DEVELOPMENT OF THE MAPPING
FUNCTION AT A CORNER

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Let D be a domain in the plane which is partially bounded by two curves Γ_1 and Γ_2 which meet at the origin and form there an interior angle $\pi\tau > 0$. Let N be an integer ≥ 2 and let α be a real number such that $0 < \alpha < 1$. Suppose that for $i = 1, 2, \Gamma_i$ admits a parametrization $x = x_i(t), y = y_i(t), 0 \leq t \leq 1$, where x_i and y_i have Nth derivatives which are uniformly α-Hölder continuous, and $|x_i'(t)| + |y_i'(t)| > 0$. Let $F(z)$ map the upper half plane conformally onto D in such a way that $F(0) = 0$. Then if τ is irrational $F(z)$ has an asymptotic expansion in powers of z and z^τ, with error term $o(z^{N\tau-\varepsilon})$. If $\tau = p/q$, a reduced fraction, then $F(z)$ has an asymptotic expansion in powers of z, z^τ, and $z^p \log z$, with error term $o(z^{N\tau-\varepsilon})$. In both cases ε is an arbitrarily small positive number. Furthermore expansions for derivatives of $F(z)$ of order $\leq N$ may be obtained by differentiating formally.

The behavior of such conformal maps at corners was first investigated by Lichtenstein [9]. Let $F^{-1}(z)$ be the function inverse to $F(z)$ which maps D onto the upper half plane. Lichtenstein showed that if Γ_1 and Γ_2 are analytic then

$$(1.1) \qquad \frac{d}{dz} F^{-1}(z) = z^{1/\tau - 1} \varphi(z)$$

where $\varphi(z)$ is continuous in $\bar{D}$ and $\varphi(0) \neq 0$. This result was later generalized in two ways. One was to weaken the requirements on Γ_1 and Γ_2. It follows from the work of Kellogg [4] and Warschawski [10] that with very modest conditions imposed on Γ_1 and Γ_2 one has

$$F^{-1}(z) = z^{1/\tau} \varphi(z)$$

where again $\varphi(z)$ is continuous in $\bar{D}$ and $\varphi(0) \neq 0$. In particular this follows if one assumes that Γ_1 and Γ_2 have continuously turning tangents in a neighborhood of the origin (though weaker conditions will suffice).

The other generalization of Lichtenstein's theorem was an improvement of the result (1.1), maintaining the analyticity requirement. For the case $\tau = 1$ Lewy [8] showed that $F(z)$ has an asymptotic expansion

in powers of z and $\log z$. Later Lehman [6] showed that expansions of the kind mentioned in the first paragraph are valid for all angles $\pi\tau > 0$, provided Γ_1 and Γ_2 are analytic. Thus in this paper we dovetail the results of the two developments. Furthermore we shall indicate some applications to the behavior at corners of solutions of elliptic partial differential equations; see [3], [5], [7], [8], [11] and [12].

2. **Principal results.** Let N be an integer ≥ 2 and let α be a real number such that $0 < \alpha < 1$. Assume that for $i = 1, 2, \Gamma_i$ admits a parametrization $x = x_i(t)$, $y = y_i(t)$ where $x_i(t)$ and $y_i(t)$ are uniformly $C^{N+\alpha}$ for $0 \leq t \leq 1$,[1] and assume that there exists a $\delta > 0$ such that $|x_i'(t)| + |y_i'(t)| \geq \delta$ for $0 \leq t \leq 1$. Let $F(z)$ map the upper half plane conformally onto D. Then $G(z) \equiv F(z^{1/\tau})$ maps the sector $0 < \arg z < \pi\tau$ onto D and we have the following theorems.

THEOREM 1. *If τ is irrational then there exists a polynomial $P(z, z^\tau)$ such that as $z \to 0$, $0 \leq \arg z \leq \pi$,*

$$F(z) = z^\tau P(z, z^\tau) + o(z^{N\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0,0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^\tau, z^p \log z)$ such that as $z \to 0$, $0 \leq \arg z \leq \pi$,

$$F(z) = z^\tau P(z, z^\tau, z^p \log z) + o(z^{N\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leq N$ may be obtained by differentiating formally.

THEOREM 2. *If τ is irrational then there exists a polynomial $P(z, z^{1/\tau})$ such that as $z \to 0$, $0 \leq \arg z \leq \pi\tau$,*

$$G(z) = zP(z, z^{1/\tau}) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0,0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^{1/\tau}, z^q \log z)$ such that as $z \to 0$, $0 \leq \arg z \leq \pi\tau$,

$$G(z) = zP(z, z^{1/\tau}, z^q \log z) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leq N$ may be obtained by differentiating formally.

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[1] This means there exists a constant K such that for $0 \leq s < t \leq 1$ and $0 \leq n \leq N$

$$\left| \frac{d^n}{dt^n} x_i(s) - \frac{d^n}{dt^n} x_i(t) \right| + \left| \frac{d^n}{dt^n} y_i(s) - \frac{d^n}{dt^n} y_i(t) \right| \leq K |s - t|^\alpha .$$

From Theorems 1 and 2 one can obtain an asymptotic expansion for the inverse function $F^{-1}(z)$ which maps D onto the upper half plane. The method is an iterative one, starting with $F(z) = o(z^{\tau-\varepsilon})$ and increasing the exponent of the error term; see, for instance, Wasow [11], pp. 49–50.

THEOREM 3. *If τ is irrational then there exists a polynomial $P(z, z^{1/\tau})$ such that as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$F^{-1}(z) = z^{1/\tau} P(z, z^{1/\tau}) + o(z^{N-1+1/\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^{1/\tau}, z^q \log z)$ such that as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,

$$F^{-1}(z) = z^{1/\tau} P(z, z^{1/\tau}, z^q \log z) + o(z^{N-1+1/\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leq N$ may be obtained by differentiating formally,

Since $G^{-1}(z) = (F(z))^\tau$, we have, by the binomial theorem.

THEOREM 4. *If τ is irrational there exists a polynomial $P(z, z^{1/\tau})$ such that as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$G^{-1}(z) = z P(z, z^{1/\tau}) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^{1/\tau}, z^q \log z)$ such that as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,

$$G^{-1}(z) = z P(z, z^{1/\tau}, z^q \log z) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leq N$ may be obtained by differentiating formally.

3. Applications to partial differential equations. The expansions of Theorems 2 and 4 have immediate applications to a previous paper of the author [12]. In particular § 4 and 5 of [12] need only be modified suitably to obtain the following theorems.

Let $U(x, y)$ be a solution in D of the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + KU = F$$

where K and F are $(N-1)$-times continuously differentiable in $D \cup \Gamma_1 \cup \Gamma_2 \cup \{0\}$, U is twice continuously differentiable in D, and U_x

and U_y are α-Hölder continuous in every compact subset of $D \cup \Gamma_1 \cup \Gamma_2$. We also assume that for $i = 1, 2, U$ satisfies on Γ_i a boundary condition

$$\delta_i \frac{\partial U}{\partial n} + A_i U = B_i$$

where $\delta_i = 0$ or 1, $\partial/\partial n$ represents the outgoing normal derivative, and A_i and B_i are $(N - 1)$-times continuously differentiable as functions of arc length, defined on $\Gamma_i \cup \{0\}$, and $A_i(0) \neq 0$, if $\delta_i = 0$. Finally, we assume that as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,

$$U(z) = o(z^\mu)$$

where $\mu > \max(-1, -1/\tau)$ if $\delta_1 = \delta_2$ and $\mu > \max(-1, -1/2\tau)$ if $\delta_1 \neq \delta_2$. Then

THEOREM 5.   *If $\delta_1 = \delta_2 = 0$ then as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$U(z) = \log z P_1 + \log \bar{z} P_2 + P_3 + o(z^{N-1-\varepsilon})$$

where P_1, P_2 and P_3 are polynomials in $z, \bar{z}, z^{1/\tau}$ and $\bar{z}^{1/\tau}$ if τ is irrational and in $z, \bar{z}, z^{1/\tau}, \bar{z}^{1/\tau}, z^q \log z$ and $\bar{z}^q \log \bar{z}$ if $\tau = p/q$; and ε is an arbitrarily small positive number. If $B_1(0)A_2(0) = B_2(0)A_1(0)$, P_1 and P_2 vanish identically. Furthermore expansions for derivatives of $U(z)$ of order $\leq N - 2$ may be obtained by differentiating formally.

THEOREM 6.   *If $\delta_1 = 0$ and $\delta_2 = 1$ (or $\delta_1 = 1$ and $\delta_2 = 0$) then as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$U(z) = P + o(z^{N-1-\varepsilon})$$

where P is a polynomial in $z, \bar{z}, z^{1/2\tau}$ and $\bar{z}^{1/2\tau}$ if τ is irrational; P is a polynomial in $z, \bar{z}, z^{1/2\tau}, \bar{z}^{1/2\tau}, z^q \log z$ and $\bar{z}^q \log \bar{z}$ if $\tau = p/q$ and q is odd; P is a polynomial in $z, \bar{z}, z^{1/2\tau}, \bar{z}^{1/2\tau}, z^{q/2} \log z$ and $\bar{z}^{q/2} \log \bar{z}$ if $\alpha = p/q$ and q is even; and ε is an arbitrarily small positive number. Furthermore expansions for derivatives of $U(z)$ of order $\leq N - 2$ may be obtained by differentiating formally.

THEOREM 7.   *If $\delta_1 = \delta_2 = 1$ then as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$U(z) = \log z P_1 + \log \bar{z} P_2 + P_3 + o(z^{N-\varepsilon})$$

where P_1, P_2 and P_3 are polynomials in $z, \bar{z}, z^{1/\tau}$ and $\bar{z}^{1/\tau}$ if τ is irrational and in $z, \bar{z}, z^{1/\tau}, \bar{z}^{1/\tau}, z^q \log z$ and $\bar{z}^q \log \bar{z}$ if $\tau = p/q$; and ε is an arbitrarily small positive number. If $U(z)$ is bounded at the origin then P_1 and P_2 vanish identically. Furthermore derivatives of $U(z)$ of order $\leq N - 1$ may be obtained by differentiating formally.

