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PROJECTIONS IN THE SPACE OF BOUNDED LINEAR OPERATORS

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Thorp has shown that for X and Y certain Banach spaces of sequences there is no continuous linear projection of the bounded linear operators from X to Y onto the compact linear operators from X to Y. In this paper, this result, as well as related results for the weakly compact linear operators, is demonstrated for cases including $(a)X$ an infinite dimensional abstract L-space and Y an infinite dimensional space whose conjugate contains a countable total set and $(b)X$ a separable B-space and $Y = C(S)$ with S either a metric space containing an infinite number of points or S a compact space which contains a one-to-one convergent sequence.

We recall that a subspace of a Banach space X is said to be complemented (in X) if there is a continuous linear projection of X onto that subspace. In [14] it is shown that for X and Y certain Banach spaces of sequences the subspace $K(X, Y)$ of compact linear operators from X to Y is not complemented in $B(X, Y)$, the space of bounded linear operators from X to Y. Here, we will prove similar results for either X an abstract L-space or Y a space of type $C(S)$ and will also consider projections on the subspace $W(X, Y)$ of weakly compact linear operators mapping X to Y.

All maps will be linear and X and Y will be Banach spaces. Abstract L-spaces are defined in [7, page 394]; $C(S)$ shall be the space of bounded continuous functions on a topological space S and we use the *sup* norm. We recall that a set in X', the conjugate of the Banach space X, is total if the only vector mapped into zero by that set is the zero vector.

Our main results are Theorems 1 and 2 below.

1. THEOREM. *Let $\mathscr{L}$ be an infinite dimensional abstract L-space and let X have a complemented subspace Y. Suppose that Y' contains a countable total set. Then*

(a) *If Y is infinite dimensional, then $K(\mathscr{L}, X)$ is not complemented in $B(\mathscr{L}, X)$. In fact, $K(\mathscr{L}, Y)$ is complemented in $B(\mathscr{L}, Y)$ if and only if these spaces are equal and this happens if and only if Y is finite dimensional.*

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(b) *If weak and norm sequential convergence are not equivalent in* Y, *then* $K(\mathscr{L}, X)$ *is not complemented in* $W(\mathscr{L}, X)$. *In fact,* $K(\mathscr{L}, Y)$ *is complemented in* $W(\mathscr{L}, Y)$ *if and only if these spaces are equal and this happens if and only if norm and weak sequential convergence are equivalent in* Y.

(c) *If* Y *is not reflexive, then* $W(\mathscr{L}, X)$ *is not complemented in* $B(\mathscr{L}, X)$. *In fact,* $W(\mathscr{L}, Y)$ *is complemented in* $B(\mathscr{L}, Y)$ *if and only if these spaces are equal and this happens if and only if* Y *is reflexive.*

2. THEOREM. *Suppose that* S *is either a (not necessarily compact) metric space which contains an infinite number of points, or that* S *is a compact space which contains a one-to-one convergent sequence. Let* X *have a complemented subspace* Y *and suppose that* Y *is separable. Then*

(a) *If* Y *is infinite dimensional, then* $K(X, C(S))$ *is not complemented in* $B(X, C(S))$. *In fact,* $K(Y, C(S))$ *is complemented in* $B(Y, C(S))$ *if and only if these spaces are equal and this happens if and only if* Y *is finite dimensional.*

(b) *If weak and norm sequential convergence are not the same in* Y', *then* $K(X, C(S))$ *is not complemented in* $W(X, C(S))$. *In fact,* $K(Y, C(S))$ *is complemented in* $W(Y, C(S))$ *if and only if these spaces are equal and this happens if and only if norm and weak sequential convergence are the same in* Y'.

(c) *If* Y *is not reflexive, then* $W(X, C(S))$ *is not complemented in* $B(X, C(S))$. *In fact,* $W(Y, C(S))$ *is complemented in* $B(Y, C(S))$ *if and only if these spaces are equal and this happens if and only if* Y *is reflexive.*

We remark, in connection with part (b) of both theorems, that weak and norm sequential convergence are the same in l [2, page 137]. In Theorem 2, part (b), the separability of Y is essential, for if $C(S)$ is separable, then $W(m, C(S)) = B(m, C(S))$, since weak* and weak sequential convergence are equivalent in m' [9, Theorem 9, page 168], yet m is not reflexive. It follows from either theorem that $K(l, m)$ is not complemented in $B(l, m)$, a result incorrectly proved in [14].

The above theorems can be extended by use of the following lemma.

