PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

INVARIANT SPLITTING IN JORDAN AND ALTERNATIVE ALGEBRAS

EARL J. TAFT

Let A be a finite-dimensional Jordan or alternative algebra over a field F of characteristic 0. Let N denote the radical of A. Then A possesses maximal semisimple subalgebras isomorphic to A/N, [5], [6], any two of which are strictly conjugate, [2], [9]. If G is a finite group of automorphisms and antiautomorphisms of A, then A possesses G-invariant maximal semisimple subalgebras, [10]. We investigate here the uniqueness question for such G-invariant maximal semisimple subalgebras. The result is that the strict conjugacy can be chosen to commute pointwise with G and to be in the enveloping associative algebra generated by the right and left multiplications in A.

Similar results have been obtained for associative algebras, [11], and Lie algebras, [12]. However, in the associative case, the conjugacy can be obtained in terms of adjoints of G-symmetric elements, i.e., elements left fixed by the automorphisms in G and sent into their negatives by the antiautomorphisms in G. In the Lie algebra case, one needs only to consider automorphisms, and the conjugacy is obtained in terms of adjoints of fixed points of G. In each case, the conjugacy is in the enveloping associative algebra of A. In both the Jordan and alternative cases, the automorphisms which occur would commute pointwise with G if the elements of A which occur in their formulation in terms of right and left multiplications were to be fixed points of G. However, we have not obtained the conjugacies in this form, and it seems to be an open question whether or not it is always possible to do so.

If G is assumed fully reducible, instead of finite, then A will also possess G-invariant maximal semisimple subalgebras. This is noted in the Jordan case in [4] when G contains only automorphisms, and the same proof can be extended to cover the alternative case, even if G also contains antiautomorphisms. We have answered the uniqueness question for the similar situation in the associative and Lie cases, [13]. For the Jordan and alternative case, the problem seems more complicated. We note here that it is easily answered if $N^2 = 0$, with the strict conjugacy commuting pointwise with G. However, the general question remains open.

Received July 15, 1964. Research supported by the National Science Foundation under grant NSF-GP1239 and by the Research Council of Rutgers, The State University.

2. Preliminaries. If $a \in A$, we let R_a and L_a stand for right and left multiplication by a, i.e., $xR_a = xa$, $xL_a = ax$. The following two lemmas are easily proved by straightforward calculation.

LEMMA 1. *Let g be an automorphism of A. Then $g^{-1}R_a g = R_{ag}$ and $g^{-1}L_a g = L_{ag}$.*

LEMMA 2. *Let g be an antiautomorphism of A. Then $g^{-1}R_a g = L_{ag}$, $g^{-1}L_a g = R_{ag}$.*

A derivation of A will be called inner if it is in the enveloping Lie algebra generated by the right and left multiplications in A, [7]. We will have occasion to use the following types of inner derivations. If A is Jordan, and $x, s \in A$, then $[R_x, R_s] = R_x R_s - R_s R_x$ is an inner derivation of A which, for $x \in N$, will be a nilpotent element of the radical of the enveloping associative algebra generated by multiplications in A by elements of A, [1], [2], [8]. If A is alternative, and $s, x \in A$, then $D_{s,x} = [R_s, R_x] + [L_s, R_x] + [L_s, L_x]$ is an inner derivation of A which, for $x \in N$, will be a nilpotent element of the radical of the enveloping associative algebra generated by the left and right multiplications of A, [7], [9].

LEMMA 3. *If A is alternative, $a, b \in A$, then $[R_a, L_b] = [L_a, R_b]$, and $D_{a,b} = -D_{b,a}$.*

Proof. $x[R_a, L_b] = b(xa) - (bx)a = -(b, x, a)$, where $(b, x, a) = (bx)a - b(xa)$ is the associator of b, x, and a. Also $x[L_a, R_b] = (ax)b - a(xb) = (a, x, b)$. The first part of Lemma 3 follows from the skew-symmetry of the associator function. Hence

$$\begin{aligned} D_{b,a} &= [R_b, R_a] + [L_b, R_a] + [L_b, L_a] \\ &= -[R_a, R_b] - [R_a, L_b] - [L_a, L_b] \\ &= -[R_a, R_b] - [L_a, R_b] - [L_a, L_b] = -D_{a,b} . \end{aligned}$$

LEMMA 4. *Let A be Jordan, and g an automorphism of A. Then $g^{-1}[R_a, R_b]g = [R_{ag}, R_{bg}]$.*

This is immediate from Lemma 1.

LEMMA 5. *Let A be alternative, and g an automorphism or antiautomorphism of A. Then $g^{-1}D_{a,b}g = D_{ag,bg}$.*

Proof. This is clear from Lemma 1 if g is an automorphism. Let g be an antiautomorphism. Then, using Lemma 2, $g^{-1}D_{a,b}g = [L_{ag}, L_{bg}] + [R_{ag}, L_{bg}] + [R_{ag}, R_{bg}] = D_{ag,bg}$ by Lemma 3.