4. **Some Lemmas.** Later we shall need some properties of functions which are Hölder continuous in a set, but whose Hölder constants diverge in a certain way near a boundary point of the set. Let S be a subset of the plane which does not contain the origin, but of which the origin is a cluster point. Let μ and β be real numbers, $0 \leq \beta < 1$, and let M be a nonnegative integer. Let $f(x, y)$ be a real or complex valued function such that $f(x, y) \in C^{M+\beta}$ for $(x, y) \in S$, and suppose that for $0 \leq n \leq M$

(i) $D^n f(x, y) = O(z^{\mu-n})$

as $z \to 0$, $z \in S$, where D^n ranges over all nth order partial derivatives, and

(ii)   there exists a constant K such that

$$\sup \frac{|D^n f(z) - D^n f(\zeta)|}{|z - \zeta|^\beta} |z|^{n+\beta-\mu} \leq K$$

where the supremum is taken over all derivatives D^n, and over all points $z, \zeta \in S$ such that $0 < |z - \zeta| < \delta |z|, \delta |\zeta|$; δ is assumed to be some positive number < 1. The totality of such functions we designate by $W_\mu^{M+\beta}(S)$. If S is the sector $\delta_1 \leq \arg z \leq \delta_2, 0 < |z| < |z_0|$, we write $W_\mu^{M+\beta}([\delta_1, \delta_2])$. We omit the dependence on z_0 because we are only concerned with properties (i) and (ii) in some neighborhood of the origin. If S is a segment $0 < x < A$ we write $W_\mu^{M+\beta}$; properties (i) and (ii) should then be modified properly for a function of one variable. We observe that if $\beta = 0$ property (ii) follows from property (i) and the condition $|z - \zeta| < \delta |z|, \delta |\zeta|$.

We now list some properties of the W-spaces. We state them for the complex case, though with suitable modifications the properties hold for the real case. Thus we assume $0 < |z - \zeta| < \delta |z|, \delta |\zeta|$, and $z, \zeta \in S$.

1. $1 - \delta < |z/\zeta| < 1 + \delta$.

2.   Let $\mu \leq -1$. If $(\partial/\partial x)f(z), (\partial/\partial y)f(z) \in W_\mu^0(S)$ then $f(z)$ differs by a constant from a function in $W_{\mu+1}^1(S)$. The proof is contained in Bourbaki [2].

3.   If $(\partial/\partial x)f(z), (\partial/\partial y)f(z) \in W_\mu^0(S)$ then f differs by a constant from a function in $W_{\mu+1}^\beta(S), 0 \leq \beta < 1$. The proof follows from property 2 above and the mean value theorem for functions of two variables.

4.   There exists a constant K depending only on μ, β and δ such that

$$|z^{\mu+\beta} - \zeta^{\mu+\beta}| \leq K |z|^\mu |z^\beta - \zeta^\beta| \leq K |z|^\mu |z - \zeta|^\beta .$$

Here we assume that S is so chosen that $z^{\mu+\beta}$ and z^β are single valued functions.

5.   Let $f(z) \equiv z^\mu$, and assume z^μ is single valued for $z \in S$. Then

for all integers $M \geqq 0$ and any β such that $0 \leqq \beta \leqq 1$, $f(z) \in W_\mu^{M+\beta}(S)$.

6. Let M and N be integers $\geqq 0$, let α and β satisfy $0 < \alpha, \beta < 1$, and let μ and ν be real numbers. Let $f(z) \in W_\mu^{M+\alpha}(S)$ and $g(z) \in W_\nu^{N+\beta}(S)$. Let $\alpha' = \min(\alpha, \beta)$, $M' = \min(M, N)$ and $\mu' = \min(\mu, \nu)$. Then

$$f(z) + g(z) \in W_{\mu'}^{M'+\alpha'}(S)$$

and

$$f(z)g(z) \in W_{\mu\nu}^{M'+\alpha'}(S) .$$

Proof. The first statement follows from the fact that $W_{\mu'}^{M'+\alpha'}(S) \supseteqq W_\mu^{M+\alpha}(S) \cap W_\nu^{N+\beta}(S)$, and because the W-spaces are linear. For the second statement we observe first that $f(z)g(z) \in C^{M'+\alpha'}(S)$. Then

$$f(z)g(z) = O(z^\mu)O(z^\nu) = O(z^{\mu+\nu}) ,$$

and

$$\begin{aligned}
|f(z)g(z) - f(\zeta)g(\zeta)| &\leqq |f(z)| \, |g(z) - g(\zeta)| + |g(\zeta)| \, |f(z) - f(\zeta)| \\
&\leqq K_1 |z|^{\mu+\nu-\beta} |z - \zeta|^\beta + K_2 |z|^{\mu+\nu-\alpha} |z - \zeta|^\alpha \\
&\leqq K_3 |z|^{\mu+\nu-\alpha'} |z - \zeta|^{\alpha'}
\end{aligned}$$

since

$$|z - \zeta|^\beta = |z - \zeta|^{\beta-\alpha'} |z - \zeta|^{\alpha'} \leqq \delta^{\beta-\alpha'} |z|^{\beta-\alpha'} |z - \zeta|^{\alpha'}$$

and

$$|z - \zeta|^\alpha \leqq \delta^{\alpha-\alpha'} |z|^{\alpha-\alpha'} |z - \zeta|^{\alpha'} .$$

The proof then follows easily from induction.

We now state three lemmas. The analogous theorems for the real case follow without difficulty.

LEMMA 1. *Let $\mu > 0$ and let $f(z) \in W_\mu^{M+\alpha}(S)$. Suppose also that $|f(z)| \geqq \delta_1 |z|^\mu$, $z \in S$, for some $\delta_1 > 0$. Let S' be the range of S and suppose $g(z) \in W_\nu^{N+\beta}(S)$. Then*

$$h(z) = g(f(z)) \in W_{\mu\nu}^{M'+\alpha\beta}(S)$$

where $M' = \min(M, N)$.

Proof. It is clear that $h(z) = O(z^{\mu\nu})$. Next, for $|z - \zeta| < \delta|z|, \delta|\zeta|$

$$\begin{aligned}
|h(z) - h(\zeta)| &= |g(f(z)) - g(f(\zeta))| \\
&\leqq K_1 \max(|f(z)|^{\nu-\beta}, |f(\zeta)|^{\nu-\beta}) |f(z) - f(\zeta)|^\beta \\
&\leqq K_2 \max(|z|^{\mu(\nu-\beta)}, |\zeta|^{\mu(\nu-\beta)}) |z|^{(\mu-\alpha)\beta} |z - \zeta|^{\alpha\beta} \\
&\leqq K_3 |z|^{\mu\nu-\alpha\beta} |z - \zeta|^{\alpha\beta}
\end{aligned}$$

provided $|f(z) - f(\zeta)| \leqq \delta\,|f(z)|,\ \delta\,|f(\zeta)|$. In the contrary case, however, suppose $|f(z)| \leqq |f(\zeta)|$ and $|f(z) - f(\zeta)| > \delta\,|f(z)|$. Then

$$\frac{|\,g(f(z)) - g(f(\zeta))\,|}{|\,z - \zeta\,|^{\alpha\beta}} \leqq K_4\,|\,g(f(z)) - g(f(\zeta))\,|\,|\,z\,|^{(\mu-\alpha)\beta}\,|\,f(z) - f(\zeta)\,|^{-\beta}$$

$$\leqq K_5 \max\,(|\,f(z)\,|^{\nu},\ |\,f(\zeta)\,|^{\nu})\,|\,f(z)\,|^{-\beta}\,|\,z\,|^{(\mu-\alpha)\beta}$$

$$\leqq K_6 \max\,(|\,z\,|^{\mu\nu},\ |\,\zeta\,|^{\mu\nu})\,|\,z\,|^{-\alpha\beta}$$

$$\leqq K_7\,|\,z\,|^{\mu\nu-\alpha\beta}$$

and thus $h(z) \in W^{\alpha\beta}_{\mu\nu}(S)$.

Writing $f = \varphi + i\psi$, $\varphi,\ \psi$ real, we have

$$\frac{\partial}{\partial x}h(z) = g_{\varphi}(f(z))\varphi_x(z) + g_{\psi}(f(z))\psi_x(z)\ .$$

Now by definition $g_{\varphi} \in W^{N-1+\beta}_{(\nu-1)}(S')$, and thus $g_{\varphi}(f(z)) \in W^{M'-1+\alpha\beta}_{(\nu-1)\mu}(S)$ as well as $\varphi_x,\ \psi_x \in W^{M-1+\alpha}_{\mu-1}(S)$, and thus, by Proposition 6,

$$\frac{\partial}{\partial x}h(z) \in W^{M'-1+\alpha\beta}_{\mu\nu-1}(S)\ .$$

The lemma follows by similar arguments.

LEMMA 2. *Let $f(z)$ map S onto a set S' in such a way that $f(z)$ is conformal on the interior of S, and suppose $f(z) \in W^{M+\alpha}_{\mu}(S)$, $\mu > 0$. Assume also that $|f'(z)| \geqq \delta_1\,|\,z\,|^{\mu-1}$, $z \in S$, for some $\delta_1 > 0$. Let $g(\zeta)$ be the function inverse to $f(z)$ which maps the interior of S' into S, and assume that $g(\zeta) \in C^{M+\alpha}(S')$ (this is the case if S and S' have sufficiently smooth boundaries).*
Then

$$g(\zeta) \in W^{M+\alpha}_{1/\mu}(S')\ .$$

Proof. Let z_0 be fixed. Then for $|z|$ sufficiently small we have $|f(z)| \leqq |f(z_0)|$. Thus

$$|f(z_0)| - |f(z)| = |\,|f(z_0)| - |f(z)|\,| \leqq |f(z_0) - f(z)|$$

$$= \left|\int_z^{z_0} f'(w)dw\right|$$

$$\leqq \int_z^{z_0} K_1\,|\,w\,|^{\mu-1}\,|\,dw\,| \leqq K_2(1 + |\,z_0\,|^{\mu} - |\,z\,|^{\mu})$$

where the path of integration is taken to be a union of paths $\arg w =$ const. and $|\,w\,| = $ const. Thus

$$|f(z)| \geqq K_3 + K_2\,|\,z\,|^{\mu} \geqq K_2\,|\,z\,|^{\mu}\ .$$

Since $|f(z)| \leqq K_4\,|\,z\,|^{\mu}$ we have

$$| g'(\zeta) | = | f'(z) |^{-1} \leqq \delta_1^{-1} | z |^{1-\mu} \leqq K_5 | \zeta |^{1/\mu-1} .$$

By Propositions 2 and 3 we have $g(\zeta) \in W_{1/\mu}^{\alpha}(S')$.