3, LEMMA. *Suppose that* X_1 *and* Y_1 *are complemented subspaces* *of, respectively, X and Y. Then*

(a) *If $K(X, Y)$ is complemented in $B(X, Y)$, then $K(X_1, Y_1)$ is complemented in $B(X_1, Y_1)$.*

(b) *If $K(X, Y)$ is complemented in $W(X, Y)$, then $K(X_1, Y_1)$ is complemented in $W(X_1, Y_1)$.*

(c) *If $W(X, Y)$ is complemented in $B(X, Y)$, then $W(X_1, Y_1)$ is complemented in $B(X_1, Y_1)$.*

Proof. Let P_1 be a projection of X onto X_1 and let P_2 project Y onto Y_1. For case (a), suppose that P is a projection of $B(X, Y)$ onto $K(X, Y)$. Define a map Q on $B(X_1, Y_1)$ by $Q(T) = [P_2 \circ P(T \circ P_1)]|_{X_1}$, where $F|_{X_1}$ is the restriction of a map F to X_1. Then Q is a projection of $B(X_1, Y_1)$ onto $K(X_1, Y_1)$. The other cases are similar.

So, for example, Theorem 1 gives useful information about maps with range in a space which contains a complemented subspace isomorphic to an abstract L-space.

Note that by Lemma 3 it suffices to prove Theorems 1 and 2 under the assumption $X = Y$. We now find canonical subspaces of $\mathscr{L}$ and $C(S)$ and reduce the problem still further.

From Corollary 4, page 221 of [12] we see that any infinite dimensional complemented subspace of an abstract L-space contains a complemented copy of l. So we may assume in the proof of Theorem 1 that $\mathscr{L} = l$.

In [1] Arens has shown that if S_0 is a metrizable compact subspace of a paracompact space S, then there is a projection of $C(S)$ onto a subspace isomorphic to $C(S_0)$; this is a generalization of Borsuk's theorem [4], in which S_0 is a separable closed subspace of a metric space S. From Arens' result we see that if S is a compact space which contains a one to one convergent sequence, then $C(S)$ contains a complemented copy of c. From Borsuk's result, if S is a metric space containing an infinite number of points, then $C(S)$ contains a complemented copy of either m or c. (We remark that a particularly nice proof of Borsuk's theorem is given in [10]). Thus it suffices to prove Theorem 2 for $C(S) = m$ and $C(S) = c$.

We have now reduced the problem to its essentials. We will need the following representation theorems [15]:

Let $T : X \to C(S)$ be a linear operator. Then T defines a function $p : S \to X'$ by $p(s)(x) = Tx(s)$ and p is continuous as a map into (X', X), i.e. into X' with the weak* topology. Then T is continuous if and only if $p(S)$ is bounded and in this case, $\| T \| = \sup \{\| p(s) \| : s \text{ in } S\}$. Also, T is compact if and only if, in addition, $p(S)$ is conditionally compact (i.e., if and only if the (norm) closure of $p(S)$ is compact) and T is weakly compact if and only if, in addition, $p(S)$ is conditionally compact in the weak topology of X'.

Let $T: l \rightarrow X$ be a linear operator. Then, letting e_i denote the characteristic function of the set $\{i\}$, T is continuous if and only if $\{Te_i : i = 1, 2, \cdots\}$ is bounded and in this case $\| T \| = \sup \| Te_i \|$. The map T is compact if and only if the set $\{Te_i : i = 1, 2, \cdots\}$ is conditionally compact and is weakly compact if and only if that set is conditionally compact in the weak topology of X.

The first representation theorem is due to Bartle [3], for compact S, and the second is due to Dunford and Pettis [6].

The following lemma is the backbone of all our proofs. We denote the space of all bounded functions from a set S to a Banach space X by $m(S : X)$ with $\| f \| = \sup \{\| f(s) \| : s \text{ in } S\}$. If the space X is the scalar field we write $m(S)$, which is also called $B(S)$, and if S is a countably infinite set we have the space m of bounded sequences. For any f in $m(S : X)$, the support of f, supt (f), is given by $\{s : f(s) \neq 0\}$.

4. LEMMA. *Let M and N, $N \subseteqq M$, be closed subspaces of $m(S : X)$ and let N contain all the functions which have finite support. Suppose that there is a function f in $m(S : X)$ and an uncountable family of functions in $m(S)$, $\{g_a : a \text{ in } A\}$, with the properties*:

(1) $\| g_a \| \leqq 1$ *for all a in A,*

(2) fg_a, *the function whose value at s is $f(s)g_a(s)$, is in $M - N$, and*

(3) $\operatorname{supt}(g_a) \cap \operatorname{supt}(g_b)$ *is finite for $a \neq b$.*

Then $(M/N)'$ does not contain a countable total subset. Hence, if M' contains a countable total subset, then N is not complemented in M.

