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EXTREME OPERATORS INTO $C(K)$

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If X and Y are real Banach spaces let $S(X, Y)$ denote the convex set of all linear operators from X into Y having norm less than or equal to 1. The main theorem is this: If K_1 and K_2 are compact Hausdorff spaces with K_1 metrizable and if T is an extreme point of $S(C(K_1), C(K_2))$, then there are continuous functions $\phi: K_2 \rightarrow K_1$ and λ in $C(K_2)$ with $|\lambda| = 1$ such that $(Tf)(k) = \lambda(k)f(\phi(k))$ for all k in K_2 and f in $C(K_1)$. There are several additional theorems which discuss the possibility of replacing $C(K_1)$ in this theorem by an arbitrary Banach space.

Suppose that K_1 is a compact Hausdorff space and that $C(K_1)$ is the Banach space of real-valued continuous functions on K_1, with supremum norm. Denote by S^* the unit ball of $C(K_1)^*$; then S^* is a weak* compact convex set and therefore the set $\operatorname{ext} S^*$ of its extreme points is nonempty, by the Krein-Milman theorem. Arens and Kelley [1] (cf. [4, p. 441]) showed that these extreme points are precisely those functionals of the form $f \rightarrow \lambda f(k) (f \in C(K_1))$, where $k \in K_1$ and $\lambda = 1$ or $\lambda = -1$. [We denote the functional "evaluation at k" by φ_k, so the extreme points of S^* are the functionals $\lambda\varphi_k$, $|\lambda| = 1$.] If we restrict our attention to the "positive face" of S^* (those functionals φ such that $\varphi(1) = 1$), then this is a weak* compact convex subset of S^* and its extreme points are those obtained from $\operatorname{ext} S^*$ by taking $\lambda = 1$. Now, the members of $C(K_1)^*$ can be regarded as continuous linear operators from $C(K_1)$ into $C(K_2)$, where K_2 consists of a single point. Thus, it is natural to consider the possible extension of the above results to the more general situation when K_2 is an arbitrary compact Hausdorff space, and S^* is replaced by the unit ball $S = S(C(K_1), C(K_2))$ of the Banach space $B = B(C(K_1), C(K_2))$ of all bounded linear operators from $C(K_1)$ into $C(K_2)$, with the usual operator norm. Corresponding to the positive face of S^* we have the convex subset S_1 of S, consisting of those T in S such that $T1 = 1$. It was shown by A. and C. Ionescu Tulcea [5] (and generalized in [8]) that $\operatorname{ext} S_1$ consists of the "composition operators", i.e., those of the form $(Tf)(k) = f(\psi(k))$ $(f \in C(K_1), k \in K_2)$ where ψ is a continuous function from K_2 into K_1. It is easily seen that these operators are also extreme in S; more generally, if $\psi: K_2 \rightarrow K_1$ is continuous and $\lambda \in C(K_2)$ with $|\lambda| = 1$,

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then every operator T of the form

$$(*) \qquad Tf = \lambda \cdot f \circ \psi , \qquad f \in C(K_1)$$

is an extreme point of S. Indeed, if k is a point of K_2, then the functional on $C(K_1)$ defined by $f \to (Tf)(k) = \lambda(k)f(\psi(k))$ is the extreme point $\lambda(k)\varphi_{\psi(k)}$ of $S^* \subset C(K_1)^*$. The above assertion follows easily from this remark. The main result of the present paper is the converse of this assertion (Theorem 1), under the additional hypothesis that K_1 be metrizable. The proof is not elementary, in sharp contrast to the simplicity of the proofs for the corresponding statement for S_1. The metrizability hypothesis may be unnecessary; we use it only to enable us to apply one of Michael's selection theorems [6, 7]. If we consider the case of *complex*-valued continuous functions on K_1 and K_2, then the obvious analogues of the Arens-Kelley and Ionescu Tulcea results are valid ([4, p. 441] and [8]), and it is still true that every operator of the form (*) is extreme. We do not know whether the converse is true, however, even with the metrizability hypothesis.

