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NOTE GENERALIZING A RESULT OF SAMUEL'S

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Let $C(A)$ denote the class group of a Krull domain A. Samuel has established (a) $C(A) \to C(A[[X_1, \cdots, X_n]])$ is injective, and (b) $C(A) \to C(A[[X_1, \cdots, X_n]])$ is bijective in Case A is a regular U.F.D. This note establishes that $C(A) \to C(A[[X_1, \cdots, X_n]])$ is bijective in Case A is a regular noetherian domain, thus adding a complement to (a) while generalizing (b). A corollary of this is that $A[[X_1, \cdots, X_n]]_S$ is a U.F.D. if A is a regular Noetherian domain and S is the set of nonzero elements of A.

In [1], Samuel gave an example of a nonregular noetherian U.F.D. A such that $A[[X]]$ is not a U.F.D. In this case certainly, the mapping of the class group $C(A)$ into $C(A[[X]])$ is not onto (since a unique factorization domain is characterized by $C(A) = 0$). In the same article, Samuel showed that $A[[X]]$ is a U.F.D. in Case A is a regular U.F.D. Here it is proved that $C(A) \to C(A[[X_1, \cdots, X_n]])$ is one-to-one onto in Case A is a regular noetherian domain. The main tool is the technical Theorem 3 below, which shows that if W is unmixed of height 1 in $A[[X_1, \cdots, X_n]]$, then there is an unmixed height 1 ideal I of A such that IW is principal. From this the result stated above follows directly.

Two lemmas are needed to facilitate the main results.

LEMMA 1. *Let B be a regular noetherian domain and let I and W be two unmixed height 1 ideals of B such that I and W have no associated prime ideals in common. Then $IW = I \cap W$.*

Proof. For each M, B_M is a regular local Noetherian ring, hence a U.F.D., so IB_M and WB_M are both principal ideals. Since I and W have no associated prime ideals in common, neither do IB_M and WB_M. It follows that $IB_M \cap WB_M = IB_M \cdot WB_M$. Thus for each M, $(I \cap W)B_M = IB_M \cap WB_M = IB_M \cdot WB_M = (IW)B_M$. This establishes the lemma.

LEMMA 2. *Let B be a regular noetherian domain and Z be an unmixed height 1 ideal of B. If X is an element of B such that (a) X is in the Jacobson radical of B and (b) $Z : XB = Z$, then $Z + XB$ is unmixed of height 2.*

Proof. Let P be an associated prime ideal of $Z + XB$. Then B_P is a regular local ring, hence is a U.F.D. [3, Thm., p. 406]. ZB_P is

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principal, so choose T in Z such that $ZB_P = TB_P$. Then PB_P is an associated prime ideal of $(Z + XB)B_P = ZB_P + XB_P = TB_P + XB_P$. But $Z : XB = Z$ implies that $ZB_P : XB_P = ZB_P$, or $TB_P : XB_P = TB_P$, so $\{T, X\}$ is a prime sequence in B_P. This implies that height of $PB_P = 2$ [3, Thm. 2, p. 397]. But height of $P =$ height of PB_P.

THEOREM 3. *Let A be a regular noetherian domain and let $B_n = A[[X_1, \cdots, X_n]]$. If W is any unmixed height 1 ideal of B_n, then there is an unmixed height 1 ideal I of A such that IW is principal and $IW = IB_n \cap W$.*

Proof. Let $B_0 = A$. The theorem will be proved for $n \geqq 0$ by induction on n.

(1) $n = 0$. W is an unmixed height 1 ideal of A. Since A is regular, it is integrally closed, so $W = P_1^{(n_1)} \cap \cdots \cap P_k^{(n_k)}$ where the P_i $i = 1, \cdots, k$ are height 1 prime ideals and $P_j^{(n_i)}$ $i = 1, \cdots, k$ is the n_ith symbolic power of P_i. Choose d an element of A so that $V_{P_i}(d) = n_i$ $i = 1, \cdots, k$ where V_{P_i} denotes the discrete valuation determined by P_i. Then dA can be written

$$dA = P_1^{(n_1)} \cap \cdots \cap P_k^{(n_k)} \cap P_{k+1}^{(n_{k+1})} \cap \cdots \cap P_l^{(n_l)}$$

where the P_{k+j} $j = 1, \cdots, l - k$ are further height 1 prime ideals of A. Let $I = P_{k+1}^{(n_{k+1})} \cap \cdots \cap P_l^{(n_l)}$. Then visibly $I \cap W = dA$ is principal. By Lemma 1, $I \cap W = IW$.