Next,

$$\frac{| g'(\zeta_1) - g'(\zeta_2) |}{| \zeta_1 - \zeta_2 |^{\alpha}} = \frac{| f'(z_2) - f'(z_1) |}{| f'(z_1) | | f'(z_2) | | z_1 - z_2 |^{\alpha}} \frac{| z_1 - z_2 |^{\alpha}}{| \zeta_1 - \zeta_2 |^{\alpha}}$$

$$\leqq K_6 | z_1 |^{1-\mu} | z_2 |^{1-\mu} | z_1 |^{\mu-1-\alpha} | f'(z) |^{-\alpha}$$

where z lies between z_1 and z_2. Since $1 - \delta \leqq | z_1/z_2 | \leqq 1 + \delta$ we have

$$\frac{| g'(\zeta_1) - g'(\zeta_2) |}{| \zeta_1 - \zeta_2 |^{\alpha}} \leqq K_7 | z_1 |^{1-\mu-\alpha-(\mu-1)\alpha} \leqq K_8 | \zeta_1 |^{(1/\mu)-1-\alpha} .$$

Thus $g(z) \in W_{1/\mu}^{1+\alpha}(S')$. The proof follows by induction.

LEMMA 3. *Let* $f(z) \in W_{\mu}^{N+\alpha}(S)$ *and let* $P(z)$ *be a polynomial of degree* $< \mu$ *with* $P(0) = 1$. *Let* γ *be a positive real number. Then there exists a function* $f_1(z) \in W_{\mu}^{N+\alpha}(S)$ *and a polynomial* $P_1(z)$ *of degree* $< \mu$ *such that*

$$(P(z) + f(z))^{\gamma} = P_1(z) + f_1(z) .$$

Proof. The proof follows easily from the binomial theorem.

In obtaining the asymptotic expansions we shall come across certain integrals which were studied in [8], [5], and [12]. To estimate these integrals we use the following lemmas. The first was proved in [8] and [5]. The second is a generalization of a theorem in [8], [5], and [12] and will be proved in § 9. The integrals are Lebesgue integrals extended over positive values of t. The variable z lies on the logarithmic Riemann surface with branch point at the origin. The kernel of the integrands is the function $\log (t - z)$ which we define in the following way. For fixed t we make cuts along the Riemann surface from $te^{2\pi i k}$ to $\infty e^{2\pi i k}, k = 0, \pm 1, \pm 2, \cdots$. The logarithm is uniquely defined, except for z lying on these cuts, as the analytic continuation of the logarithm which is real for $0 < | z | < t, \arg z = 0$.

LEMMA 4. *Let* A *be a positive number,* μ *a real number* > -1, *and* n *a nonnegative integer. For* $0 < \arg z < 2\pi$, *let*

$$f(z) = \int_0^A t^{\mu} (\log t)^n \log (t - z) dt .$$

Then there exists a polynomial $P(\log z)$ *and a power series* $p(z)$ *which converges for* $| z | < A$, *such that*

$$f(z) = z^{\mu+1}P(\log z) + p(z) \ .$$

If μ is an integer the polynomial P is of degree $n + 1$; otherwise it is of degree n.

LEMMA 5. Let μ be a real number > -1 which is not an integer, and let $\beta(t) \in W_\mu^{N-1+\alpha}$ for $0 < t \leqq A$. For $0 < \arg z < \pi$, let

$$g(z) = \int_0^A \beta(t) \log (t - z) dt \ .$$

Then there exists a polynomial $q(z)$ of degree $< \mu + 1$ such that

$$\varphi(z) \equiv g(z) - q(z) \in W_{\mu+1}^{N+\alpha}((0, \pi)) \ .$$

A similar result obtains for $-\pi < \arg z < 0$, with the same polynomial $q(z)$.

5. **Preliminary results.** It follows from Warschawski [10] that $F^{-1}(z)$, which maps D onto the upper half plane, satisfies the relation

(5.1) $$F^{-1}(z) = z^{1/\tau}\varphi(z)$$

where $\varphi(z)$ is continuous in $D \cup \Gamma_1 \cup \Gamma_2 \cup \{0\}$ and $\varphi(0) \neq 0$. We shall show in this section that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D \cup \Gamma_1 \cup \Gamma_2)$.

It follows easily from the Cauchy integral theorem that

$$F^{-1}(z) \in W_{1/\tau}^{N+\alpha}([\lambda, \pi\tau - \lambda])$$

where λ is a small positive number: one simply examines the integral

$$\frac{d^n}{dz^n}F^{-1}(z) = \frac{n!}{2\pi i} \oint \frac{F^{-1}(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

taken over a circle about z of radius $\delta |z|$, bearing in mind that $F^{-1}(z) = 0(z^{1/\tau})$ as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$. Thus it will suffice to show that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$ and $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D'')$ where $D' = D \cap \{z \colon \arg z \geqq \pi\tau - 2\lambda\}$ and $D'' = D \cap \{z \colon \arg z \leqq 2\lambda\}$. Because of the symmetry between Γ_1 and Γ_2 we need only show that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$.

Next, if we have $V(z) = Im\,F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$, then, by Warschawski's result above and the Cauchy-Riemann equations, we have $Re\,F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$, and thus $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$. Thus we shall show $V(z) \in W_{1/\tau}^{N+\alpha}(D')$.

Now we make a transformation which has the effect of straightening out Γ_2. Let $y = \beta(x)$ be a parametrization of Γ_2 (if $\tau = 1/2$ or $3/2$ this is impossible; but a small rotation about the origin would take care of this difficulty). Then it can be shown that $\beta(x) \in C^{N+\alpha}$ for $0 \leqq x \leqq A$, where A is a small positive number; furthermore, by

the hypotheses of § 2, for $0 \leqq n \leqq N$, $0 \leqq x_1, x_2 \leqq x_0$

$$\sup_{x_1 \neq x_2} \frac{|\beta^{(n)}(x_1) - \beta^{(n)}(x_2)|}{|x_1 - x_2|^\alpha} < \infty .$$

We make the transformation $\xi = x$, $\eta = y - \beta(x)$, and set $v(\xi, \eta) = V(x, y)$. Then v is defined (at least) for $0 < \xi^2 + \eta^2 < A_1$, $-\xi \tan \delta \leqq \eta < 0$, provided A_1 and δ are chosen small enough. The points (ξ, η) are images of a subset of the points (x, y) such that $(\pi\tau - \delta_1)x \leqq y \leqq \beta(x)$, where δ_1 is a small positive number. Since $\beta(x) = O(x)$, we find that $\delta_2 \leqq y/x \leqq 1/\delta_2$ for some $\delta_2 > 0$, and thus, since

$$\xi^2 + \eta^2 = x^2 + y^2 - 2y\beta(x) + (\beta(x))^2 ,$$

$$\delta_3 \leqq \frac{x^2 + y^2}{\xi^2 + \eta^2} \leqq 1/\delta_3$$

for some $\delta_3 > 0$. Since $V(x, y) = O(z^{1/\tau})$, we have $v(\xi, \eta) = O(\zeta^{1/\tau})$, where $\zeta = \xi + i\eta$.

We now state a lemma which is a special case of a theorem of Agmon, Douglis and Nirenberg ([1], pp. 657–660). Let $0 < R < 1$ and let S be the semicircle $\xi^2 + \eta^2 < R$, $\eta \leqq 0$. For $\zeta \in S$ let d_ζ denote the distance from ζ to the circular part of the boundary of S.

LEMMA 6. *Let $u(\xi, \eta)$ be a solution of a uniformly elliptic partial differential equation*

$$Lu = au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = 0 ,$$

whose coefficients are $C^{N-2+\alpha}$ in S with uniform α-Hölder constants. Let $u(\xi, 0) = 0$ for $-R < \xi < R$. If $u \in C^{2+\alpha}(S)$ then $u \in C^{N+\alpha}(S)$, and there exists a constant K, independent of u and R, such that

$$|u|_{N+\alpha} \leqq K \sup_{\zeta \in S} |u(\zeta)|$$

where

$$|u|_{N+\alpha} = \sup_{4|\zeta_1 - \zeta_2| < d_{\zeta_1}, d_{\zeta_2}} d_{\zeta_1}^{N+\alpha} \frac{|D^N u(\zeta_1) - D^N u(\zeta_2)|}{|\zeta_1 - \zeta_2|^\alpha}$$

$$+ \sum_{k=0}^{N} \sup_{\zeta \in S} d_\zeta^k |D^k u(\zeta)| ;$$

the suprema are taken over all kth and Nth order derivatives of u.