Proof. Let f_a be the coset in M/N which contains fg_a and note that $f_a \neq 0$. To show that $(M/N)'$ does not contain a countable total subset it will suffice to show that a functional x' in $(M/N)'$ can fail to annihilate only countably many elements in the set $\{f_a : a \text{ in } A\}$, so it will suffice to show that the set $C(n) = \{f_a : |x'(f_a)| \geqq 1/n\}$ is finite for each natural number n. To see this let $h_1, h_2, \cdots, h_m$ be in $C(n)$, set $b_i = \overline{x'(h_i)}/|x'(h_i)|$ and let $x = \Sigma b_i h_i$. The critical point is to note that $\| x \| \leqq \| f \|$. Then, since $\| x' \| \, \| f \| \geqq |x'(x)| \geqq m/n$, we see that $C(n)$ is finite.

If the subspace N is complemented in M we have $M = N \oplus R$ where R is a closed subspace of M. Then, since R' contains a countable total subset whenever M' does, and M/N is isomorphic to R, we see that $(M/N)'$ contains a countable total subset if M' does.

We use the next lemma in constructing functions g_a which are as described in Lemma 4.

5. LEMMA. *Let I be a countable set. Then there is an uncountable family $\{U_a : a \text{ in } A\}$ of infinite subsets of I with $U_a \cap U_b$ finite for $a \neq b$.*

Proof. See problem 6Q, page 97 of [8].

The above lemmas are a generalization of the method of [16] and the basic idea can be found in [11] and [13].

As we have noted, Theorem 1 is reduced to the following lemma:

6. LEMMA. *Theorem* 1 *holds in the special case $X = Y$ and $\mathscr{L} = l$.*

Proof. Let $I = \{1, 2, \cdots\}$. By the representation theorem, $B(l, X)$ corresponds to $m(I : X)$ and $K(l, X)[W(l, X)]$ to the subspace of $m(I : X)$ consisting of those functions whose range is [weakly] conditionally compact.

Let $\{U_a : a \text{ in } A\}$ be a family of subsets of I as in Lemma 5 and let g_a be the characteristic function of U_a.

For case (a), suppose that X is infinite dimensional and select a sequence $\{x_i\}$ of points from the unit sphere of X so that $\{x_i\}$ contains no convergent subsequence. We define f in $m(I : X)$ by $f(i) = x_i$. Now we apply Lemma 4 to see that $B(l, X)/K(l, X) = m(I : X)/K$, where K is the subspace of functions with conditionally compact range, is a space whose conjugate contains no countable total set. But $m(I : X)$ does have a conjugate which contains a countable total set, since we are assuming that this is true of $Y = X$; so by Lemma 4 $K(l, X)$ is not complemented in $B(l, X)$.

For case (b) let $\{x_i\}$ be a weakly convergent sequence of points on the unit sphere of X which contains no norm convergent subsequence, assuming that weak and norm sequential convergence are not the same in X, and proceed as above.

For case (c), assume that X is not reflexive and let $\{x_i\}$ be a bounded sequence which contains no weakly convergent subsequences.

That the spaces are equal under the conditions given follows directly. This completes the proof of Theorem 1.

Now Theorem 2 has been reduced to the case $X = Y$ and either $C(S_1) = m$ or $C(S_2) = c$; since S_1 and S_2 are separable Hausdorff spaces which contain a countably infinite number of isolated points, the following lemma will suffice:

7. LEMMA. *Let S_0 be a separable Hausdorff space which contains a countably infinite number of isolated points. Then Theorem 2 holds if $X = Y$ and $S = S_0$.*

Proof. The proof is quite a bit like the proof of Lemma 6. Let $I = \{s_1, s_2, \cdots\}$ be the countably infinite set of isolated points of $S = S_0$ and let U_a be a family of subsets of I as in Lemma 5. Let g_a be the characteristic function of the set U_a.

