim f$, such that g is fixed point free.*

Proof. By (10) of § 3, $L(1, f) = (-1)^n L_f \mu$, where $n = \dim M$. If $L_f = 0$, then $L(1, f) = 0$ and $f \sim g$ such that 1 and g have no coincidences, i.e., g has no fixed points.

Corollary (4.1) may be reformulated as follows. Let α denote a homotopy class of maps $M \rightarrow M$. Call α *fixed point free* if α contains a fixed point free representative. Also define L_α by $L_\alpha = L_f, f \in \alpha$.

COROLLARY (4.2). *If M is a compact, 1-connected topological manifold and $\alpha: M \rightarrow M$ is a homotopy class of maps, then α is fixed point free if, and only if, $L_\alpha = 0$.*

5. Appendix.

LEMMA (5.1). *Let (ξ, ξ_0) denote a locally trivial fibered pair where $\xi = (E, p, B, F)$, $\xi_0 = (E_0, p_0, B, F_0)$. Suppose we are given a commutative diagram of maps*

$$\begin{array}{ccc}
 & & E \\
 & \nearrow F & \downarrow p \\
 X & \xrightarrow{f} & B
 \end{array}$$

and a map $G: X \rightarrow E_0$ such that $F \sim G$. Then, there is a homotopy $H: X \times I \rightarrow E$ such that $H_1 = F, pH_t = f, 0 \leq t \leq 1$, and $H_0: X \rightarrow E_0$.

Proof. Since (ξ, ξ_0) is a fibered pair in the sense of [1], there is a regular lifting function λ for ξ which works simultaneously for ξ_0 , i.e., if $e_0 \in E_0$ and $\omega \in B^I$, then $\lambda(e_0, \omega)$ is a path in E_0 . By assumption there is a homotopy $K: X \times I \rightarrow E$ such that $K_0 = G$ and $K_1 = F$. K induces a map $\tilde{K}: X \rightarrow E^I$ in the usual manner with $\tilde{K}(x)(t) = K(x, t)$, $0 \leq t \leq 1$. Let $\alpha(x) \in B^I$ denote the path $p(x)$, and observe that $\alpha(x)(t) = p(K(x, t))$, $0 \leq t \leq 1$, and, in particular, $\alpha(x)(1) = f(x)$. For $0 \leq s \leq t$, let $\alpha(x)_s$ denote the path given by $\alpha(x)_s(t) = \alpha(x)(s + t - st)$, $0 \leq t \leq 1$. $\alpha(x)_s$ is a path with initial point $p(H(x, s))$ and terminal point $f(x)$. The required homotopy H is given by

$$H(x, s) = \lambda[K(x, s)\alpha(x)_s](1), \quad 0 \leq s \leq 1.$$

Bibliography

1. G. Allaud and E. Fadell, *A fiber homotopy extension theorem*, Trans. Amer. Math. Soc. **104** (1962), 239-251.
2. D. G. Bourgin, *Modern Algebraic Topology*, MacMillan (1963).
3. F. E. Browder, *On the fixed point index for continuous mappings of locally connected spaces*, Summa Brasil. Math. **4** (1960), 253-293.
4. E. Fadell, *Generalized normal bundles for locally flat imbeddings*, to appear in Trans. Amer. Math. Soc. **114** (1965), 488-513.
5. F. B. Fuller, *The homotopy theory of coincidences*, Ann. of Math. **59** (1954), 219-226.
6. ———, *The Homotopy Theory of Coincidences*, Princeton Thesis, 1951.
7. J. Kister, *Microbundles are bundles*, Ann. Math. **80**(1964), 190-199.
8. A. T. Lundell, *Obstruction theory of principal fiber bundles*, Trans. Amer. Math. Soc. **97** (1960), 161-192.
9. J. Milnor, *Lectures on Characteristic Classes* (Mimeographed Notes), Princeton, 1957.
10. D. Puppe, *Topologie II*, Universität Bonn, 1960.
11. F. Wecken, *Fixpunktklassen III*, Math. Annalen **118** (1941-43), 544-577.
12. J. Weier, *Fixpunkttheorie in topologischen Mannigfaltigkeiten*, Math. Zeit. **59** (1953), 171-190.

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