It is possible to formulate our characterization of the extreme operators in S in another way: An operator T in $S(C(K_1), C(K_2))$ (K_1 metrizable) is extreme in S if and only if the functional $F_T(k): f \to (Tf)(k)$ ($f \in C(K_1)$) is extreme in $S^* \subset C(K_1)^*$ for a dense set of k's in K_2. (Simply use the fact that ext S^* is weak* compact and that $k \to F_T(k)$ is continuous.) In this form, the statement makes sense for operators T from an aribtrary Banach space X into $C(K)$, so that it is conceivable that the following is true: Suppose that X is a Banach space, that K is a compact Hausdorff space and that T is an operator in $S = S(X, C(K))$, the unit ball of bounded operators from X to $C(K)$. Then:

(A) *If T is extreme in S, then the functional $F_T(k)$ is extreme in the unit ball S^* of X^* for a dense set of points k in K.*

The converse of this assertion is clearly true. The assertion itself, however, is false; Theorem 2 shows that (in particular) if K has no isolated points, then there exists a space X and an extreme operator T in $S(X, C(K))$ such that for each k in K, $F_T(k)$ is *not* extreme in S^*. The hypothesis concerning isolated points is unavoidable, since it is easily verified that if T is extreme in $S(X, C(K))$ and k is an isolated point of K, then $F_T(k)$ is extreme in S^*. (Thus, if K has a dense set of isolated points, then assertion (A) is true. This observation let Arterburn and Whitley [2] to conjecture that it is generally true.) Assertion (A) is true if some strong restrictions are made on X; as noted in [2], (A) holds if every point in the boundary of S^* is extreme (i.e. if X^* strictly convex). Assertion (A) is also true if X is three-dimensional (cf. Theorem 3). An example shows that (A) is no longer true for all four-dimensional spaces.

In order to prove Theorem 1 we make the usual identification between $C(K_1)^*$ and the space of finite signed Borel measures μ on K_1. For f in $C(K_1)$, $\mu(f)$ denotes the integral of f with respect to μ. Recall that $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are nonnegative and have disjoint Borel supports; furthermore, $\|\mu\| = \mu^+(1) + \mu^-(1)$. Now, to each operator T in $B(C(K_1), C(K_2))$ there corresponds a unique function $k \to \mu_k$ from K_2 into $C(K_1)^*$ such that

$$(Tf)(k) = \mu_k(f) \qquad (k \in K_2, f \in C(K_1))$$

and

$$\|T\| = \sup\{\|\mu_k\| : k \in K_2\} .$$

This function is continuous from K_2 into $C(K_1)^*$ in its weak* topology (briefly: $k \to \mu_k$ is weak* continuous). Any weak* continuous function $k \to \nu_k$ for which $\|\nu_k\|$ is bounded defines an operator in B (by means of the above equation) with norm equal to $\sup \|\nu_k\|$. This correspondence between norm bounded weak* continuous functions and operators in B is linear; in particular, we have the following useful fact.

If $k \to \mu_k$ is related to T as above, then T is extreme in $S(C(K_1), C(K_2))$ if and only if the following is true: Whenever $k \to \nu_k$ is weak* continuous and $\|\mu_k \pm \nu_k\| \leqq 1$ for each k in K_2, then $\nu_k = 0$ for all k.

We can now prove our main result.

THEOREM 1. *Suppose that K_1 and K_2 are compact Hausdorff spaces, that K_1 is metrizable, and that T is an operator in $S(C(K_1), C(K_2))$. Then T is an extreme point of this set if and only if there exist continuous functions $\psi: K_2 \to K_1$ and λ in $C(K_2)$, $|\lambda| = 1$, such that*

$$(Tf)(k) = \lambda(k)f(\psi)) \qquad (k \text{ in } K_2, f \text{ in } C(K_1)) .$$

Proof. We have already noted that any operator of the above form is extreme, so suppose T is extreme in $S(C(K_1), C(K_2))$ and let $k \to \mu_k$ be the corresponding weak* continuous map from K_2 into the weak* compact unit ball S^* of $C(K_1)^*$. We distinguish two cases:

Case I. For each k in K_2, either $\mu_k \geqq 0$ or $\mu_k \leqq 0$.

