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EMBEDDING A CIRCLE OF TREES IN THE PLANE

H. C. WISER

Concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold, it is shown here that there is an embedding if the continuum is decomposable and the manifold is orientable. The embedding is obtained by constructing an annulus on the manifold containing the continuum; in the nonorientable case an annulus or a Möbius strip containing the continuum may be found. Similar results are obtained for continua on a 2-manifold which have a decomposition into continua with zero 1-dimensional Betti numbers such that the decomposition space is a finite planar graph.

This extends the results of [2] concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold. Definitions and a summary of other results may be found in [2].

THEOREM 1. *A decomposable homogeneous proper subcontinuum of an orientable 2-manifold can be embedded in the plane.*

Proof. Let X be a decomposable proper subcontinuum of an orientable 2-manifold M. By Theorem 11 of [2] there is a continuous collection G of disjoint continua filling X such that the decomposition space X' is a simple closed curve and the elements of G are mutually homeomorphic, homogeneous, and treelike. Consider the upper semi-continuous decomposition of M whose nondegenerate elements are the elements of G. By Theorem 1 of [1], the decomposition space M' is homeomorphic to M. Let A' be a closed annular neighborhood of the simple closed curve X' on M'. Let f be the projection map from M to M' and A be $f^{-1}(A')$. Consider A' to be filled by a continuous collection of simple closed curves $\{J_\alpha\}$, $\alpha \in [01]$, where $J_{1/2} = X'$. Then f is one-to-one on A-X and $f^{-1}(J_\alpha)$ must be compact since J_α is compact; thus $f/f^{-1}(J_\alpha)$ is a continuous one-to-one map of a compact set for $\alpha \neq 1/2$ and therefore a homeomorphism. For a closed subinterval I of $[01]$ not containing $1/2, f^{-1}(\Sigma_{\alpha \in I} J_\alpha)$ is a closed annulus. If $p \in A$-$f^{-1}(J_0 + J_1)$ then $f(p)$ is in the interior of A' and thus p is interior to A. The continuum A must then be a 2-manifold with boundary consisting of $f^{-1}(J_0) + f^{-1}(J_1)$. Fit 2-cells C_0 and C_1 to the boundary curves of A to make a 2-manifold M_1 without boundary. Considering an upper semi-continuous decomposition of M_1 with the elements of G as the nondegenerate elements, we have M_1' as merely

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A' with 2-cells C'_0 and C'_1 fitted to J_0 and J_1; i.e., M'_1 is a 2-sphere. Using Theorem 1 of [1] again, M_1 must be homeomorphic to M'_1 and A must be a closed annulus; X is thus planar.

In the same manner we have the following results:

THEOREM 2. *A decomposable homogeneous proper subcontinuum of a nonorientable 2-manifold is contained in an open annulus or open Möbius strip on the manifold.*

THEOREM 3. *If a subcontinuum of an orientable 2-manifold has an upper semi-continuous decomposition into continua with zero mod 2 1-dimensional Betti numbers*[1] *such that the decomposition space is a finite planar graph then the continuum can be embedded in the plane.*

In view of Theorem 1, a nonplanar homogeneous subcontinuum of an orientable 2-manifold would have to be in the class of nontreelike indecomposable continua. No planar homogeneous continuum in this class is known, although the pseudo-circle is a candidate. It would be nice to eliminate the condition of orientability in Theorem 1.

REFERENCES

1. J. H. Roberts and N. E. Steenrod, *Monotone transformations of two-dimensional manifolds*, Ann. of Math. **39** (1938), 851-863.

2. H. C. Wiser, *Decomposition and homogeneity of continua on a 2-manifold*, Pacific. J. Math. **12** (1962), 1145-1162.

UNIVERSITY OF UTAH
AND
WASHINGTON STATE UNIVERSITY

[1] As in [**1**], the statement "the continuum X on the 2-manifold M has a zero mod 2, 1-dimensional Betti number" means that for any complex K of M containing X there is a smaller complex L of M containing X such that each of its 1-cycles bounds in K.