For case (a), assume that X is infinite dimensional and choose a sequence $\{x'_i\}$ of elements in X' which converge to zero in the weak* topology of X' and yet contain no norm convergent subsequence. Define f to be zero on $S - I$ and $f(s_i) = x'_i$. Now, via the representation theorem, $B(X, C(S))$ corresponds to the subspace B of $m(S : X')$ which consists of those functions in $m(S : X')$ which are continuous as maps from S to X' with the weak* topology, and $K(X, C(S))$ corresponds to the subspace of B which consists of those functions which have conditionally compact range. So the proof for case (a) will be completed by Lemma 4 if we can show that the function fg_a is continuous as a map from S to (X', X). To see this, suppose that $\{s(\alpha)\}$ is a net in S which converges to s. If s is isolated the net is eventually s and then $\{f(s(\alpha))g_a(s(\alpha))\}$ is eventually $f(s)g_a(s)$ and so fg_a is continuous at s, so we may suppose that s is not isolated. Since s is not isolated, $f(s)g_a(s)$ is zero and so we must show that $\{f(s(\alpha))g_a(s(\alpha))\}$ converges to zero; this net will converge to zero if for each natural number N there is an α_0 such that $s(\alpha)$ is not in $\{s_1, s_2, \cdots, s_N\}$ for $\alpha \geqq \alpha_0$, if there is no such α_0 for some N we find that s is isolated, a contradiction.

For case (b) we assume that weak and norm sequential convergence are not the same in X' and choose a sequence $\{x'_i\}$ which converges weakly to zero but has no norm convergent subsequence. Since x'_i converges weakly to zero it converges to zero in the weak* topology and so fg_a, as above, is continuous.

For case (c), let $\{x'_i\}$ be a sequence which converges to zero in the weak* topology but contains no weakly convergent subsequence. A bit of caution is necessary here. We are assuming that $X = Y$ is separable and so the weak* topology on the unit sphere in X' is metrizable [7, Theorem 1, page 426] and so if weak* and weak sequential convergence are the same, then the sphere is weak sequentially compact and hence weakly sequentially compact, hence weakly compact and so X is reflexive. However, if X is not separable, we may have weak* and weak sequential convergence the same in X' without X being reflexive; for example, $X = m$ [9, Theorem 9, page 168].

It follows from the representation theorem that the spaces are equal under the given conditions.

This completes the proof of Theorem 2.

There is no known example where $K(X, Y)$ is complemented in $B(X, Y)$ and yet is not equal to $B(X, Y)$, ditto for the subspace $W(X, Y)$ and for $K(X, Y)$ as a subspace of $W(X, Y)$.

A simple case which remains open is whether $K(m, c)$ is complemented in $B(m, c)$. If m had a separable complemented subspace which was infinite dimensional, then Theorem 2 would solve this problem; but m does not [12, Theorem 6, page 221].

Added in proof. The argument following Lemma 3, which relies on references [1] and [4] to show that for certain S the space $C(S)$ contains a complemented subspace isomorphic to either m or c, can be replaced by the elementary Corollary 6 of D. W. Dean's paper Subspaces of $C(H)$ Which Are Direct Factors of $C(H)$ (Proc. Amer. Math. Soc. 16 (1965), 237–242).

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REFERENCES

1. R. Arens, *Extensions of functions on fully normal spaces*, Pacific J. Math. **2** (1952), 11-22.

2. S. Banach, *Opérations Linéaires*, Chelsea, 1955.

3. R. Bartle, *On compactness in fuctional analysis*, Trans. Amer. Math. Soc. **79** (1955), 35-57.

4. K. Borsuk, *Über isomorphie der funktionalräume*, Bull. Int. Acad. Pol. Sci. (1933), 1-10.

5. M. Day, *Normed Linear Spaces*, Springer, 1962.

6. N. Dunford, and B. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323-392.

7. N. Dunford, and J. Schwartz, *Linear Operators*, Vol. I, Interscience, 1958.

8. L. Gillman, and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960.

9. A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espace du type* $C(K)$, Canadian J. Math. **5** (1953), 129-173.

10. S. Kakutani, *Simultaneous extension of continuous functions considered as a positive linear operation*, Japan J. Math. **17** (1940), 1-4.

11. S. Kakutani, and M. Nakamura, *Banach limits and the Čech compactification of a countable discrete set*, Proc. Imp. Acad. Japan **19** (1943), 224-229.

12. A. Pełczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209-227.

13. A. Pełczynski, and V. Sudakov, *Remark on non-complemented subspaces of the space* $m(S)$, Colloquim Math. **9** (1962), 85-88.

14. E. Thorp, *Projections onto the subspace of compact operators*, Pacific J. Math. **10** (1960), 693-696.

15. E. Thorp, and R. Whitley, *Operator representation theorems,* Illinois J. Math. (to appear).
16. R. Whitley, *Projecting m onto* c_0, Amer. Math. Monthly (to appear).

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