Case II. For some k in K_2, the decomposition $\mu_k = \mu_k^+ - \mu_k^-$ is nontrivial.

We take care of Case I with the help of the Ionescu Tulcea theorem. Note that in this case, $\|\mu_k\| = |\mu_k(1)|$. Furthermore, the real valued function $k \to |\mu_k(1)|$ is continuous on K_2, hence if μ is any nontrivial measure in S^*, then the function $k \to (1 - \|\mu_k\|)\mu$ is weak* continuous from K_2 into S^*. Since

$$\|\mu_k \pm (1 - \|\mu_k\|)\mu\| \leqq \|\mu_k\| + (1 - \|\mu_k\|)\|\mu\| \leqq 1 ,$$

we see that for each k, $|\mu_k(1)| = \|\mu_k\| = 1$. Thus, if we let $\lambda(k) = \mu_k(1)$, then $\lambda \in C(K_2)$ and $|\lambda| = 1$. Consider the operator $U: C(K_1) \to C(K_2)$ defined by $(Uf)(k) = \lambda(k)^{-1}(Tf)(k)$. It is easily verified that U is extreme in S and that $U1 = 1$. It follows from the Ionescu Tulcea theorem [5; 8, Theorem 1] that there exists a continuous function $\psi: K_2 \to K_1$ such that $Uf = f \circ \psi$, and hence T has the required form.

We next show that if Case II holds, then T can not be extreme. Assume, then, that there exists a point k_0 in K_2 such that $\mu_{k_0} = \mu_{k_0}^+ - \mu_{k_0}^-$ and $\mu_{k_0}^+ \neq 0$, $\mu_{k_0}^- \neq 0$. For each k in K_2, let

$$\Sigma(k) = \{\mu : \mu \in S^*, 0 \leqq \mu \leqq \mu_k^+\} .$$

This is a nonempty convex weak* closed subset of S^*, hence is weak* compact. We shall prove that the map $k \to \Sigma(k)$ from K_2 into the set of all subsets of $C(K_1)^*$ is lower semicontinuous in the sense of Michael [6] (where we take the weak* topology in $C(K_1)^*$). Thus, we must show the following: Given a point k_1 in K_2, a measure μ_1 in $\Sigma(k_1)$ and a weak* neighborhood V of μ_1, then there is a neighborhood U of k_1 such that $V \cap \Sigma(k)$ is nonempty for each k in U. We can assume that $V = \{\mu : \mu \in C(K_1)^*, |\mu(f_i) - \mu_1(f_i)| \leqq 1, f_i \in C(K_1), i = 1, 2, \cdots, n\}$. Suppose that there is a net k_α converging to k_1 such that $V \cap \Sigma(k_\alpha)$ is empty for each α. Since $\mu_{k_\alpha}^+ \in S^*$ for each α we may assume (by taking a subnet if necessary) that $\mu_{k_\alpha}^+ \to \mu_2$ for some $\mu_2 \geqq 0$ in S^*; similarly $\mu_{k_\alpha}^- \to \mu_3 \geqq 0$. Since $\mu_{k_\alpha} \to \mu_{k_1}$ we have $\mu_{k_1} = \mu_2 - \mu_3 = \mu_{k_1}^+ - \mu_{k_1}^-$ and it follows from the definition of the Hahn decomposition that $\mu_2 \geqq \mu_{k_1}^+$. Now, $\mu_{k_1}^+ \geqq \mu_1 \geqq 0$, so by the Radon-Nikodym theorem we can write $d\mu_1 = g_1 d\mu_2$, where g_1 is a Borel function on K_1 with $0 \leqq g_1 \leqq 1$. We can choose a continuous function g on K_1 such that $0 \leqq g \leqq 1$ and

$$\int_{K_1} |g - g_1| \, d\mu_2 \leqq (2 \max\{\|f_i\| : 1 \leqq i \leqq n\})^{-1} .$$

If we define μ_α by $d\mu_\alpha = g d\mu_{k_\alpha}^+$ then $\mu_\alpha \in \Sigma(k_\alpha)$ for each α. Furthermore, for each i we have

$$\mu_\alpha(f_i) = \int_{K_1} f_i d\mu_\alpha = \int_{K_1} f_i g d\mu_{k_\alpha}^+ \to \int_{K_1} f_i g d\mu_2$$

and hence

$$\limsup |\mu_\alpha(f_i) - \mu_1(f_i)| \leqq \int_{K_1} |f_i g - f_i g_1| \, d\mu_2 \leqq 1/2 ,$$

so that μ_α is eventually in $V \cap \Sigma(k_\alpha)$, a contradiction.