Since $V(x, y)$ is harmonic, we have

$$Lv \equiv v_{\xi\xi} + (1 + \beta'(\xi)^2)v_{\eta\eta} - 2\beta'(\xi)v_{\xi\eta} - \beta''(\xi)v_\eta = 0$$

for $0 < \xi^2 + \eta^2 < A_1$, $-\xi \tan \delta \leqq \eta \leqq 0$. Also

$$v(\xi, 0) = V(x, \beta(x)) = 0 .$$

We now apply the lemma to v and the semicircles

$$(\xi - \xi_0)^2 + \eta^2 \leqq \xi_0^2 \sin^2 \delta, \eta \leqq 0$$

where $0 < \xi_0 < (1/2)A_1$; these semicircles are tangent to the rays $\eta = 0$, and $\eta = -\xi \tan \delta$. In each semicircle we have, for some $K_1 > 0$,

$$\sup |v(\zeta)| \leqq K_1 |\zeta|^{1/\tau} .$$

In the semicircle $(\xi - \xi_0)^2 + \eta^2 \leqq ((1/2)\xi_0 \sin \delta)^2, \eta \leqq 0$, we have $d_\zeta \geqq (1/2)\xi_0 \sin \delta$. Thus for $(\xi - \xi_0)^2 + \eta^2 \leqq ((1/2)\xi_0 \sin \delta)^2, \eta \leqq 0$, we have

$$\zeta^k |D^k v(\zeta)| \leqq \left(\xi_0\left(1 + \frac{1}{2}\sin \delta\right)\right)^k |D^k v(\zeta)|$$

$$\leqq \left(\frac{2}{\sin \delta}\left(1 + \frac{1}{2}\sin \delta\right)d_\zeta\right)^k |D^k v(\zeta)| \leqq K_2 |\zeta|^{1/\tau}$$

for $0 \leqq k \leqq N$. Thus $v(\zeta) \in W_{1/\tau}^N([-\delta_4, 0])$ where δ_4 is small. By the mean value theorem $v(\zeta) \in W_{1/\tau}^{N-1+\alpha}([-\delta_4, 0])$. To estimate $|D^N v(\zeta_1) - D^N v(\zeta_2)|$ we use the lemma again; the details are similar to those above. Thus we can conclude that $v(\zeta) \in W_{1/\tau}^{N+\alpha}([-\delta_4, 0])$. Since

$$\delta_3 \leqq \frac{x^2 + y^2}{\xi^2 + \eta^2} \leqq \frac{1}{\delta_3} ,$$

it follows, by easy calculations, that for some small positive λ,

$$V(z) \in W_{1/\tau}^{N+\alpha}([\pi\tau - \lambda, \pi\tau]) .$$

Thus we conclude that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D \cup \Gamma_1 \cup \Gamma_2)$.

6. A preliminary transformation. From now on for the sake of definiteness we will assume that Γ_1 is tangent to the positive x-axis at the origin and that Γ_2 is tangent to the ray $\arg z = \pi\tau$ at the origin.

We set $H(z) = (F(z))^{1/\tau}$. Then $H(z)$ maps the upper half plane conformally onto a domain D' which is the image of D under the transformation $z \rightarrow z^{1/\tau}$. D' is partially bounded by curves Γ_1' and Γ_2' which have horizontal tangents at the origin. From the binomial theorem it is clear that theorem 1 is equivalent to an asymptotic expansion

$$(6.1) \quad \begin{aligned} H(z) &= zP(z, z^\tau) + o(z^{(N-1)\tau-\varepsilon}) & (\alpha \text{ irrational}) \\ H(z) &= zP(z, z^\tau, z^p \log z) + o(z^{(N-1)\tau-\varepsilon}) & (\alpha = p/q) \end{aligned}$$

as $z \rightarrow 0, 0 \leqq \arg z \leqq \pi$, where $\varepsilon > 0$ can be chosen arbitrarily small

and the polynomial P has a nonvanishing constant term; furthermore we must show that we can differentiate (6.1) N times. Since Theorems 2, 3 and 4 follow directly from Theorem 1, we need only prove (6.1).

By Lemma 1, and since $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D \cup \Gamma_1 \cup \Gamma_2)$, we have

$$H^{-1}(z) = F^{-1}(z^\tau) \in W_1^{N+\alpha^2}(D' \cup \Gamma_1' \cup \Gamma_2') \; .$$

By Lemma 2, $H(z) \in W_1^{N+\alpha^2}([0, \pi])$.

7. **An integral representation.** We will now construct an integral representation for $H(z)$ based on the equations for Γ_1 and Γ_2. Let $F(z) = \xi + i\eta$. Then we have

$$\eta = \sum_{n=1}^{N-1} c_n \xi^n + \varphi_1(\xi)$$

where $\varphi_1 \in W_N^{N+\alpha}$; this is merely the Taylor series for Γ_1, and is valid for $0 \leqq \xi \leqq \xi_0$.

We will now adopt the convention of dropping subscripts on coefficients whose value is unimportant; then we have

$$\eta = \sum_{n=1}^{N-1} c\xi^n + \varphi_1(\xi) \; .$$

With $w = H(z) = u + iv$, we have

$$w = (\xi + i\eta)^{1/\tau} = \xi^{1/\tau}\Big(1 + i \sum_{n=1}^{N} c\xi^{n-1} + \frac{\varphi_1(\xi)}{\xi}\Big)^{1/\tau}$$

$$= \xi^{1/\tau}\Big(a + \sum_{n=1}^{N} c\xi^n + \varphi_2(\xi)\Big) \; ,$$

and by Lemma 3, $\varphi_2 \in W_{N-1}^{N+\alpha}$. It is readily seen that $Re\ a \neq 0$. Then we have, by separating real and imaginary parts,

(7.1) $u = a_1 \xi^{1/\tau}(1 + c\xi + c\xi^2 + \cdots + c\xi^{N-2} + \varphi_3(\xi))$

(7.2) $v = \xi^{1/\tau}(c + c\xi + c\xi^2 + \cdots + c\xi^{N-2} + \varphi_4(\xi))$

with $\varphi_3, \varphi_4 \in W_{N-1}^{N+\alpha}$. Next,

$$u^\tau = a_1^\tau \xi(1 + c\xi + c\xi^2 + \cdots + c\xi^{N-2} + \varphi_5(\xi))$$

with $\varphi_5 \in W_{N-1}^{N+\alpha}$. As $a_1^\tau \neq 0$, we have, by the inverse function theorem,

(7.3) $\xi = u^\tau(c + cu^\tau + cu^{2\tau} + \cdots + cu^{(N-2)\tau} + \varphi_6(u^\tau))$

where φ_6, considered as a function of u^τ, belongs to $W_{N-1}^{N+\alpha}$. Thus by Lemma 1, $\varphi_7(u) = \varphi_6(u^\tau) \in W_{(N-1)\tau}^{N+\alpha^2}$. Substituting (7.3) in the right side of (7.2), we obtain

$$(7.4) \quad v = u\Big(\sum_{j=0}^{N-2} cu^{j\tau} + \varphi_7(u)\Big)^{1/\tau}$$
$$\times \Big(\sum_{j=0}^{N-2} cu^{j\tau}\Big(\sum_{k=0}^{N-2} cu^{k\tau} + \varphi_7(u)\Big)^j + \varphi_4(\xi)\Big).$$

We set

$$\varphi_8(u) = \varphi_4(\xi) = \varphi_4\Big(u^\tau\Big(\sum_{j=0}^{N-2} cu^{j\tau} + \varphi_7(u)\Big)\Big).$$

It is easily checked that $\xi(u) \in W_\tau^{N+\alpha^2}$ as a function of u, and thus $\varphi_8(u) \in W_{(N-1)\tau}^{N+\alpha^2}$. Thus, expanding the right side of (7.4), it follows that

$$v = u(c + cu^\tau + \cdots + cu^{(N-1)\tau} + \varphi_9(u))$$

with $\varphi_9 \in W_{(N-1)\tau}^{N+\alpha^2}$. Finally, $(dv/du)\,|_{u=0} = 0$, and thus

$$(7.5) \qquad v = u(cu^\tau + cu^{2\tau} + \cdots + cu^{(N-1)\tau} + \varphi_9(u)).$$

This equation is valid for v and u defined on the segment $y = 0$, $0 \leq x \leq A$, provided A is chosen small enough.

If $0 < \tau < 1/2$ or $3/2 < \tau \leq 2$ we can repeat the same argument on Γ_2: note that we never used the fact that Γ_1 has a horizontal tangent, but only that Γ_1' (and Γ_2') has a horizontal tangent at the origin. If $1/2 < \tau < 3/2$, we replace ξ by $|\xi|$; and for $0 < \tau \leq 2$, we replace u by $|u|$.

Finally, if $\tau = 1/2$ or $3/2$ we begin with the equation

$$\xi = \sum_{n=1}^N c\eta^n + \varphi_{10}(\eta)$$

and carry through with the roles of ξ and η reversed. Thus we have, for $-A \leq x \leq 0, y = 0$,

$$(7.6) \qquad v = u(cu^\tau + cu^{2\tau} + \cdots + cu^{(N-1)\tau} + \varphi_{11}(u))$$

with $\varphi_{11} \in W_{(N-1)\tau}^{N+\alpha^2}$.

We now consider the Green's function for the upper half plane

$$(7.7) \qquad G(t, z) = -\frac{1}{2\pi}\{\log|t - z| + \log|t - \bar{z}|\},$$

where $t = x_t + iy_t$. It is easily seen that $(\partial/\partial y_t)G(x_t, z) = 0$. We apply Green's theorem to the functions $G(t, z)$ and $u(t) = Re\, H(t)$ on the semi-circle $0 < |t| < A, y_t > 0$, and obtain

$$u(z) = \int_{-A}^A G(t, z)\frac{\partial}{\partial y_t}u(t)dt + \int_{\substack{y_t>0 \\ |t|=A}} (uG_{n_t} - Gu_{n_t})ds_t$$

where s_t represents arc length and n_t the outward normal. By (7.7) we have

$$\int_{\substack{y_t > 0 \\ |t| = A}} (uG_{n_t} - Gu_{n_t})ds_t = p(z) + p(\bar{z})$$

where $p(z)$ is a power series which converges for $|z| < A$. Also, for $y_t = 0$,

$$G(t, z) = -\frac{1}{2\pi}\{\log|t - z| + \log|t - \bar{z}|\}$$

$$= -\frac{1}{2\pi}\log|t - z|^2$$

$$\equiv -\frac{1}{2\pi}\{\log(t - z) + \log(t - \bar{z})\} .$$

Here we define $\log(t - z)$ as the analytic continuation of the logarithm which is real for $0 < |z| < t$, $\arg z = 0$. The congruence holds modulo $2\pi i$; however, each of the logarithms on the right side has imaginary part $> -\pi$ and $< \pi$. Thus we may replace the congruence by equality. With these observations in mind, we obtain

$$(7.8) \qquad u(z) = -\frac{1}{2\pi}\int_{-A}^{A}\frac{\partial}{\partial y_t}u(t)$$
$$\times \{\log(t - z) + \log(t - \bar{z})\}dt + p(z) + p(\bar{z}) .$$

Since $u(z) = Re\, H(z)$ and $p(z)$ has real coefficients, we replace (7.8) by the equation of which it is the real part, namely

$$H(z) = -\frac{1}{\pi}\int_{-A}^{A}\frac{\partial}{\partial y_t}u(t)\log(t - z)dt + p(z) + \text{const.} ,$$

where the constant takes care of the nonuniqueness of the conjugate harmonic function of $u(z)$. We now drop this constant, changing $p(z)$ if necessary, and use (7.5) and (7.6), together with

$$\frac{\partial}{\partial y_t}u(t) = -\frac{\partial}{\partial x_t}v(t) ,$$

to obtain

$$(7.9) \qquad \begin{aligned} H(z) &= \frac{1}{\pi}\int_{-A}^{0}u_t(t, 0)\left\{\sum_{j=1}^{N-1}cu^{j\tau} + \varphi(u)\right\}\log(t - z)dt \\ &+ \frac{1}{\pi}\int_{0}^{A}u_t(t, 0)\left\{\sum_{j=1}^{N-1}cu^{j\tau} + \psi(u)\right\}\log(t - z)dt + p(z) . \end{aligned}$$

Here

$$\varphi(u) = \frac{d}{du}(u\varphi_{11}(u)) \in W_{(N-1)\tau}^{N-1+\alpha^2}$$

and

$$\psi(u) = \frac{d}{du}(u\varphi_9(u)) \in W_{(N-1)\tau}^{N-1+\alpha^2}.$$

Furthermore, (7.9) is valid for $0 \leqq \arg z \leqq \pi, 0 < |z| < A$.