If D is a nilpotent derivation of A, then $\exp D = I + D + (D^2/2!) + \cdots$ is an automorphism of A. We assume familiarity with the Campbell-Hausdorff formula, [3], $(\exp D_1)(\exp D_2) = \exp D_3$, where D_3 is in the Lie algebra generated by D_1 and D_2.

3. The Jordan case.

THEOREM 1. *Let A be a finite-dimensional Jordan algebra over a field F of characteristic 0. Let G be a finite group of automorphisms of A. Let S be a G-invariant maximal semisimple subalgebra of A. Let T be a G-invariant semisimple subalgebra of A. Then there exists an automorphism $U = \exp D$ of A such that*

(1) *U maps T into S,*

(2) *D (and hence U) commutes pointwise with G,*

(3) *D is a nilpotent inner derivation of A which is in the radical of the enveloping associative algebra of A.*

Proof. Let N denote the radical of A. Let s and n denote the projections of the vector space $A = S \oplus N$ onto S and N respectively. Then s and n are linear mappings such that

(i) $s(t_1t_2) = s(t_1)s(t_2)$

(ii) $n(t_1t_2) = s(t_1)n(t_2) + n(t_1)s(t_2) + n(t_1)n(t_2)$

(iii) $s(tg) = s(t)g, \qquad n(tg) = n(t)g$

for $t_1, t_2, t \in T$, $g \in G$.

(i) and (ii) follow since N is an ideal. (iii) follows from the invariance of T, S and N under G.

Now set $N_1 = N$, $N_i = N_{i-1}^2 + AN_{i-1}^2$. By [5], the N_i form a nonincreasing sequence of ideals terminating in 0. Now $T_1 = T \subseteqq A = S + N_1$. Suppose that we have found automorphisms $U_0 = \exp 0$, $U_1 = \exp D_1, \cdots, U_{i-1} = \exp(D_{i-1})$ of A satisfying (2) and (3) of Theorem 1 such that $T_i = TU_0U_1 \cdots U_{i-1} \subseteqq S + N_i$. Then we will show that there exists an automorphism U_i of A satisfying (2) and (3) of Theorem 1 such that $T_iU_i \subseteqq S + N_{i+1}$. Hence if $N_k = 0$, then $U = U_0U_1 \cdots U_{k-1}$ will be the desired automorphism by the Campbell-Hausdorff formula.

Now T_i is a G-invariant semisimple subalgebra of A, so that (i), (ii), (iii) hold for $t_1, t_2, t \in T_i$. Consider the space $N_i \mid N_{i+1}$. We consider this as a T_i-module by defining $t \cdot \bar{n} = \bar{n} \cdot t = \overline{ns(t)}$ for $n \in N_i$, $t \in T_i$. Then by (ii), we have

(iv) $\overline{n(t_1t_2)} = \overline{n(t_1)} \cdot t_2 + t_1 \cdot \overline{n(t_2)}$.

(iv) says that the map $t \rightarrow \overline{n(t)}$ is a derivation of T_i into the module $N_i \mid N_{i+1}$. Hence, by [2], there exist elements $x_1, \cdots, x_p$ in N_i, $t_1, \cdots, t_p \in T_i$ such that

(v) $\overline{n(t)} = \sum_{j=1}^{p} ((\bar{x}_j \cdot t) \cdot t_j - \bar{x}_j \cdot (tt_j))$ for $t \in T_i$ i.e.,

$$\overline{n(t)} = \sum_{j=1}^{p} \overline{(x_j s(t))s(t_j)} - \overline{x_j s(tt_j)} .$$

Using (i), we have

(vi) $n(t) \equiv s(t) \sum_{j=1}^{p} [R_{x_j}, R_{s(t_j)}] \pmod{N_{i+1}}$ for $t \in T_i$.

Let $g \in G$. Then

$$[R_{x_j g}, R_{s(t_j)g}] = g^{-1}[R_{x_j}, R_{s(t_j)}]g$$

by Lemma 4. Hence

$$\begin{aligned} s(t) \sum_{j=1}^{p} [R_{x_j g}, R_{s(t_j)g}] &= s(t)g^{-1}\Big(\sum_{j=1}^{p} [R_{x_j}, R_{s(t_j)}]\Big)g \\ &= s(tg^{-1})\Big(\sum_{j=1}^{p} [R_{x_j}, R_{s(t_j)}]\Big)g \equiv n(tg^{-1})g = n(t) \pmod{N_{i+1}} . \end{aligned}$$

It follows that if we set $D_i = -(1/m) \sum_{g \in G} (\sum_{j=1}^{p} [R_{x_j g}, R_{s(t_j)g}])$, where m is the order of G, then

(vii) $n(t) \equiv -s(t)D_i \pmod{N_{i+1}}$ for $t \in T_i$.