Thus, the set-valued map $k \to \Sigma(k)$ is lower semicontinuous and since $C(K_1)$ is separable—this is the only place where we use the metrizability of K_1—we may apply a selection theorem of Michael ([6, Th. 3.2*], [6, Ex. 1.3*] and [7]) to conclude the existence of a continuous selection for this map taking at k_0 the preassigned value $\mu_{k_0}^+$, that is, there exists $\sigma: K_2 \to C(K_1)^*$ which is weak* continuous such that $\sigma_{k_0} = \mu_{k_0}^+$ and $\sigma_k \in \Sigma(k)$ for each k, i.e. $0 \leqq \sigma_k \leqq \mu_k^+$. Similarly, there exists a continuous map $\tau: K_2 \to C(K_1)^*$ such that $\tau_{k_0} = \mu_{k_0}^-$ and $0 \leqq \tau_k \leqq \mu_k^-$ for each k. We define the map $\nu: K_2 \to C(K_1)^*$ by

$$\nu_k = \sigma_k \| \tau_k \| + \tau_k \| \sigma_k \| .$$

Since $\| \sigma_k \| = \sigma_k(1)$, the function $k \to \| \sigma_k \|$ is continuous (similarly for τ) and hence $k \to \nu_k$ is weak* continuous. Furthermore, $\nu_{k_0} \neq 0$. (This last assertion is the reason we needed two cases; in case I, $\nu_k \equiv 0$.) Finally, $\| \mu_k \pm \nu_k \| \leqq 1$ for each k. Indeed,

$$\begin{aligned} \| \mu_k + \nu_k \| &= \| (\mu_k^+ + \sigma_k \| \tau_k \|) \| + \| (\mu_k^- - \tau_k \| \sigma_k \|) \| \\ &= \| \mu_k^+ \| + \| \sigma_k \| \, \| \tau_k \| + \| \mu_k^- \| - \| \sigma_k \| \, \| \tau_k \| \\ &= \| \mu_k \| \leqq 1 ; \end{aligned}$$

similarly for $\| \mu_k - \nu_k \|$. It follows that T is not extreme, a contradiction which completes the proof.

It is clear that the above proof would be valid without the hypothesis that K_1 be metrizable if there were a selection theorem for the more general situation. Unfortunately, however, it is known (see e.g. [3]) that there exist compact Hausdorff space K_1 and K_2 (K_1 not metrizable, K_2 metrizable) and a weak* lower semicontinuous map from K_2 into the weak* compact convex subsets of the unit ball of $C(K_1)^*$ which does *not* have a continuous selection.

It is worth noting that we can eliminate the metrizability hypothesis in Theorem 1 if we put an additional hypothesis on the operator T. Namely, *if T is extreme in $S(C(K_1), C(K_2))$ and if the function $k \to \| \mu_k \|$ is continuous, then T is of the form* $(*)$. We will sketch a proof of this fact. Since the proof of Theorem 1 used metrizability only in Case II, we will show that this case cannot occur. First, note that the continuity of $k \to \| \mu_k \|$ implies the weak* continuity of the functions $k \to \mu_k^+$ and $k \to \mu_k^-$. (This may be proved by contradiction, using the weak* compactness of S^*, the weak* continuity of $k \to \mu_k$ and the fact that if $\mu_k = \mu_1 - \mu_2$, $\mu_1, \mu_2 \geqq 0$, then $\mu_1 \geqq \mu_k^+$, $\mu_2 \geqq \mu_k^-$.) It follows that $k \to \nu_k = \mu_k^+(1)\mu_k^- + \mu_k^-(1)\mu_k^+$ is weak* continuous, and $\| \mu_k \pm \nu_k \| \leqq 1$ for each k. Since T is extreme, $\nu_k \equiv 0$; in particular, $0 = \nu_k(1) = 2\mu_k^+(1)\mu_k^-(1)$ for each k, which implies that Case I holds.