8. Obtaining the asymptotic expansions. We have, for $-A \leqq t < 0$, $H(t) \in W_1^{N+\alpha^2}$ and thus $u(t) \in W_1^{N+\alpha^2}$. Hence

$$(u(t))^{n\tau} \in W_{n\tau}^{N+\alpha^3}$$

$$u_t \in W_0^{N-1+\alpha^2}$$

and thus

$$(8.1) \qquad u_t(t, 0)\left\{\sum_{j=1}^{N-1} cu^{j\tau} + \varphi(u)\right\} \in W_\tau^{N-1+\alpha^3}$$

as a function of $t, -A \leqq t < 0$. Similarly

$$(8.2) \qquad u_t(t, 0)\left\{\sum_{j=1}^{N-1} cu^{j\tau} + \psi(u)\right\} \in W_\tau^{N-1+\alpha^3}$$

for $0 < t \leqq A$. Thus by Lemma 5, if $\tau \neq 1, 2$,

$$H(z) = az + bz^2 + \chi_1(z)$$

where $\chi_1(z) \in W_{1+\tau}^{N+\alpha^3}((0, \pi))$. As $H(z)$ has α-Hölder continuous Nth derivatives for $0 \leqq \arg z \leqq \pi$, we must have $\chi_1(z) \in W_{1+\tau}^{N+\alpha^3}([0, \pi])$.

If $\tau = 1$ or 2 Lemma 5 will not apply. However, if ε is any small positive number we can replace the $W_\tau^{N-1+\alpha^3}$ of (8.1) and (8.2) with $W_{\tau-\varepsilon}^{N-1+\alpha^3}$, and thus we can always write

$$(8.3) \qquad H(z) = az + bz^2 + \chi_1(z)$$

where $\chi_1(z) \in W_{1+\tau-\varepsilon}^{N+\alpha^3}([0, \pi])$.

We now prove Theorem 1 by induction. In the future we shall use the symbol α to represent any number between 0 and 1, and ε to represent an arbitrarily small positive number such that $n\tau - \varepsilon$ is not an integer for $0 \leqq n \leqq N$. In particular we write $\chi_1(z) \in W_{1+\tau-\varepsilon}^{N+\alpha}([0, \pi])$.

First let α be irrational. Assume that for some m, with $0 < m < N - 1$, that

$$H(z) = zP_m(z, z^\tau) + \chi_m(z)$$

where $P_m(z, z^\tau)$ is a polynomial in its arguments such that $P_m(0, 0) \neq 0$ and $\chi_m(z) \in W_{1+m\tau-\varepsilon}^{N+\alpha}([0, \pi])$. That this is the case for $m = 1$ follows

from the fact that the constant a of (8.3) is not equal to zero; this follows from (5.1) and the definition of $H(z)$.

Then by the inductive hypothesis we have, for $-A \leqq t < 0$,

$$u(t, 0) = t\left(\sum_{k+l\tau<m\tau} c_{kl}t^{k+l\tau} + \varphi_{12}(t) \right)$$

with $\varphi_{12}(t) \in W_{m\tau-\varepsilon}^{N+\alpha}$ and $c_{00} \neq 0$. A similar equation holds for $0 < t \leqq A$. Then

$$u^{n\tau} = t^{n\tau}\left\{ \sum_{k+l\tau<m\tau} ct^{k+l\tau} + \varphi_{13}^{n}(t) \right\}$$

with $\varphi_{13}^{n} \in W_{m\tau-\varepsilon}^{N+\alpha}$, and

$$u_t(t, 0) = \sum_{k+l\tau<m\tau} c(k+1+l\tau)t^{k+l\tau} + \varphi_{14}(t)$$

with $\varphi_{14} \in W_{m\tau-\varepsilon}^{N-1+\alpha}$. Also, since $\varphi(u) \in W_{(N-1)\tau}^{N+\alpha}$ as a function of u, $\varphi(u(t)) \in W_{(N-1)\tau}^{N-1+\alpha}$ as a function of t. Thus, cross-multiplying, collecting terms, and using Lemmas 1; 2 and 3, we obtain

$$u_t(t, 0)\left\{ \sum_{j=1}^{N-1} cu^{j\tau} + \varphi(u) \right\} = \sum_{k+l\tau<m\tau} ct^{k+l\tau} + \varphi_{14}(t) + \varphi_{15}(t)$$

with $\varphi_{14}(t) \in W_{(m+1)\tau-\varepsilon}^{N-1+\alpha}$ and $\varphi_{15} \in W_{(N-1)\tau-\varepsilon}^{N-1+\alpha}$. By the inductive hypothesis $m + 1 \leqq N - 1$ and we may write $\varphi_{16} = \varphi_{14} + \varphi_{15} \in W_{(m+1)\tau-\varepsilon}^{N-1+\alpha}$. Clearly a similar equation holds for $0 < t \leqq A$, and, applying Lemmas 4 and 5 we obtain

$$H(z) = \sum c_{kl}z^{k+l\tau} + \chi_{m+1}(z)$$

with $\chi_{m+1}(z) \in W_{(m+1)\tau+1-\varepsilon}^{N+\alpha}((0, \pi))$. As H has continuous Nth derivatives, $\chi_{m+1} \in W_{(m+1)\tau+1-\varepsilon}^{N+\alpha}([0, \pi])$. By Warschawski's results $c_{10} \neq 0$. Finally, setting $m = N - 2$, and $\chi_{N-1}(z) = o(z^{(N-1)\tau+1-2\varepsilon})$, we have, with 2ε replaced by ε,

$$H(z) = zP_{N-1}(z, z^\tau) + o(z^{(N-1)\tau+1-\varepsilon})$$

as $z \to 0$, $0 \leqq \arg z \leqq \pi$, and, for $0 \leqq n \leqq N$

$$\frac{d^n}{dz^n}(H(z) - zP_{N-1}(z, z^\tau)) = o(z^{(N-1)\tau+1-n-\varepsilon})$$

as $z \to 0$, $0 \leqq \arg z \leqq \pi$.

Now let $\tau = p/q$, a reduced fraction. For $0 < m < N - 1$ we assume that

$$H(z) = zP_m(z, z^\tau, z^p \log z) + \chi_m(z)$$

with $\chi_m(z) \in W_{1+m\tau-\varepsilon}^{N+\alpha}([0, \pi])$, and $P_m(0, 0, 0) \neq 0$. Then, for $-A \leqq t \leqq 0$,

$$u(t, 0) = t \cdot \sum ct^{j+k\tau}(t^p \log t)^l + \varphi_{17}(t)$$

$$u_t(t, 0) = \sum ct^{j+k\tau}(t^p \log t)^l + \varphi_{18}(t) \cdot$$

$$u^{n\tau} = t^{n\tau}\{\sum ct^{j+k\tau}(t^p \log t)^l + \varphi_{19}(t)\}$$

where $\varphi_{17}, \varphi_{19} \in W_{1+m\tau-\varepsilon}^{N+\alpha}$ and $\varphi_{18} \in W_{m\tau-\varepsilon}^{N-1+\alpha}$. Thus

$$u_t(t, 0)\Big\{\sum_{j=1}^{N-1} cu^{j\tau} + \varphi(u(t))\Big\} = \sum ct^{j+k\tau}(\log t)^l + \varphi_{20}(t)$$

where $j \geq 0, 1 \leq k \leq q, 0 \leq l \leq j/p, j + k\tau < (m + 1)\tau$ and $\varphi_{20} \in W_{(m+1)\tau}^{N-1+\alpha}$.
A similar equation obtains for $0 < t \leq A$. Applying Lemmas 4 and 5
we obtain

$$H(z) = \sum a_{jkl} z^{j+1+k\tau}(\log z)^{l'} + \chi_{m+1}(z)$$

with $\chi_{m+1} \in W_{(m+1)\tau+1-\varepsilon}^{N+\alpha}([0, \pi])$. Terms of the form $t^{j+k\tau}(\log t)^l$, with
$k < q$, contribute terms of the form $z^{j+1+k\tau}(\log z)^{l'}$ with $l' \leq l < l + 1$.
With $k = q$, however, higher powers of the logarithm appear, and we
must then show $j + 1 + k\tau \geq l' + 1$, where $l' \leq l + 1$. But then

$$j + 1 + k\tau = j + 1 + p$$
$$\geq pl + 1 + p \geq p(l + 1) + 1 \geq l + 2 \geq l' + 1 .$$

Thus we can write

$$H(z) = zP_{m+1}(z, z^\tau, z^p \log z) + \chi_{m+1}(z) ,$$

and, for $m = N - 2$ and $0 \leq n \leq N$,

$$\frac{d^n}{dz^n} H(z) = \frac{d^n}{dz^n}(zP_{N-1}(z, z^\tau, z^p \log z)) + o(z^{(N-1)\tau+1-n-\varepsilon})$$

as $z \to 0, 0 \leq \arg z \leq \pi$.