(2) Suppose the theorem has been established for $n - 1 (n \geqq 1)$. Let W be an unmixed height 1 ideal of B_n and write $W = ZX_n^k$ where $Z: X_nB_n = Z$. If $Z = B_n$, the theorem follows trivially. If $Z \neq B_n$, then Z is also unmixed of height 1. Thus $Z + X_nB_n$ is unmixed of height 2 by Lemma 2. Let $Z_0 = Z + X_nB_n/X_nB_n$. Z_0 is unmixed of height 1 in B_{n-1}. By induction, there is an ideal I of A such that $IZ_0 = IB_{n-1} \cap Z_0$ is principal, say $IZ_0 = u_0(X_1, \cdots, X_{n-1}) \cdot B_{n-1}$. Choose an element $u(X_1, \cdots, X_{n-1}, X_n)$ in IZ whose leading coefficient when written as a power series in X_n is $u_0(X_1, \cdots, X_{n-1})$.

Let $f(X_1, \cdots, X_n)$ be any element of $IB_n \cap Z$. Then $f(X_1, \cdots, X_{n-1}, 0)$ is in $IB_{n-1} \cap Z_0$. This implies that $f - g_0u = X_n \cdot f_1$, where f_1 is in B_n and g_0 is in B_{n-1}. Since f and u are both in Z, f_1 is in $Z: X_nB_n = Z$. Clearly f_1 is in IB_n. So repeating, we can find an f_2 in B_n and a g_1 in B_{n-1} such that $f_1 - g_1u = X_n \cdot f_2$. Continuing, we get that

$$f = u \cdot \sum_{i=0}^{\infty} g_i X_n^i$$

showing simultaneously that $IB_n \cap Z$ is principal and that $IB_n \cap Z = IZ$.

To conclude, $IW = IZX_n^k B_n = u \cdot X_n^k B_n$ is principal. If v is in $IB_n \cap W$, from v in W it follows that $v = X_n^k \cdot v'$, where v' is in Z. But then v' is in IB_n so v' is in $IB_n \cap Z = IZ$. This gives that $v = X_n^k v'$ is in $IZX_n^k = IW$, showing that $IB_n \cap W \subseteqq IW$. The opposite inclusion is trivial, so the induction is complete.

COROLLARY 4. *The map $C(A) \to C(A[[X_1, \cdots, X_n]])$ of the class group of A into the class group of $A[[X_1, \cdots, X_n]]$ is one-to-one onto if A is a regular noetherian domain.*

Proof. Samuel [2, Prop. 1, p. 156 and Prop. 3, p. 138] has shown that the map is one-to-one. Theorem 3 proves that it is onto in the present case.

COROLLARY 5. *Let A be a regular noetherian domain. Let M be a multiplicative set of $A[[X_1, \cdots, X_n]]$. Then $C(A[[X_1, \cdots, X_n]]_M)$ is a homomorphic image of $C(A)$.*

Proof. Samuel [2, Prop. 2, p. 157] shows that $C(R) \to C(R_S)$ is always onto. Corollary 4 supplies the rest.

COROLLARY 6. *Let A be a regular noetherian domain. Let S denote the nonzero elements of A. Then $B' = A[[X_1, \cdots, X_n]]_S$ is a U.F.D.*

Proof. Let W be an unmixed height 1 ideal of $A[[X_1, \cdots, X_n]]$. Then there is an unmixed height 1 ideal I of A such that IW is principal, say $IW = (U)$. Then $UB' = IB' \cdot WB'$, but $IB' = B'$, so $WB' = UB'$ is principal.

REMARKS. (1) Samuel [2] has established the analogue of Corolary 4 for $A[X_1, \cdots, X_n]$. This implies that Corollaries 5 and 6 also hold for $A[X_1, \cdots, X_n]$, Corollary 6 of course being trivial.

(2) As originally submitted, this note established Theorem 3 and its corollaries only in the case that A is a Dedekind domain. In the original presentation, Corollary 6 was the main tool for the proofs of Corollaries 4 and 5. I wish to express my gratitude to the referee for bringing Samuel's results [2] to my attention and for suggesting the generalization to regular Noetherian domains. Lemma 1 is the only addition necessary to effect the generalization.

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