The above hypothesis on T is fulfilled in case T is a *compact* operator; indeed, T is compact if and only if $k \to \mu_k$ is *norm* continuous [4, p. 490]. (Examination of the above proof and that in [8, Theorem 1] shows that in this case it suffices to assume only that T is extreme in the unit ball of compact operators.)

Suppose, now, that X is a Banach space and that T is an extreme point of $S(X, C(K))$ (K compact Hausdorff). As noted in the introduction, the functional $F_T(k)$ is extreme in the unit ball S^* of X^* whenever k is an isolated point of K. The next result shows that for any K there exists a space X such that these are the *only* such functionals which are extreme.

THEOREM 2. *Suppose that K is a compact Hausdorff space. Then there exists a Banach space X and an extreme operator T in $S(X, C(K))$ such that $F_T(k) \in \operatorname{ext} S^*$ (if and) only if k is an isolated point of K.*

Proof. Let M be the compact set of all points in K which are not isolated and let $l_2(M)$ be the Banach space of all real-valued functions y on M such that $\|y\|^2 = \Sigma_{k \in M} y^2(k) < \infty$. Let $X_0 = C(K) \oplus l_2(M)$, with norm $\|(f, y)\| = \max(\|f\|, \|y\|)$ ($f \in C(K), y \in l_2(M)$). As is well known, the conjugate space X_0^* of such a product space is linearly isometric to $C(K)^* \oplus l_2(M)^*$, with $\|(\mu, y^*)\| = \|\mu\| + \|y^*\|$. Since $l_2(M)$ is a Hilbert space, we may identify $l_2(M)^*$ and $l_2(M)$. The space X_0^* in its weak* topology is linearly homeomorphic with the product space $C(K)^* \times l_2(M)^*$ (where we take the weak* topologies in each of these spaces). Let $S_1 = S_{C(K)^*} \times \{0\}$ and $S_2 = \{0\} \times S_{l_2(M)^*}$; these sets are weak* compact and convex, and the set $\operatorname{conv}(S_1 \cup S_2)$ is identical with the unit ball of X_0^*. (Consider $(\mu, y^*) = \|\mu\|(\mu/\|\mu\|, 0) + \|y^*\|(0, y^*/\|y^*\|)$.) For k in M, let $e_k \in l_2(M)$ be the function which is 1 at k, 0 elsewhere. Let $Q = \{\pm(\varphi_k, \pm e_k) : k \in M\} \cup \{\pm(\varphi_k, 0) : k \in M\}$ (as usual, $\varphi_k \in C(K)^*$ is evaluation at k). Note that Q is weak* compact. Indeed, it follows from the definition of $l_2(M)$ that any y in $l_2(M)$ gets arbitrarily small outside finite sets, so if $\{k_\alpha\}$ is a net in M with infinite range, then the set $\{e_{k_\alpha}\}$ has a subnet which converges to 0 in the weak* topology. This fact (together with compactness of M and weak* continuity of $k \to \varphi_k$) makes it possible to show that Q is weak* compact. Let A be the weak* closed convex hull of Q; then A is norm bounded, hence weak* compact, and the set $S' = \operatorname{conv}(S_1 \cup S_2 \cup A)$ is weak* compact and convex. Furthermore, S' is symmetric with respect to 0 and contains the unit ball $\operatorname{conv}(S_1 \cup S_2)$ of X_0^*. From standard results on duality, then, S' is the polar of a convex body in X_0 which defines a norm $\|\cdot\cdot\|'$ on X_0, equivalent to the original norm. Furthermore, if we let X denote X_0 in the norm $\|\cdot\cdot\|'$, then S' is the unit ball of the space X^*. We have defined our space X; to define the operator

T, let $F_T(k) = (\varphi_k, 0)$, $k \in K$. It is clear that F_T is weak* continuous; our next task is to show that T has the required properties.