9. Proof of Lemma 5. Suppose that $q(z)$ exists and $\varphi(z) = g(z) - q(z) \in W_{\mu+1}^N((0, \pi])$. Then it follows that $\varphi(z) \in W_{\mu+1}^{N-1+\alpha}((0, \pi])$.
Hence we need only show that there exists a polynomial $q(z)$ such
that

$$\varphi^{(N)}(z) \equiv g^{(N)}(z) - q^{(N)}(z) \in W_{\mu-N+1}^\alpha((0, \pi]) .$$

We break the proof into three parts, numbered I, II and III.

I. First we assume $0 < \arg z \leq \delta$. We have

$$g^{(N)}(z) = -\int_0^A (N - 1)! \frac{\beta(t)}{(t - z)^N} dt .$$

We write, with $r = |z|$,

$$\int_0^A = \int_0^{x-r/2} + \int_{x-r/2}^{x+r/2} + \int_{x+r/2}^A = I_1 + I_2 + I_3 \ .$$

Throughout the proof we shall use constants $C_1, C_2, C_3, \cdots$, which are independent of z; to simplify notation we shall use one symbol C to denote all such constants. I_1 is bounded in absolute value by

$$C \int_0^{x-r/2} t^\mu (\delta x)^{-N} dt \leqq C r^{\mu+1-N} \ ,$$

where we have used $|\beta(t)| \leqq C t^\mu$

For I_2 we expand $\beta(t)$ in a Taylor series about the point x and get

(9.1)
$$\begin{aligned}
I_2 = \sum_{k=0}^{N-2} \frac{(N-1)!}{k!} \beta^{(k)}(x) \int_{x-r/2}^{x+r/2} \frac{(t-x)^k}{(t-z)^N} dt \\
+ \int_{x-r/2}^{x+r/2} \frac{\beta^{(N-1)}(\tau)(t-x)^{N-1}}{(t-z)^N} dt
\end{aligned}$$

where τ lies between x and t.

The integral term J_k arising from the kth term of (9.1) can be written in the form

$$\begin{aligned}
J_k &= C \beta^{(k)}(x) \int_{-r/2}^{r/2} \frac{t^k}{(t-iy)^N} dt \\
&= C \sum_{j=0}^{k} \binom{k}{j} \beta^{(k)}(x) \int_{-r/2}^{r/2} \frac{(t-iy)^j (iy)^{k-j}}{(t-iy)^N} dt \\
&= \sum_{j=0}^{k} C_j \beta^{(k)}(x)(iy)^{k-j} \left[\left(\frac{r}{2} - iy\right)^{j-N+1} - \left(-\frac{r}{2} - iy\right)^{j-N+1} \right],
\end{aligned}$$

and thus, since $|\beta^{(k)}(x)| \leqq C x^{\mu-k}$,

$$|J_k| \leqq C \sum_{j=0}^{k} x^{\mu-k} y^{k-j} 2 \left(\frac{r}{2}\right)^{j-N+1} \leqq C r^{\mu-N+1} \ .$$

The last integral on the right side of (9.1) we write in the form

$$\begin{aligned}
\int_{-r/2}^{r/2} \frac{\beta^{(N-1)}(\tau_1 + x) t^{N-1}}{(t-iy)^N} dt \\
= \int_0^{r/2} t^{N-1} \left\{ \frac{\beta^{(N-1)}(\tau_1 + x)}{(t-iy)^N} - \frac{\beta^{(N-1)}(\tau_2 + x)}{(t+iy)^N} \right\} dt
\end{aligned}$$

where $0 < \tau_1, -\tau_2 < t$. We write the term in brackets in two parts, and get

(9.2)
$$\int_0^{r/2} t^{N-1} \left\{ \frac{\beta^{(N-1)}(\tau_1 + x)}{(t-iy)^N} - \frac{\beta^{(N-1)}(\tau_1 + x)}{(t+iy)^N} \right\} dt$$

$$+ \int_0^{r/2} t^{N-1} \left\{ \frac{\beta^{(N-1)}(\tau_1 + x)}{(t + iy)^N} - \frac{\beta^{(N-1)}(\tau_2 + x)}{(t + iy)^N} \right\} dt .$$

The first integral is equal to

$$2i \, Im \int_0^{r/2} t^{N-1} \frac{\beta^{(N-1)}(\tau_1 + x)}{(t - iy)^N} dt$$

$$= 2i \, Im \int_0^{r/2} \frac{t^{N-1}(t + iy)^N \beta^{(N-1)}(\tau_1 + x) dt}{(t^2 + y^2)^N} .$$

We make the change of variables $t = sy$ to obtain

$$2i \, Im \int_0^{r/2y} \frac{s^{N-1}(s + i)^N \beta^{(N-1)}(\sigma + x) ds}{(s^2 + 1)^N}$$

$$= 2i \, Im \sum_{k=1}^{N} \binom{N}{k} \int_0^{r/2y} \frac{s^{2N-1-k} i^k \beta^{(N-1)}(\sigma + x) ds}{(s^2 + 1)^N}$$

where $\sigma < r/2$; this is bounded in absolute value by

$$C \sum_{k=1}^{N} \int_0^{\infty} \frac{s^{2N-1-k}}{(s^2 + 1)^N} \, | \, r/2 \pm x \, |^{\mu-N+1} \, ds \leq C r^{\mu-N+1} .$$

The second integral on the right side of (9.2) is bounded absolutely by

$$C \int_0^{r/2} | \, 2t \, |^{\alpha} \, | \, x \pm r/2 \, |^{\mu-N+1-\alpha} \, t^{-1} dt \leq C r^{\mu+1-N} .$$

To handle I_3 we observe that

$$\frac{1}{(t - z)^N} = \sum_{k=0}^{\infty} \binom{k + N - 1}{k} z^k t^{-N-k} .$$

Let m be the integer such that $\mu < m < \mu + 1$ and assume $m \geqq N$. We have

$$I_3 = \sum_{k=0}^{m-N} \frac{(k + N - 1)!}{k!} z^k \int_0^A \beta(t) t^{-N-k} \, dt$$

(9.3)

$$- \sum_{k=0}^{m-N} \frac{(k + N - 1)!}{k!} z^k \int_0^{x+r/2} \beta(t) t^{-N-k} \, dt$$

$$+ \int_{x+r/2}^A \sum_{k=m-N+1}^{\infty} \frac{(k + N - 1)!}{k!} z^k \beta(t) t^{-N-k} \, dt .$$

We set

$$q_1(z) \equiv - \sum_{k=0}^{m-N} \frac{(k + N - 1)!}{k!} z^k \int_0^A \beta(t) t^{-N-k} dt .$$

If $m < N$ we set $q_1(z) \equiv 0$, and the last sum of (9.3) begins with

$k = 0$. In any event, $q(z)$ will be taken such that $q^{(N)}(z) \equiv q_1(z)$; its exact form is given in [12].

Thus to prove that $\varphi^{(N)}(z) \equiv g^{(N)}(z) - q_1(z) \in W_{\mu-N+1}((0, \delta))$, we need only estimate the last two terms on the right side of (9.3). We have

$$\left| \sum_{k=0}^{m-N} \frac{(k + N - 1)!}{k!} z^k \int_0^{x+r/2} \beta(t)t^{-N-k} dt \right|$$

$$\leq C \sum_{k=0}^{m-N} r^k \int_0^{x+r/2} t^{\mu-N-k} dt = Cr^{\mu+1-N} ,$$

and

$$\left| \int_{x+r/2}^A \sum_{k=m-N+1}^\infty \frac{(k + N - 1)!}{k!} z^k \beta(t)t^{-N-k} dt \right|$$

$$\leq C \sum_{k=m-N+1}^\infty \frac{(k + N - 1)!}{k!} r^k \int_{x+r/2}^\infty t^{\mu-N-k} dt$$

$$\leq C \sum_{k=m-N+1}^\infty \frac{(k + N - 1)!}{k!} \frac{r^{\mu+1-N}}{|\mu - N - k + 1|} \left(\cos \delta + \frac{1}{2} \right)^{\mu-N-k+1}$$

$$= Cr^{\mu+1-N}$$

where we assume $\cos \delta > 1/2$. Thus $\varphi^{(N)}(z) \in W_{\mu-N+1}((0, \delta))$.

II.   For $\delta \leq \arg z \leq \pi$, observe that $\varphi(z)$ is analytic for $|z| > 0$. That $\varphi(z) \in W_{\mu+1}$ follows from [12]. By Cauchy's theorem

(9.4) $$\varphi^{(N)}(z) = \frac{N!}{2\pi i} \int \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^{N+1}}$$

where the integral is taken around a circle with z as center and radius $\delta_1 |z|$, where δ_1 is a small positive number. Then

$$|\varphi^{(N)}(z)| \leq C(2\pi \delta_1 |z|)(|z| (1 \pm \delta_1))^{\mu+1}(\delta_1 |z|)^{-N-1} = Cr^{\mu+1-N} .$$

III.   We will now show that

$$|\varphi^{(N)}(z) - \varphi^{(N)}(\zeta)| \leq C |z|^{\mu+1-N-\alpha} |z - \zeta|^\alpha ,$$

for $|z - \zeta| \leq \delta |z|, \delta |\zeta|$. First, this inequality follows immediately from (9.4) for $\delta \leq \arg z, \arg \zeta \leq \pi$. Thus we will restrict ourselves to the range $0 < \arg z, \arg \zeta \leq \delta$. We have

$$\varphi^{(N)}(z) - \varphi^{(N)}(\zeta) = -\int_0^{x-r/2} (N - 1)! \beta(t) \left\{ \frac{1}{(t - z)^N} - \frac{1}{(t - \zeta)^N} \right\} dt$$

$$- \int_{x-r/2}^{x+r/2} (N - 1)! \beta(t) \left\{ \frac{1}{(t - z)^N} - \frac{1}{(t - \zeta)^N} \right\} dt$$

$$+ \sum_{k=0}^{m-N} \frac{(k + N - 1)!}{k!} (z^k - \zeta^k) \int_0^{x+r/2} \beta(t)t^{-N-k} dt$$

$$- \int_{x+r/2}^{A} \sum_{\substack{k=m-N+1 \\ k \geq 0}}^{\infty} \frac{(k+N-1)!}{k!} (z^k - \zeta^k) \beta(t) t^{-N-k} dt$$

$$= I^1 + I^2 + I^3 + I^4 .$$

Then

$$I^1 = -(N-1)! \int_0^{x-r/2} \beta(t) \sum_{k=1}^{N} \frac{t^{N-1}(-1)^k [\zeta^k - z^k] dt}{(t-z)^N (t-\zeta)^N} .$$

