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# SYMMETRIC DUAL NONLINEAR PROGRAMS

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**Consider a function $K(x, y)$ continuously differentiable in $x \in R^n$ and $y \in R^m$. We form two problems:**

**PRIMAL: Find $(x, y) \geqq 0$ and Min $F$ such that**

$$F = K(x, y) - y^T D_y K(x, y), \; D_y K(x, y) \leqq 0$$

**DUAL: Find $(x, y) \geqq 0$ and Max $G$ such that**

$$G = K(x, y) - x^T D_x K(x, y), \; D_x K(x, y) \geqq 0$$

**where $D_y K(x, y)$ and $D_x K(x, y)$ denote the vectors of partial derivatives $D_{y_i} K(x, y)$ and $D_{x_j} K(x, y)$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Our main result is the existence of a common extremal solution $(x_0, y_0)$ to both the primal and dual systems when (i) an extremal solution $(x_0, y_0)$ to the primal exists, (ii) $K$ is convex in $x$ for each $y$, concave in $y$ for each $x$ and (iii) $K$, twice differentiable, has the property at $(\mathbf{x}_0, y_0)$ that its matrix of second partials with respect to $y$ is negative definite.**

$F$ and $G$ are related to the conjugate functions considered by Fenchel [6] (see also [9]) and to the Legendre transforms of $K$.

*Special Cases.*

A. If we set $K(x, y) = c^T x + b^T y - y^T A x$, we obtain von Neumann's symmetric formulation of primal and dual linear programs: see [3] or [9].

PRIMAL: Find $x \geqq 0$, Min $F$ such that $F = c^T x$, $Ax \geqq b$

DUAL: Find $y \geqq 0$, Max $G$ such that $G = b^T y$, $A^T y \leqq c$ .

B. Symmetric dual quadratic programs [2] can be obtained by setting

$$K(x, y) = c^T x + b^T y - y^T A x + \frac{1}{2}(x^T D x - y^T E y) .$$

The dual quadratic programs of Dorn [4] result when $E = 0$.

C. If we set $K(x, y) = F_0(x) - \sum_{i=1}^{m} y_i F_i(x)$, we obtain the nonlinear statements of duality by Wolfe [12] and Huard [7]. For $y$ not sign restricted, the problems may be simplified to

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PRIMAL: Find $x \geqq 0$ and Min $F_0(x)$ such that
$F_i(x) = 0,\ i = 1, \cdots, m$ .

DUAL: Find $x \geqq 0$, $y$ and Max $K(x, y)$ such that
$D_{x_j}K(x, y) \geqq 0,\ j = 1, \cdots, n$ .

*A "Weak" Duality Theorem.*

With additional conditions on $K(x, y)$ we can state a relation between solutions of the primal and dual systems.

THEOREM 1. *If $K(x, y)$ is convex in $x$, for each $y$, and concave in $y$, for each $x$, then any (not necessarily extremal) solutions $(x, y)$ of the primal and $(u, v)$ of the dual satisfy the inequality*

$$F(x, y) \geqq G(u, v)$$

*where equality holds only if $(x, v)$ is orthogonal to*

$$[D_u K(u, v), - D_y K(x, y)] .$$

*Proof.* By the assumptions of convexity, concavity, and differentiability

$$K(x, v) - K(u, v) \geqq (x - u)^T D_u K(u, v)$$
$$K(x, v) - K(x, y) \leqq (v - y)^T D_y K(x, y) .$$

Subtracting and rearranging, we get

$$[K(x, y) - y^T D_y K(x, y)] - [K(u, v) - u^T D_u K(u, v)]$$
$$\geqq x^T D_u K(u, v) - v^T D_y K(x, y) \geqq 0 .$$

REMARK. The significance of Theorem 1 is apparent when in addition it is also true that $F(x^0, y^0) = G(u^0, v^0)$ for some $(x^0, y^0)$ and $(u^0, v^0)$; in that case, $(x^0, y^0)$ and $(u^0, v^0)$ are extremal solutions of the primal and dual, respectively. This is illustrated in the theorem below.

*A Strong Duality Theorem.*

Under certain assumptions, we can show that if one of the systems can be extremized, the two systems possess a solution in common. (References on duality, in addition to those mentioned above, are [11], [5], [1].)

THEOREM 2. *If $(x^0, y^0)$ is an extremal solution for the primal where $K(x, y)$ is twice differentiable and $- D_{yy}K(x^0, y^0)$ is positive definite, then $(x^0, y^0)$ satisfies the dual constraints with $F(x^0, y^0) = G(x^0, y^0)$. If, in addition, $K(x, y)$ is convex in $x$, for each $y$, and concave in $y$, for each $x$, then*

$$\text{Min}\, F(x, y) = F(x^0, y^0) = G(x^0, y^0) = \text{Max}\, G(x, y) .$$

*Proof.* By a result of John [8, Theorem 1], see also Kuhn and Tucker [10], there exist "multipliers" $v_0 \geqq 0$ (scalar) and $v \geqq 0$ ($m$-vector) such that $(v_0, v) \neq 0$, $v^T D_y(x^0, y^0) = 0$, and

$$\begin{aligned} &v_0 D_y F(x^0, y^0) + D_{yy} K(x^0, y^0) v \geqq 0 \\ &(y^0)^T [v^0 D_y F(x^0, y^0) + D_{yy} K(x^0, y^{0)} v] = 0 \\ &v_0 D_x F(x^0, y^0) + D_{xy} K(x^0, y^0) v \geqq 0 \\ &(x^0)^T [v_0 D_x F(x^0, y^0) + D_{xy} K(x^0, y^0) v] = 0 . \end{aligned}$$

Replacing $F$ by its definition, $K(x, y) - y^T D_y K(x, y)$, yields

$$D_{yy} K(x^0, y^0)(v - v_0 y^0) \geqq 0 \tag{1}$$

$$(y^0)^T D_{yy} K(x^0, y^0)(v - v_0 y^0) = 0 \tag{2}$$

$$v_0 D_x K(x^0, y^0) + D_{xy} K(x^0, y^0)(v - v_0 y^0) \geqq 0 \tag{3}$$

$$(v_0 x^0)^T D_x K(x^0, y^0) + (x^0)^T D_{xy} K(x^0, y^0)(v - v_0 y^0) = 0 . \tag{4}$$

Multiplying (1) by $v^T \geqq 0$, (2) by $v_0$, and then subtracting, we obtain

$$(v - v_0 y^0)^T D_{yy} K(x^0, y^0)(v - v_0 y^0) \geqq 0 .$$

But since the matrix $-D_{yy} K(x^0, y^0)$ is positive definite, we have $v - v_0 y^0 = 0$. Relations (3) and (4) yield

$$v_0 D_x K(x^0, y^0) \geqq 0 , \tag{5}$$

$$(v_0 x^0)^T D_x K(x^0, y^0) = 0 . \tag{6}$$

Moreover, since $v^T D_y K(x^0, y^0) = 0$, we have

$$(v_0 y^0)^T D_y K(x^0, y^0) = 0 . \tag{7}$$

But $v_0 > 0$ for otherwise $v = v_0 y^0 = 0$ and $(v_0, v) = 0$, a contradiction. Dividing (5), (6), and (7) by the positive scalar $v_0$ produces relations which imply that $(x^0, y^0)$ satisfies the dual constraints as well as the equation $F(x^0, y^0) = G(x^0, y^0)$. The final assertion of the theorem is an application of Theorem 1.

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