Suppose, first, that $k \in K$ and that $(\varphi_k, 0) \pm (\mu, y^*) \in S'$. It follows immediately that $\|\varphi_k \pm \mu\| \leqq 1$, so that $\mu = 0$ and hence $(\varphi_k, \pm y^*) \in S'$. We will show that if $k \in K \sim M$, then $y^* = 0$, while if $k \in M$, then $y^* = \lambda e_k$ for some $|\lambda| \leqq 1$. To this end, let $\Phi_k = \{f : f \in C(K), f(k) = 1 = \|f\|\}$. If $f \in \Phi_k$, let $H_f = \{(\nu, z^*) : (\nu, z^*) \in X_0^*, \nu(f) = 1\}$; this is a weak* closed hyperplane and, since $(\nu, z^*) \in S'$ implies $\|\nu\| \leqq 1$ (hence $\nu(f) \leqq 1$), H_f is a supporting hyperplane to S'. In particular, $H_f \cap S'$ is an extremal subset of S'. Since the intersection of extremal sets is an extremal set, the set $J_k = \bigcap_{f \in \Phi_k} H_f$ is an extremal subset of S'; furthermore, it is clear that $(\varphi_k, y^*) \in S' \cap J_k$. Now, since J_k is extremal, $\operatorname{ext}(J_k \cap S') \subset J_k \cap \operatorname{ext} S'$, and from the definition of S' we see that $\operatorname{ext} S' \subset \operatorname{ext} S_1 \cup \operatorname{ext} S_2 \cup \operatorname{ext} A$. Also, since Q is compact, the Milman theorem implies that $\operatorname{ext} A \subset Q$. Since $J_k \cap S_2$ is empty, we have $\operatorname{ext}(J_k \cap S') \subset (J_k \cap \operatorname{ext} S_1) \cup (Q \cap J_k)$. Recall that if $(\nu, z^*) \in J_k$, $\|\nu\| \leqq 1$, then $\nu(f) = 1$ for all $f \in \Phi_k$, so $\nu = \varphi_k$. Thus, $J_k \cap S_1 = \{(\varphi_k, 0)\}$, and $Q \cap J_k = \{(\varphi_k, \pm e_k), (\varphi_k, 0)\}$ if $k \in M$; otherwise $Q \cap J_k$ is empty. Now, $(\varphi_k, y^*) \in J_k \cap S'$ and the latter is the closed convex hull of its extreme points. From what we have just shown, $(\varphi_k, y^*) = (\varphi_k, 0)$ if $k \in K \sim M$, while (φ_k, y^*) is a convex combination of $(\varphi_k, 0)$ and $(\varphi_k, \pm e_k)$ if $k \in M$; these remarks show that y^* has the form indicated earlier. It is clear that $(\varphi_k, 0) \in \operatorname{ext} S'$ if and only if $k \in K \sim M$. It remains to show that if there exists a weak* continuous map G: $K \to X_0^*$ such that $F_T(k) \pm G(k) \in S'$ for each k, then $G = 0$. Such a map would have the form $G(k) = (\mu_k, y_k^*)$, where $k \to \mu_k$, $k \to y_k^*$ are weak* continuous in the appropriate spaces, and we would have $(\varphi_k, 0) \pm (\mu_k, y_y^*) \in S'$. From what we have shown above, $\mu_k = 0$ for all k, $y_k^* = 0$ for $k \in K \sim M$, and $y_k^* = \lambda(k) e_k$ for some $|\lambda(k)| \leqq 1$ if $k \in M$. Since $k \to y_k^*$ is continuous, $y_k^* = 0$ for k in the closure of $K \sim M$. Suppose that k is an interior point of M and choose an infinite net of distinct points k_α in M with $k_\alpha \to k$. We know that $e_{k_\alpha} \to 0$ in the weak* topology, so $y^* k_\alpha = \lambda(k_\alpha) e_{k_\alpha} \to 0$ and hence $y_k^* = 0$. Thus, $y_k^* = 0$ for all k in K and the proof is complete.