Since $|\zeta - z| \leq \delta |z|$ and $|\zeta| \leq (1 + \delta)|z|$, we have

$$|\zeta^k - z^k| \leq |\zeta - z|^\alpha |\zeta - z|^{1-\alpha} \sum_{j=0}^{k-1} |\zeta|^{k-j-1} |z|^j$$

$$\leq C |z|^{k-\alpha} |\zeta - z|^\alpha .$$

Hence

$$|I^1| \leq C \sum_{k=1}^{N} |z - \zeta|^\alpha |z|^{k-\alpha} \int_0^{x-r/2} t^{\mu+N-k} \left(\frac{r}{2}\right)^{-N} \left(r\left(\frac{1}{2} - \delta\right)\right)^{-N} dt$$

$$\leq C |z - \zeta|^\alpha |z|^{\mu-N+1-\alpha} .$$

Similarly, we can assume the sum in I_3 begins with $k = 1$, and we obtain

$$|I^3| \leq C |z - \zeta|^\alpha \sum_{k=1}^{m-N} |z|^{k-\alpha} \int_0^{x+r/2} t^{\mu-N-k} dt$$

$$\leq C |z - \zeta|^\alpha |z|^{\mu-N+1-\alpha} .$$

Likewise

$$|I^4| \leq C |z - \zeta|^\alpha \sum_{\substack{k=m-N+1 \\ k>0}}^{\infty} \frac{(k+N-1)!}{k!} |z|^{k-\alpha} \int_{x+r/2}^{\infty} t^{\mu-N-k} dt$$

$$\leq C |z - \zeta|^\alpha \sum_{\substack{k=m-N+1 \\ k>0}}^{\infty} \frac{(k+N-1)!}{k!} |z|^{\mu-N+1-\alpha} \frac{\left(\cos \delta + \frac{1}{2}\right)^{\mu-N-k+1}}{|\mu - N - k + 1|}$$

$$= C |z|^{\mu-N+1-\alpha} |z - \zeta|^\alpha .$$

Finally we must evaluate I^2. We write

$$\beta(t) = \sum_{k=0}^{N-2} \frac{\beta^{(k)}(x)}{k!} (t-x)^k + \psi(x, t)$$

where

$$\psi(x, t) = \int_x^t \frac{(t-\sigma)^{N-2}}{(N-2)!} \beta^{(N-1)}(\sigma) d\sigma ,$$

and

$$\beta(t) = \sum_{k=0}^{N-2} \frac{\beta^{(k)}(\xi)}{k!}(t-\xi)^k + \psi(\xi, t) \; .$$

Then we have

$$
I^2 = -\sum_{k=0}^{N-2} \frac{(N-1)!}{k!} \int_{x-r/2}^{x+r/2} \left\{ \frac{\beta^{(k)}(x)(t-x)^k}{(t-z)^N} - \frac{\beta^{(k)}(\xi)(t-\xi)^k}{(t-\zeta)^N} \right\} dt
$$

$$
- (N-1) \int_{x-r/2}^{x+r/2} \left\{ \frac{\psi(x, t)}{(t-z)^N} - \frac{\psi(\xi, t)}{(t-\zeta)^N} \right\} dt
$$

$$
= \sum_{k=0}^{N-2} J^k + J^{N-1} \; .
$$

With

$$(t-x)^k = \sum_{j=0}^{k} \binom{k}{j}(t-z)^j(z-x)^{k-j}$$

we have, for $k \leqq N - 2$,

$$
J^k = -\frac{(N-1)!}{k!} \sum_{j=0}^{k} \binom{k}{j}
$$

$$
\times \int_{x-r/2}^{x+r/2} [\beta^{(k)}(x)(t-z)^{j-N}(z-x)^{k-j} - \beta^{(k)}(\xi)(t-\zeta)^{j-N}(\zeta-x)^{k-j}] dt
$$

$$
= -\frac{(N-1)!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j-N+1}
$$

$$
\times \left\{ \beta^{(k)}(x)(z-x)^{k-j}\left(\left(x+\frac{r}{2}-z\right)^{j-N+1} - \left(x-\frac{r}{2}-z\right)^{j-N+1}\right) \right.
$$

$$
\left. - \beta^{(k)}(\xi)(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right) \right\}
$$

To the term in brackets we add and subtract

$$
\beta^{(k)}(x)(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right) .
$$

Then the jth bracketed term becomes

(9.5)

$$
\beta^{(k)}(x)\left[(z-x)^{k-j}\left(\left(x+\frac{r}{2}-z\right)^{j-N+1} - \left(x-\frac{r}{2}-z\right)^{j-N+1}\right)\right.
$$

$$
- (\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right)\right]
$$

$$
+ (\beta^{(k)}(x) - \beta^{(k)}(\xi))
$$

$$
\times \left[(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right)\right].
$$

Thus to evaluate J^k, $k \leqq N - 2$, it suffices to evaluate each term of (9.5).

Since $|\zeta - \xi| = |\eta| \leqq |\zeta| \leqq |z| (1 + \delta)$, and

$$\left| x + \frac{r}{2} - \zeta \right| \geqq r\left(\cos \delta - \delta - \frac{1}{2} \right)$$

$$\left| x - \frac{r}{2} - \zeta \right| \geqq r\left(\frac{1}{2} - \delta \right),$$

the second term of (9.5) is bounded absolutely by

$$Cx^{\mu-k-\alpha} |x - \xi|^{\alpha} r^{k-N+1} \leqq Cr^{\mu-N+1-\alpha} |z - \zeta|^{\alpha} .$$

To evaluate the first term of (9.5), consider the function

$$f(p, q) = (iq)^{k-j}\left(\left(x + \frac{r}{2} - p - iq \right)^{j-N+1} - \left(x - \frac{r}{2} - p - iq \right)^{j-N+1} \right)$$

and its first partial derivatives, $f_1(p, q) = f_p(p, q)$, and $f_2(p, q) = f_q(p, q)$. Then we must evaluate

$$\beta^{(k)}(x)(f(x, y) - f(\xi, \eta)) .$$

By the mean value theorem we have, for some λ with $0 < \lambda < 1$,

$$f(x, y) - f(\xi, \eta) = (x - \xi)f_1(x + \lambda(\xi - x), y + \lambda(\eta - y))$$
$$+ (y - \eta)f_2(x + \lambda(\xi - x), y + \lambda(\eta - y)) .$$

Then using previously mentioned inequalities for $|\zeta - \xi|, |x \pm r/2 - \zeta|$, etc., it follows easily that for $k \leqq N - 2$

$$|J^k| \leqq Cr^{\mu-N+1-\alpha} |z - \zeta|^{\alpha} .$$

Thus to finish the proof we must evaluate

$$J^{N-1} = -(N - 1)\int_{x-r/2}^{x+r/2} \left\{ \frac{\psi(x, t)}{(t - z)^N} - \frac{\psi(\xi, t)}{(t - \zeta)^N} \right\} dt$$

$$= -(N - 1)\int_{-r/2}^{r/2} \frac{\psi(x, x + t)}{(t - iy)^N}$$

$$+ (N - 1)\int_{x-\xi-r/2}^{x-\xi+r/2} \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} dt .$$

We will assume $x \geqq \xi$; were $x < \xi$ another similar argument would prevail. Then

$$J^{N-1} = (N - 1)\int_{-r/2}^{r/2} \left\{ \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} - \frac{\psi(x, x + t)}{(t - iy)^N} \right\} dt$$

(9.6)
$$- (N - 1)\int_{-r/2}^{x-\xi-r/2} \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} dt$$

$$+ (N - 1)\int_{r/2}^{x-\xi+r/2} \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} dt .$$

By definition, for $t \geqq 0$,

$$|\psi(\xi, \xi + t)| \leqq \int_{\xi}^{\xi+t} \frac{(\xi + t - \sigma)^{N-2}}{(N-2)!} |\beta^{(N-1)}(\sigma)| \, d\sigma \leqq C\xi^{\mu-N+1}t^{N-1},$$

and thus the third integral on the right side of (9.6) is bounded absolutely by

$$C |z|^{\mu-N+1} \int_{r/2}^{x-\xi+r/2} t^{-1} dt \leqq C |z|^{\mu-N+1} |x + \xi| \left(\frac{r}{2}\right)^{-1}$$

$$\leqq C |z - \zeta|^{\alpha} |z|^{\mu-N+1-\alpha}.$$

We handle the second integral of (9.6) in the same fashion.
Thus we have left to evaluate

$$\int_{-r/2}^{r/2} \left\{ \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} - \frac{\psi(x, x + t)}{(t - iy)^N} \right\} dt$$

$$= \int_0^{r/2} \left\{ \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} + (-1)^N \frac{\psi(\xi, \xi - t)}{(t + i\eta)^N} \right.$$

(9.7)
$$\left. - \frac{\psi(x, x + t)}{(t - iy)^N} - (-1)^N \frac{\psi(x, x - t)}{(t + iy)^N} \right\} dt$$

$$= \int_0^{r/2} dt \int_0^t \frac{(t - s)^{N-2}}{(N-2)!} \left\{ \frac{\beta^{(N-1)}(\xi + s)}{(t - i\eta)^N} - \frac{\beta^{(N-1)}(\xi - s)}{(t + i\eta)^N} \right.$$

$$\left. - \frac{\beta^{(N-1)}(x + s)}{(t - iy)^N} + \frac{\beta^{(N-1)}(x - s)}{(t + iy)^N} \right\} ds$$

where we have recalled the definition of ψ.
We write

$$\beta^{(N-1)}(\xi + s) = \beta^{(N-1)}(\xi) + K_1(\xi, s)s^{\alpha}$$

$$\beta^{(N-1)}(\xi - s) = \beta^{(N-1)}(\xi) + K_2(\xi, s)s^{\alpha}$$

$$\beta^{(N-1)}(x + s) = \beta^{(N-1)}(x) + K_3(x, s)s^{\alpha}$$

$$\beta^{(N-1)}(x - s) = \beta^{(N-1)}(x) + K_4(x, s)s^{\alpha}$$

where $|K_i(s)| \leqq C |z|^{\mu-N+1-\alpha}$. Also

(9.8) $$s^{\alpha} | K_1(\xi, s) - K_3(x, s)| \leqq C |z|^{\mu-N+1-\alpha} |z - \zeta|^{\alpha}$$

and

$$s^{\alpha} | K_1(\xi, s) - K_3(x, s)| \leqq C |z|^{\mu-N+1-\alpha} s^{\alpha}.$$