Now D_i clearly satisfies (3) of the Theorem, since the $x_j g \in N$. To see that D_i satisfies (2) of the Theorem, we fix a value of j. Then $\sum_{g \in G} [R_{x_j g}, R_{s(t_j)g}] = \sum_{g \in G} g^{-1}[R_{x_j}, R_{s(t_j)}]g$ clearly commutes pointwise with G. Hence so does D_i, which is a linear combination of such mappings.

Finally, set $U_i = \exp D_i$. If $t \in T_i$, then $tU_i = t + tD_i + (t/2)D_i^2 + \cdots = s(t) + n(t) + s(t)D_i + n(t)D_i + (t/2)D_i^2 + \cdots$.

Now $n(t) \in N_i$, so that $n(t)D_i \in N_{i+1}$. Also, since the $x_1, \cdots, x_p \in N_i$, we have that $(t/2)D_i^2 + \cdots \in N_{i+1}$. Therefore

$$\begin{aligned} tU_i &\equiv s(t) + n(t) + s(t)D_i \pmod{N_{i+1}} \\ &\equiv s(t) \pmod{N_{i+1}} \text{ by (vii).} \end{aligned}$$

Hence $T_i U_i \subseteq S + N_{i+1}$. This completes the proof of the Theorem.

COROLLARY 1. *Let A be a finite-dimensional Jordan algebra over a field of characteristic* 0. *Let G be a finite group of automorphisms of A. Let S and T be G-invariant maximal semisimple subalgebras of A. Then S and T are strictly conjugate via an automorphism of A of the type described in Theorem* 1.

COROLLARY 2. *Let A and G be as in Corollary* 1. *Let T be any G-invariant semisimple subalgebra of A. Then T is contained in a G-invariant maximal semisimple subalgebra of A.*

Corollary 1 is an immediate consequence of Theorem 1. Corollary 2 follows from the existence of a G-invariant maximal semisimple subalgebra S of A. For then if U is an automorphism of A which maps T into S, and which commutes with G pointwise, it follows that SU^{-1} is a G-invariant maximal semisimple subalgebra of A which contains T.

4. The alternative case.

THEOREM 2. *Let A be a finite-dimensional alternative algebra over a field F of characteristic* 0. *Let G be a finite group of automorphisms and antiautomorphism of A. Let S be a G-invariant maximal semisimple subalgebra of A. Let T be a semisimple subalgebra of A. Then there exists an automorphism $U = \exp D$ of A such that*

(1) *U maps T into S,*

(2) *D (and hence U) commutes pointwise with G,*

(3) *D is a nilpotent inner derivation of A which is in the radical of the enveloping associative algebra of A.*

Proof. The proof is similar to Theorem 1. We define s and n as in Theorem 1, but use $N_i = N^i$ instead. We consider $N^i \mid N^{i+1}$ as a two-sided T_i-module by $t \cdot \bar{n} = \overline{s(t)n}$ and $\bar{n} \cdot t = \overline{ns(t)}$. Then (i), (ii), (iii) and (iv) are valid. Hence, by [9], there exist elements $x_1, \cdots, x_p \in N^i$ and $t_1, \cdots, t_p \in T_i$ such that

(v) $\overline{n(t)} = t \sum_{j=1}^{p} D_{t_j, \bar{x}_j}$ for $t \in T_i$

where $D_{t_j \bar{x}_j}$ is the inner derivation $[R_{t_j}, R_{\bar{x}_j}] + [L_{t_j}, R_{\bar{x}_j}] + [L_{t_j}, L_{\bar{x}_j}]$ of T_i into its two-sided module $N^i \mid N^{i+1}$. As in Theorem 1, we obtain

(vi) $n(t) \equiv s(t) \sum_{j=1}^{p} D_{s(t_j), x_j} (\bmod N^{i+1})$ for $t \in T_i$,

where $D_{s(t_j), x_j}$ is the inner derivation $[R_{s(t_j)}, R_{x_j}] + [L_{s(t_j)}, R_{x_j}] + [L_{s(t_j)}, L_{x_j}]$ of A.

Now let $g \in G$. Then by Lemma 5, we have $g^{-1}(D_{s(t_j), x_j})g = D_{s(t_j)g, x_j g}$. Hence, for any $g \in G$, $s(t) \sum_{j=1}^{p} D_{s(t_j)g, x_j g} = s(t)g^{-1}(\sum_{j=1}^{p} D_{s(t_j), x_j})g = s(tg^{-1})(\sum_{j=1}^{p} D_{s(t_j), x_j})g \equiv n(t) (\bmod N^{i+1})$ by (iii) and (v).