Our next result shows that the conjecture of Arterburn and Whitley is true for two or three dimensional spaces X, as well as for certain other finite dimensional spaces.

THEOREM 3. *Suppose that X is a finite dimensional normed linear space such that*

(i) $\dim X \leqq 3$

or

(ii) *the unit ball of X is a polyhedron.*

If K is a compact Hausdorff space and T is an extreme operator in $S(X, C(K))$, then $\{k: k \in K$ and $F_T(k) \in \operatorname{ext} S^*\}$ is dense in K.

Proof. We shall give the proof for the case $\dim X = 3$, since the proof for the remaining cases is similar but much simpler. The idea of the proof (in any of the three cases) is this: We show that for each k, $F_T(k)$ must lie on the surface of the unit ball S^* of X^*. We then show that $F_T(k)$ cannot be in the (relative) interior of any face of S^*. In the two dimensional case, this completes the proof, while in the polyhedral case we can apply the same argument to faces of one less dimension, and the proof proceeds by induction. (Recall that in case (ii), the dual ball S^* is *also* a polyhedron.) In the three dimensional case we need a special argument, since the interior of a one dimensional face need not be isolated from other one dimensional faces. Suppose, then, that $\dim X = 3$ and that $T \in \operatorname{ext} S(X, C(K))$. Let $y^* \neq 0$ be a fixed element of S^*. The map $k \to (1 - \| F_T(k) \|)y^*$ is continuous, and $\| F_T(k) \pm (1 - \| F_T(k) \|)\, y^* \| \leqq \| F_T(k) \| + (1 - \| F_T(k) \|)\, \| y^* \| \leqq 1$; since T is extreme, this implies that $\| F_T(k) \| = 1$ for each k, i.e., $F_T(k)$ is on the boundary of S^* for each k. We next show that $F_T(k)$ cannot be an interior point of a two dimensional face of S^*, for any k in K. Indeed, if $F_T(k_0)$ were an interior point of such a face, the same would be true for all points of k in some neighborhood of k_0. We could then easily find a continuous real valued function g on K, not identically zero, and $x^* \in X^*$, $x^* \neq 0$, such the $\| F_T(k) \pm g(k)x^* \| \leqq 1$ for each k, contradicting the fact that T is extreme. Suppose, finally, that there exists nonempty open set $U \subset K$ such that $F_T(k) \notin \operatorname{ext} S^*$ for each k in U. From what we have shown it follows that for each x^* in $F_T(U)$ there exists a vector, $u(x^*)$ in X^*, unique up to sign, such that $\| u(x)^* \| = 1$ and $\| x^* \pm \varepsilon u(x^*) \| = 1$ for some $\varepsilon > 0$. For every x^* in $F_T(U)$, let $\lambda(x^*) = \sup\{\varepsilon : \| x^* \pm \varepsilon u(x^*) \| = 1\}$. Suppose that x_n^*, x^* are in $F_T(U)$ and that $x_n^* \to x^*$. The sequence $u(x_n^*)$ has a subsequence converging to some y^* in S^*. If $\lambda(x_n^*) \geqq \delta$ for some $\delta > 0$ and all n we see that $\| x^* \pm \delta y^* \| = 1$ and hence $y^* = \pm u(x^*)$ and $\lambda(x^*) \geqq \delta$. This shows that λ and therefore $\lambda \circ F_T$ are upper semi-continuous; in particular, $\lambda \circ F_T$ has points of continuity and hence there exist a nonempty open subset M of U and an $\varepsilon > 0$ such that $\lambda(x^*) \geqq \varepsilon$ for x^* in $F_T(M)$. We may assume that for some x in X, $|(u(x^*), x)| > 0$ for all x^* in $F_T(M)$ (otherwise replace M by

$$M \cap F_T^{-1}\{x^*; |(u(x^*), x)| > 0\} ,$$

where x is chosen so that this intersection is nonempty). For x^* in $F_T(M)$ let $e(x^*)$ be the sign (i.e. 1 or -1) such that $e(x^*)(u(x^*), x) > 0$. The compactness argument used above shows that the map $x^* \to$

$e(x^*)u(x^*)$ is continuous on $F_T(M)$. Choose a real valued continuous function g on K such that $g \neq 0, 0 \leqq g \leqq \varepsilon$ and $g(k) = 0$ if $k \notin M$. Then for all k,

$$\| F_T(k) \pm g(k)e(F_T(k))u(F_T(k)) \| \leqq 1 .$$

(We define $g(k)e(F_T(k))u(F_T(k))$ to be zero for $k \notin M$.) Since T is extreme, we conclude that the set U must have been empty and the proof is complete.