Similar inequalities hold for $s^{\alpha} | K_2 - K_4|$.
Next, (9.7) becomes

$$\int_0^{r/2} dt \int_0^t \frac{(t-s)^{N-2}}{(N-2)!} \left\{ \beta^{(N-1)}(\xi) \left(\frac{1}{(t-i\eta)^N} - \frac{1}{(t+i\eta)^N} \right) \right.$$

$$- \beta^{(N-1)}(x) \left. \left(\frac{1}{(t-iy)^N} - \frac{1}{(t+iy)^N} \right) \right\} ds$$

(9.9)

$$+ \int_0^{r/2} dt \int_0^t \frac{(t-s)^{N-2}}{(N-1)!} \left\{ \frac{K_1(\xi,s)s^\alpha}{(t-i\eta)^N} \right.$$

$$\left. - \frac{K_2(\xi,s)s^\alpha}{(t+i\eta)^N} - \frac{K_3(x,s)s^\alpha}{(t-iy)^N} + \frac{K_4(x,s)s^\alpha}{(t+iy)^N} \right\} ds .$$

Notice that

$$\int_0^{r/2} dt \int_0^t \frac{(t-s)^{N-2}}{(N-2)!} \left(\frac{1}{(t-i\eta)^N} - \frac{1}{(t+i\eta)^N} \right) ds$$

$$= \int_0^{r/2} \frac{t^{N-1}}{(N-1)!} \left(\frac{1}{(t-i\eta)^N} - \frac{1}{(t+i\eta)^N} \right) dt$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{1}{(N-1)!}$$

$$\times \int_0^{r/2} \{ (t-i\eta)^{-1-k}(i\eta)^k - (t+i\eta)^{-1-k}(-i\eta)^k \} dt$$

$$= \sum_{k=0}^{N-1} C_k \left\{ \left(\frac{r}{2} - i\eta \right)^{-k} (i\eta)^k - \left(\frac{r}{2} + i\eta \right)^{-k} (-i\eta)^k \right\} .$$

Thus the first term of (9.9) is bounded absolutely by terms of the form

$$\beta^{(N-1)}(\xi) \left\{ \left(\frac{i\eta}{r/2 - i\eta} \right)^k - \left(\frac{-i\eta}{r/2 + i\eta} \right)^k \right\}$$

$$- \beta^{(N-1)}(x) \left\{ \left(\frac{iy}{r/2 - iy} \right)^k - \left(\frac{-iy}{r/2 + iy} \right)^k \right\}$$

$$= 2i\beta^{(N-1)}(\xi) \, Im \left[\left(\frac{i\eta}{r/2 - i\eta} \right)^k - \left(\frac{iy}{r/2 - iy} \right)^k \right]$$

$$+ 2i \, Im \left[\left(\frac{iy}{r/2 - iy} \right)^k \right] (\beta^{(N-1)}(\xi) - \beta^{(N-1)}(x)) .$$

The second term is bounded by $C \, |z|^{\mu-N+1-\alpha} \, |z-\zeta|^\alpha$. The first, using the mean value theorem, is bounded by

(9.10) $$C \, |z|^{\mu-N+1} \, |y-\eta| \left| Im \frac{\sigma^{k-1} r i^k}{(r/2 - i\sigma)^{k+1}} \right|$$

where σ lies between y and η. But then $\sigma \le Cr$, $|r/2 - i\sigma| \le r/2$, and hence (9.10) is $\le$

$$C \, |z|^{\mu-N+1} \, |y-\eta|^\alpha \, r^{1-\alpha} r^{-1} \le C \, |z|^{\mu-N+1-\alpha} \, |z-\zeta|^\alpha .$$

Thus we are left with the second term of (9.9). By symmetry we need only consider

$$\int_0^{r/2} dt \int_0^t (t - s)^{N-2} s^\alpha \left\{ \frac{K_1(\xi, s)}{(t - i\eta)^N} - \frac{K_3(x, s)s^\alpha}{(t - iy)^N} \right\} ds$$

$$(9.11) \quad = \int_0^{r/2} dt \int_0^t (t - s)^{N-2} s^\alpha (K_1(\xi, s) - K_3(x, s))(t - i\eta)^{-N} ds$$

$$+ \int_0^{r/2} dt \int_0^t (t - s)^{N-2} s^\alpha K_3(x, s) \left(\frac{1}{(t - i\eta)^N} - \frac{1}{(t - iy)^N} \right) ds \ .$$

By (9.8), for any $\varepsilon > 0$ we have

$$s^\alpha \, | \, K_1(\xi, s) - K_3(x, s) \, | \, \leqq C_1^\varepsilon C_1^{1-\varepsilon} \, | \, z \, |^{\mu - N + 1 - \alpha} \, s^{\alpha\varepsilon} \, | \, x - \xi \, |^{\alpha(1-\varepsilon)} \ .$$

Thus the first term on the right side of (9.11) is bounded in absolute value by

$$CC_1^\varepsilon C_2^{1-\varepsilon} \, | \, z - \zeta \, |^{\alpha(1-\varepsilon)} \, | \, z \, |^{\mu - N + 1 - \alpha} \int_0^{r/2} dt \int_0^t t^{N-2} s^{\alpha\varepsilon} \, | \, t - i\eta \, |^{-N} \, ds$$

$$\leqq CC_1^\varepsilon C_2^{1-\varepsilon} \, | \, z - \zeta \, |^{\alpha(1-\varepsilon)} \, | \, z \, |^{\mu - N + 1 - \alpha} \int_0^{r/2} t^{\alpha\varepsilon - 1} dt$$

$$= CC_1^\varepsilon C_2^{1-\varepsilon} \, | \, z - \zeta \, |^{\alpha(1-\varepsilon)} \, | \, z \, |^{\mu - N + 1 - \alpha + \alpha\varepsilon} \ .$$

Now let $\varepsilon \to 0$.

Finally, to evaluate the second term on the right side of (9.11), we observe that

$$\left| \frac{1}{(t - i\eta)^N} - \frac{1}{(t - iy)^N} \right|$$

$$\leqq \sum_{k=1}^N \binom{N}{k} t^{N-k} \, | \, y^k - \eta^k \, | \, | \, t - i\eta \, |^{-N} \, | \, t - iy \, |^{-N}$$

$$\leqq | \, z - \zeta \, |^\alpha \sum_{k=1}^N t^{N-k} y^{k-\alpha} \, | \, t - i\eta \, |^{-N} \, | \, t - iy \, |^{-N}$$

where we have assumed that $y \geqq \eta$. Then

$$\left| \int_0^{r/2} dt \int_0^t (t - s)^{N-2} s^\alpha K_3(x, s)((t - i\eta)^{-N} - (t - iy)^{-N}) ds \right|$$

$$\leqq \sum_{k=1}^N C_k \int_0^{r/2} t^{2N-k-1+\alpha} \, | \, z \, |^{\mu - N + 1 - \alpha} \, | \, z - \zeta \, |^\alpha y^{k-\alpha} \, | \, t - i\eta \, |^{-N} \, | \, t - iy \, |^{-N} dt$$

$$\leqq C \, | \, z \, |^{\mu - N + 1 - \alpha} \, | \, z - \zeta \, |^\alpha \sum_{k=1}^N \int_0^{r/2} t^{N-k-1+\alpha} y^{k-\alpha} \, | \, t - iy \, |^{-N} \, dt \ .$$

With the change of variables $t = \tau y$, the kth integral becomes

$$\int_0^{r/2y} \tau^{N-k-1+\alpha} \, | \, \tau - i \, |^{-N} \, dt \leqq \int_0^\infty \tau^{N-k-1+\alpha} \, | \, \tau - i \, |^{-N} \, dt = C$$

since $1 \leqq k \leqq N$. This completes the evaluation of J^{N-1} and the theorem is proved.

References

1. S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623-727.

2. N. Bourbaki, *Fonctions D'Une Variable Reelle*, vol. XII, no. 1132, pp. 83.

3. D. S. Carter, *Local behavior of plane gravity flows at the confluence of free boundaries and analytic fixed boundaries*, J. Math. Mech. **10** (1961), 441-450.

4. O. D. Kellogg, *Harmonic functions and Green's integral*, Trans. Amer. Math. Soc, **13** (1912), 109-132.

5. R. S. Lehman, *Developments in the neighborhood of the beach of surface waves over an inclined bottom*, Comm. Pure Appl. Math. **7** (1954), 393-439.

6. ————, *Development of the mapping function at an analytic corner*, Pacific J. Math. **7** (1957), 1437-1449.

7. ————, *Developments at an analytic corner of solutions of elliptic partial differential equations*, J. Math. Mech. **8** (1959), 727-760.

8. Hans Lewy, *Developments at the confluence of analytic boundary condition*, Univ. of Calif. Publ. in Math. **1**, no. 7 (1950), 247-280; also Proc. Int. Cong. Math. (1950), 601-605.

9. L. Lichtenstein, *Über die konforme Abbildung ebener analytischer Gebiete mit Ecken*, J. Reine Angew. Math. **140** (1911), 100-119.

10. S. Warschawski, *Über das Verhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung*, Math A. **35** (1932), 321-456.

11. W. Wasow, *Asymptotic development of the solution of Dirichlet's problem at analytic corners*, Duke Math. J. **24** (1957), 47-56.

12. N. W. Wigley, *Asymptotic expansions at a corner of solutions of mixed boundary value problems*, J. Math. Mech. **13** (1964) 549-576.

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