Now set $D_i = -(1/m) \sum_{g \in G} (\sum_{j=1}^{p} D_{s(t_j)g, x_j g})$, where m is the order of G. Then we have

(vii) $n(t) \equiv -s(t)D_i (\bmod N^{i+1})$ for $t \in T_i$.

D_i satisfies (3) of the Theorem since the $x_j g \in N$. To see that D_i satisfies (2) of the Theorem, we fix a value of j. Then $\sum_{g \in G} D_{s(t_j)g, x_j g} = \sum_{g \in G} g^{-1} D_{s(t_j), x_j} g$ commutes pointwise with G. Hence so does D_i, which is a linear combination of such mappings.

Now we set $U_i = \exp D_i$, and get that $T_i U_i \subseteqq S + N^{i+1}$ as in Theorem 1. Finally, we put $U = U_0 U_1 \cdots U_{k-1}$, where $N^k = 0$, and use the Campbell-Hausdorff formula to complete the proof of the

Theorem.

As in the Jordan case, we have the following two corollaries of Theorem 2.

COROLLARY 1. *Let A be a finite-dimensional alternative algebra over a field of characteristic* 0. *Let G be a finite group of automorphisms and antiautomorphisms of A. Let S and T be G-invariant maximal semisimple subalgebras of A. Then S and T are strictly conjugate via an automorphism of A of the type described in Theorem* 2.

COROLLARY 2. *Let A and G be an in Corollary* 1. *Let T be any G-invariant semisimple subalgebra of A. Then T is contained in a G-invariant maximal semisimple subalgebra of A.*

5. **The fully reducible case.** Let A be a finite-dimensional Jordan or alternative algebra over a field of characteristic zero. If G is a fully reducible group of automorphisms and antiautomorphisms of A, then it follows from [4] that G will leave invariant a maximal semisimple subalgebra of A. The analogue of Corollaries 1 has not been answered as yet for this case. However, if $N^2 = 0$, then any automorphism of the form described in the proofs of Theorems 1 and 2 which carries a G-invariant maximal semisimple subalgebra T onto another one, S, is unique, and hence will commute pointwise with G.

For let $U_1 = \exp D_1$, $U_2 = \exp D_2$ be of this form and both map T onto S. Then $D_1^2 = D_2^2 = 0$, so that $U_1 = I + D_1$, $U_2 = I + D_2$. If $t \in T$, then $tU_1 = t + tD_1 \in S$ and $tU_2 = t + tD_2 \in S$. Hence their difference $tD_1 - tD_2 \in S \cap N = 0$, since D_1 and D_2 have range in N. Hence $D_1 = D_2$ on T. Also D_1 and D_2 are both 0 on N since $N^2 = 0$. Hence $D_1 = D_2$ since $A = T + N$.

Now let $g \in G$. Then $g^{-1}U_1g = I + g^{-1}D_1g$ will map T onto S and $g^{-1}D_1g$ is a derivation of square zero having range in N. Hence, by the above, $g^{-1}D_1g = D_1$, that is, D_1, and hence U_1, commutes pointwise with G.

REFERENCES

1. N. Jacobson, *Derivation algebras and multiplication algebras of semisimple Jordan algebras*, Ann. Math. (2), **50** (1949), 866-874.

2. ———, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc. **70** (1951), 509-530.

3. ———, *Lie Algebras*, Interscience, New York, 1962.

4. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. **78** (1956), 200-221.

5. A. J. Penice, *The Wedderburn principal theorem for Jordan algebras*, Trans. Amer. Math. Soc. **70** (1951), 404-421.

6. R. D. Schafer, *The Wedderburn principal theorem for alternative algebras*, Bull. Amer. Math. Soc. **55** (1949), 604-614.
7. ———, *Inner derivations of non-associative algebras*, Bull. Amer. Math. Soc. **55** (1949), 769-776.
8. ———, *A theorem on the derivations of Jordan algebras*, Proc. Amer. Math. Soc. **2** (1951), 290-294.
9. ———, *Representations of alternative algebras*, Trans. Amer. Math. Soc. **72** (1952), 1-17.
10. E. J. Taft, *Invariant Wedderburn factors*, Illinois J. Math. **1** (1957), 565-573.
11. ———, *Uniqueness of invariant Wedderburn factors*, Illinois J. Math. **6** (1962), 353-356.
12. ———, *Invariant Levi factors*, Michigan Math. J. **9** (1962), 65-68.
13. ———, *Orthogonal conjugacies in associative and Lie algebras*, Trans. Amer. Math. Soc. **113** (1964), 18-29.

RUTGERS, THE STATE UNIVERSITY