Our final result is an example which shows that the above theorem fails for a certain four dimensional space X.

EXAMPLE. There exists a four dimensional normed linear space X and an extreme operator T in $S(X, C[0, 1])$ such that $F_T(t) \notin \operatorname{ext} S^*$ for each t in $[0, 1]$.

Proof. Let g be a monotonic increasing function on $[0, 1]$ such that g is discontinuous at every rational point t and continuous elsewhere. Let Σ be the subset of R^4 consisting of the points

$$\begin{aligned}
&\pm(\cos t, \sin t, \cos g(t + 0), \sin g(t + 0))\\
&\pm(\cos t, \sin t, -\cos g(t + 0), -\sin g(t + 0))\\
&\pm(\cos t, \sin t, \cos g(t - 0), \sin g(t - 0))\\
&\pm(\cos t, \sin t, -\cos g(t - 0), -\sin g(t - 0)) ,
\end{aligned}$$

where $0 \leqq t \leqq 1$. The set Σ is symmetric with respect to the origin and it is also compact: If $\{t_n\}$ is a sequence in $[0, 1]$ there is a subsequence $\{t_{n_k}\}$ such that either $t_{n_k} \uparrow t$ or $t_{n_k} \downarrow t$ for some t in $[0, 1]$.

Let S' be the convex hull of Σ; then S' is a compact, convex, symmetric subset of R^4 having nonempty interior, and hence is the unit ball of X^*, where X is the four-dimensional Banach space which has the polar S^0 of S' as its unit ball. Suppose that t is an irrational number in $[0, 1]$. The point $x = (\cos t, \sin t, 0, 0)$ is in S^0; indeed, for each (u, v, w, z) in Σ, $u \cos t + v \sin t \leqq 1$ and equality occurs only if $u = \cos t$, $v = \sin t$ and $(w, z) = \pm(\cos g(t), \sin g(t))$. It follows that if $(\cos t, \sin t, 0, 0) \pm (u, v, w, z) \in S'$, then (since the inner product of these two points with $x = (\cos t, \sin t, 0, 0)$ equals 1) $u = v = 0$ and the point $(\cos t, \sin t, w, z)$ is in the convex hull of the subset of Σ whose inner product with x equals 1, i.e. $w = \lambda \cos g(t)$, $z = \lambda \sin g(t)$, $|\lambda| \leqq 1$.

Let T be the operator from X into $C[0, 1]$ defined by $F_T(t) = (\cos t, \sin t, 0, 0)$, $t \in [0, 1]$. For each t, $F_T(t) \pm (0, 0, \cos g(t), \sin g(t)) \in \Sigma$, and hence $F_T(t) \in S'$ but $F_T(t) \notin \operatorname{ext} S'$. Suppose, however, that there exists $G\colon [0, 1] \to X^*$ which is continuous and satisfies $F_T(t) \pm G(t) \in S'$ for every t. Then, as we have seen, for every irrational t, $G(t) =$

$(0, 0, \lambda(t) \cos g(t), \lambda(t) \sin g(t))$ with $|\lambda(t)| \leqq 1$. Let t_0 be a rational point in $(0, 1)$. Since G is continuous, $\lim_{t \uparrow t_0} G(t) = \lim_{t \downarrow t_0} G(t)$ and since g is discontinuous at t_0 it follows that if $t_n \to t_0$ with t_n irrational, then $\lambda(t_n) \to 0$. Thus $G(t_0) = 0$, and therefore G vanishes on the rationals in $(0, 1)$ and hence identically on $[0, 1]$. It follows that T is extreme and the proof is complete.

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